IMT Toulouse

Skript:

Connection between Martingale Problems and Markov Processes

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Index of Notation

Probability/measure theory

$\sim$, $\overset{d}{=}$	distributed as
$\mathbb{P}_X$	distribution of X w.r.t. $\mathbb{P}$
$\mathcal{B}(\mathbb{R}^d)$	Borel σ-algebra on $\mathbb{R}^d$
δ_x	Dirac measure centered at $x \in \mathbb{R}$

Spaces of functions

$\mathcal{B}_b(\mathbb{R}^d)$	space of bounded Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}$
$C_b(\mathbb{R}^d)$	space of bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$
$C_\infty(\mathbb{R}^d)$	space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity $\lim_{\lvert x\rvert\to\infty} \lvert f(x)\rvert = 0$
$C_c^\infty(\mathbb{R}^d)$	space of smooth functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support
$D[0,\infty)$	Skorohod space = space of càdlàg functions $f : [0,\infty) \to \mathbb{R}^d$

Analysis

$\wedge$	minimum
$\lVert \cdot \rVert_\infty$	uniform norm
$\mathcal{D}(A)$	domain of operator A
càdlàg	finite left limits and right-continuous
$\mathrm{tr}(A)$	trace of A
$\nabla^2 f$	Hessian of f

1 Introduction

Central question: How to characterize stochastic processes in terms of martingale properties? Start with two simple examples: Brownian motion and Poisson process.

1.1 Definition A stochastic process $(B_t)_{t\geq 0}$ is a *Brownian motion* if

- $B_0 = 0$ almost surely,
- $B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent for all $0 = t_0 < t_1 < \ldots < t_n$ (*independent increments*),
- $B_t - B_s \sim N(0, (t-s)\,\mathrm{id})$ for all $s \leq t$,
- $t \mapsto B_t(\omega)$ is continuous for all ω (*continuous sample paths*)

1.2 Theorem (Lévy) Let $(X_t)_{t\geq 0}$ be a 1-dim. stochastic process with continuous sample paths and $X_0 = 0$. $(X_t)_{t\geq 0}$ is a Brownian motion if, and only, if $(X_t)_{t\geq 0}$ and $(X_t^2 - t)_{t\geq 0}$ are (local) martingales.

Beautiful result! Only 2 martingales needed to describe **all** the information about Brownian motion.

Remark Assumption on continuity of sample paths is needed (consider $X_t = N_t - t$ for Poisson process with intensity 1).

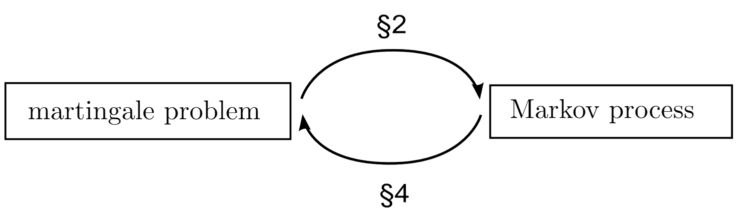
1.3 Definition A stochastic process $(N_t)_{t\geq 0}$ is a *Poisson process with intensity* $\lambda > 0$ if

- $N_0 = 0$ almost surely,
- $(N_t)_{t\geq 0}$ has independent increments,
- $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$ for all $s \leq t$,
- $t \mapsto N_t(\omega)$ is càdlàg (=right-continuous with finite left-hand limits).

1.4 Theorem (Watanabe) Let $(N_t)_{t\geq 0}$ be a counting process, i.e. a process with $N_0 = 0$ which is constant except for jumps of height +1. Then $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ if and only if $(N_t - \lambda t)_{t\geq 0}$ is a martingale.

Outline of this lecture series:

- Set up general framework to describe processes via martingales (→ martingale problems, ▸ §2)
- study connection between martingale problems and Markov processes



- application: study solutions to stochastic differential equations ▸ §3)

2
Martingale Problems

2.1 Definition Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(i). A *filtration* $(\mathcal{F}_t)_{t\geq 0}$ is a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

(ii). A stochastic process $(M_t)_{t\geq 0}$ is a *martingale (with respect to* $(\mathcal{F}_t)_{t\geq 0}$ *and* $\mathbb{P}$*)* if

- $(M_t, \mathcal{F}_t)_{t\geq 0}$ is an adapted process, i.e. X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$,
- $M_t \in L^1(\mathbb{P})$ for all $t \geq 0$,
- $\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s$ for all $s \leq t$.

Standing assumption: processes have càdlàg sample paths $t \mapsto X_t(\omega) \in \mathbb{R}^d$. Skorohod space:

$$D[0,\infty) := \{f : [0,\infty) \to \mathbb{R}^d; f \text{ càdlàg } \}.$$

Fix a linear operator $L : \mathcal{D} \to \mathcal{B}_b(\mathbb{R}^d)$ with domain $\mathcal{D} \subseteq \mathcal{B}_b(\mathbb{R}^d)$. Notation: $(L, \mathcal{D})$.

2.2 Definition Let μ be a probability measure on $\mathbb{R}^d$. A (càdlàg) stochastic process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^\mu)$ is a *solution to the* $(L, \mathcal{D})$*-martingale problem with initial distribution* μ if $\mathbb{P}^\mu(X_0 \in \cdot) = \mu(\cdot)$ and

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$$

is for any $f \in \mathcal{D}$ a martingale w.r.t to the canonical filtration $\mathcal{F}_t := \sigma(X_s; s \leq t)$ and $\mathbb{P}^\mu$.

2.3 Remark (i). For $\mu = \delta_x$ we write $\mathbb{P}^x$ instead of $\mathbb{P}^{\delta_x}$. Moreover, $\mathbb{E}^x := \int \, d\mathbb{P}^x$.

(ii). Sometimes it is convenient to fix the measurable space. On chooses $\Omega = D[0,\infty)$ Skorohod space and $X(t,\omega) = \omega(t)$ canonical process (possible WLOG, cf. Lemma 2.9 below).

Next: existence and uniqueness of solutions. Rule of thumb: Existence is much easier to prove than uniqueness.

2.1 Existence of solutions

2.4 Definition A linear operator $(L, \mathcal{D})$ satisfies *the positive maximum principle* if

$$f \in \mathcal{D}, f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geq 0 \implies Lf(x_0) \leq 0.$$

2.5 Lemma Let $(L, \mathcal{D})$ be a linear operator on $C_b(\mathbb{R}^d)$ (i.e. $\mathcal{D} \subseteq C_b(\mathbb{R}^d)$ and $L(\mathcal{D}) \subseteq C_b(\mathbb{R}^d)$). If there exists for any $x \in \mathbb{R}^d$ a solution to the $(L, \mathcal{D})$-martingale problem with initial distribution $\mu = \delta_x$, then $(L, \mathcal{D})$ satisfies the positive maximum principle.

Proof. Fix $f \in \mathcal{D}$ and $x_0 \in \mathbb{R}^d$ with $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geq 0$. Let $(X_t)_{t\geq 0}$ be a solution with $X_0 = x_0$. Then

$$0 \geq \mathbb{E}^{x_0}(f(X_t)) - f(x_0) = \mathbb{E}^{x_0}\left(\int_0^t Lf(X_s)\, ds\right),$$

and therefore it follows from the right-continuity of $s \mapsto \mathbb{E}^{x_0} Lf(X_s)$ that

$$Lf(x_0) = \lim_{t\to 0} \frac{1}{t} \int_0^t \mathbb{E}^{x_0}(Lf(X_s))\, ds \le 0. \qquad \square$$

Denote by $C_\infty(\mathbb{R}^d)$ the space of continuous functions vanishing at infinity, $\lim_{|x|\to\infty} f(x) = 0$.

2.6 Theorem ([4]) Suppose that

(i). $\mathcal{D} \subseteq C_\infty(\mathbb{R}^d)$ is dense in $\mathcal{C}_\infty(\mathbb{R}^d)$ and $L : \mathcal{D} \to C_\infty(\mathbb{R}^d)$,

(ii). L satisfies the positive maximum principle on $\mathcal{D}$,

(iii). there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{and} \quad \lim_{n\to\infty} Lf_n(x) = 0$$

for all $x \in \mathbb{R}^d$ and

$$\sup_{n\ge 1} (\|f_n\|_\infty + \|Lf_n\|_\infty) < \infty.$$

Then there exists for every probability measure μ a solution to the $(L, \mathcal{D})$-martingale problem with initial distribution μ.

2.7 Remark If (iii) fails to hold then there exists a solution which might explode in finite time.

DIY Define an operator L on the smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ by

$$\begin{aligned} Lf(x) &:= b(x)\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\nabla^2 f(x)) \\ &\quad + \int_{y\neq 0} (f(x+y) - f(x) - y\cdot\nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu(x,dy) \end{aligned}$$

where for each $x \in \mathbb{R}^d$, $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ is positive semidefinite and $\nu(x,dy)$ is a measure satisfying $\int \min\{1, |y|^2\}\,\nu(x,dy) < \infty$. Show that L satisfies the positive maximum principle. (Extra: Find sufficient conditions on b, Q, ν such that Theorem 2.6 is applicable.)

2.2 Uniqueness

Next: introduce notion of uniqueness. What is the proper notion in this context?

2.8 Definition Let $(X_t)_{t\ge0}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $(\tilde{X}_t)_{t\ge0}$ is another stochastic process (possibly on a different probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$), then $(X_t)_{t\ge0}$ equals in distribution $(\tilde{X}_t)_{t\ge0}$ if both processes have the same finite dimensional distributions, i.e.

$$\mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = \tilde{\mathbb{P}}(\tilde{X}_{t_1} \in B_1, \ldots, \tilde{X}_{t_n} \in B_n)$$

for any measurable sets B_i and any times t_i. Notation: $(X_t)_{t\ge0} \overset{d}{=} (\tilde{X}_t)_{t\ge0}$.

Mind: In general, $X_t \overset{d}{=} \tilde{X}_t$ for all $t \ge 0$ does *not* imply $(X_t)_{t\ge0} \overset{d}{=} (\tilde{X}_t)_{t\ge0}$. Consider for instance a Brownian motion $(X_t)_{t\ge0}$ and $\tilde{X}_t := \sqrt{t}X_1$.

2.9 Lemma Let $(X_t)_{t\ge0}$ be a solution to the $(L,\mathcal{D})$-martingale problem with initial distribution μ. If $(Y_t)_{t\ge0}$ is another càdlàg process such that $(X_t)_{t\ge0} \overset{d}{=} (Y_t)_{t\ge0}$, then $(Y_t)_{t\ge0}$ is a solution to the $(L,\mathcal{D})$-martingale problem with initial distribution μ.

Idea of proof. A càdlàg process $(Z_t)_{t\ge0}$ (with initial distribution μ) solves the $(L,\mathcal{D})$-martingale problem (with initial distribution μ) iff

$$\mathbb{P}^\mu\left(\left[f(Z_t) - f(Z_s) - \int_s^t Lf(Z_r)\,dr\right]\prod_{j=1}^k h_j(Z_{t_i})\right) = 0$$

for any $t_1 < \ldots < t_k \le s \le t$ and $h_j \in \mathcal{B}_b(\mathbb{R}^d)$. This is an assertion on the fdd's! (Put $Z = X$ and then replace X by Y.) $\square$

Lemma 2.9 motivates the following definition.

2.10 Definition (i). Uniqueness holds for the $(L,\mathcal{D})$-martingale problem with initial distribution μ if any two solutions have the same finite-dimensional distributions.

(ii). The $(L,\mathcal{D})$-martingale problem is *well posed* if for any initial distribution μ there exists a unique solution to the $(L,\mathcal{D})$-martingale problem with initial distribution μ.

Next result: uniqueness of the finite-dimensional distributions follows from the uniqueness of the one-dimensional distributions. Reminds us of Markov processes.

2.11 Proposition ([4, Theorem 4.4.2]) Assume that for any initial distribution μ and any two solutions $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ to the $(L,\mathcal{D})$-martingale problem with initial distribution μ it holds that

$$X_t \overset{d}{=} Y_t \qquad \text{for all} \quad t \geq 0.$$

Then uniqueness holds for any initial distribution μ, i.e. $(X_t)_{t\geq 0} \overset{d}{=} (Y_t)_{t\geq 0}$ for any two solutions $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ of the martingale problem with initial distribution μ.

2.3 Markov property

2.12 Theorem ([4, Theorem 4.4.2]) Assume that the $(L,\mathcal{D})$-martingale problem is well posed. If $(X_t)_{t\geq 0}$ is a solution to the $(L,\mathcal{D})$-martingale problem with initial distribution μ, then $(X_t)_{t\geq 0}$ satisfies the simple Markov property

$$\mathbb{E}^\mu(f(X_{s+t}) \mid \mathcal{F}_t) = \mathbb{E}^\mu(f(X_{s+t}) \mid X_t), \qquad s,t\geq 0, f\in \mathcal{B}_b(\mathbb{R}^d). \tag{2.1}$$

2.13 Remark A bit more is true. Consider the canonical framework, i.e. $\Omega = D[0,\infty)$ and $X_t(\omega) = \omega(t)$. Assume additionally to 2.12 that $x \mapsto \mathbb{P}^x(B)$ is measurable for all B. Then

$$\mathbb{E}^\mu(f(X_{s+t}) \mid \mathcal{F}_s) = (P_s f)(X_t) \quad \mathbb{P}^\mu\text{-a.s.} \tag{2.2}$$

where

$$P_s f(x) := \mathbb{E}^x f(X_s).$$

Interpretation: $(X_t)_{t\geq 0}$ is a Markov process with semigroup $(P_t)_{t\geq 0}$ (chapter 4). Note: (2.2) implies (2.1). If $\mathcal{D} \subset C_b(\mathbb{R}^d)$ we even get a strong Markov process (i.e. $t \rightsquigarrow$ finite stopping time τ).

2.14 Remark What if there is no uniqueness? Markovian selection! Consider again canonical framework,

$$\Pi^\mu := \{\mathbb{P}^\mu; (X_t,\mathbb{P}^\mu) \text{ is solution to } (L,\mathcal{D})\text{-martingale problem with initial distribution } \mu\}$$

set of solutions. Under quite weak assumptions, it is possible to choose $\mathbb{P}^\mu \in \Pi^\mu$ such that $(X_t,\mathbb{P}^\mu)$ is Markovian in the sense of (2.2), see e.g. [4, Section 4.5]. Has useful applications in analysis (e.g. for Harnack inequalities), cf. [8].

Conclusion: martingale problems are a useful tool to *construct* Markov processes.

3 Connection between SDEs and martingale problems

Aim: characterize weak solutions to SDEs of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \qquad X_0 \sim \mu \tag{3.1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion and μ a probability measure. Here for simplicity only dimension 1. Everything generalizes to higher dimensions! Standing assumption:

- $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are locally bounded,

3.1 Definition A weak solution to (3.1) is a triple (X, B), $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)_{t\geq 0}$ consisting of

- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
- a complete filtration $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$,
- a Brownian motion $(B_t, \mathcal{F}_t)_{t\geq 0}$,
- a continuous adapted process $(X_t, \mathcal{F}_t)_{t\geq 0}$

such that $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$ and

$$X_t - X_0 = \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s \quad \mathbb{P}\text{-a.s.}$$

Important: We are free to choose the probability space and the Brownian motion.

Example The SDE

$$dX_t = \operatorname{sgn}(X_t)\,dB_t, \qquad X_0 = 0,$$

has a weak solution but not a strong solution.

Idea. Existence of weak solution: Let $(W_t)_{t\geq 0}$ be a Brownian motion (on some probability space) and set

$$B_t := \int_0^t \operatorname{sgn}(W_s)\,dW_s \qquad X_t := W_t \qquad \mathcal{F}_t := \overline{\mathcal{F}_t^W}$$

By Lévy's characterization, $(B_t, \mathcal{F}_t)_{t\geq 0}$ is a Brownian motion. Moreover,

$$dB_t = \operatorname{sgn}(X_t)\,dX_t \implies dX_t = (\operatorname{sgn}(X_t))^2\,dX_t = \operatorname{sgn}(X_t)\,dB_t.$$

If there were a strong solution $(X_t)_{t\geq 0}$, then $(X_t)_{t\geq 0}$ would be a Brownian motion and $\mathcal{F}_t^X \subseteq \mathcal{F}_t^{|X|}$. This is impossible, see e.g. [11, Example 19.16]. □

Good source for further (counter)examples: [2].

Let $(X_t)_{t\geq 0}$ be a solution to (3.1). For $f \in C_b^2(\mathbb{R})$ it follows from Itô's formula that

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s \\ &= \int_0^t f'(X_s)\sigma(X_s)\,dB_s + \int_0^t Lf(X_s)\,ds \end{aligned}$$

where

$$Lf(x) := b(x)f'(x) + \frac{1}{2}a(x)f''(x)\left(= b(x)\nabla f(x) + \frac{1}{2}\mathrm{tr}(a(x)\nabla^2 f(x))\right) \tag{3.2}$$

with $a(x) := \sigma(x)^2 (= \sigma(x)\sigma(x)^T)$. Consequently,

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds = \int_0^t f'(X_s)\sigma(X_s)\,dB_s.$$

Right-hand side is martingale e.g. if f has compact support or σ is bounded. Hence:

3.2 Lemma If $(X_t)_{t\geq 0}$ is a weak solution to (3.1), then $(X_t)_{t\geq 0}$ solves the $(L, C_c^2(\mathbb{R}))$-martingale problem with initial distribution μ (with L defined in (3.2)).

Converse is also (almost) true!

3.3 Lemma If $(X_t)_{t\geq 0}$ is a continuous process which is a solution to the $(L, C_c^2(\mathbb{R}))$-martingale problem with initial distribution μ, then there exists a weak solution $(\tilde{X}_t)_{t\geq 0}$ to (3.1) with $(\tilde{X}_t)_{t\geq 0} \overset{d}{=} (X_t)_{t\geq 0}$.

Proof for $\sigma(\cdot)^2$ bounded away from 0.

(i). Since $M_t^u := u(X_t) - u(X_0) - \int_0^t Lu(X_s)\,ds$ is a martingale for $u \in C_c^2(\mathbb{R})$ it follows that $(M_t^u)_{t\geq 0}$ is a local martingale for any $u \in C^2(\mathbb{R})$. Idea: Define

$$\tau_k := \inf\{t > 0; |X_t| \geq k\}$$

and pick $f \in C_c^2(\mathbb{R})$ such that $f(x) = u(x)$ for $|x| \leq k$; then $M_{t\wedge\tau_k}^u = M_{t\wedge\tau_k}^f$ is a martingale.

(ii). By Step 1,

$$U_t := X_t - X_0 - \int_0^t b(X_s)\,ds \quad \text{and} \quad V_t := X_t^2 - X_0^2 - \int_0^t (2X_s b(X_s) + \sigma^2(X_s))\,ds$$

are local martingales. Aim: Show $\langle U\rangle_t = \int_0^t \sigma^2(X_s)\,ds$, i.e. that $U_t^2 - \int_0^t \sigma(X_s)^2\,ds$ is local martingale. Write $U_t^2 - \int_0^t \sigma^2(X_s)\,ds = N_t - R_t$ where

$$N_t := V_t - 2X_0 U_t \quad \text{local martingale!}$$

and

$$\begin{aligned} R_t &:= 2\int_0^t (X_t - X_s)b(X_s)\,ds - \left(\int_0^t b(X_s)\,ds\right)^2 \\ &= 2\int_0^t (X_t - X_s)b(X_s)\,ds - 2\int_0^t \int_s^t b(X_s)b(X_r)\,dr\,ds \\ &= 2\int_0^t (U_t - U_s)b(X_s)\,ds \\ &\overset{\text{parts}}{=} 2\int_0^t \int_0^s b(X_u)\,du\,dU_s \end{aligned}$$

is a local martingale which is of bounded variation. Hence $R = 0$, and so $U_t^2 - \int_0^t \sigma^2(X_s)\,ds = N_t$ is local martingale.

(iii). Define $W_t := \int_0^t 1/\sigma(X_s)\,dU_s$ then, by (ii),

$$\langle W\rangle_t = \int_0^t \frac{1}{\sigma^2(X_s)}\,d\langle U\rangle_s = t.$$

It follows from Lévy's characterization that $(W_t)_{t\geq 0}$ is a Brownian motion. Moreover,

$$\int_0^t b(X_s)\,ds + \underbrace{\int_0^t \sigma(X_s)\,dW_s}_{U_t} \overset{\text{Def. } U}{=} X_t - X_0. \qquad \square$$

Remark Used $\sigma^2 > 0$ only in (iii). For general σ apply martingale representation theorem to find Brownian motion $(W_t)_{t\geq 0}$ with $U_t = \int_0^t \sigma(X_s)\,dW_s$, see e.g. [4] or [6].

3.4 Corollary Let μ be a probability measure. TFAE:

(i). There exists a weak solution to the SDE (3.1).

(ii). There exists a continuous solution to the $(L, C_c^2(\mathbb{R}))$-martingale problem with initial distribution μ.

Proof. Immediate from Lemma 3.2 and 3.3. □

3.5 Corollary TFAE:

(i). For any initial distribution μ there exists a unique weak solution to the SDE (3.1).

(ii). The 'continuous' $(L, C_c^2(\mathbb{R}))$-martingale problem is well-posed.

Proof. Fix some probability measure μ.
(i) $\Longrightarrow$ (ii): Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be continuous solutions to the martingale problem. By Lemma 3.3, there exist $(\tilde{X}_t)$ and $(\tilde{Y}_t)$ solving the SDE (3.1) such that $(X_t)_{t\geq 0} \overset{d}{=} (\tilde{X}_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0} \overset{d}{=} (\tilde{Y}_t)_{t\geq 0}$. Hence, by (i),

$$(X_t)_{t\geq 0} \overset{d}{=} (\tilde{X}_t)_{t\geq 0} \overset{(i)}{=} (\tilde{Y}_t)_{t\geq 0} \overset{d}{=} (Y_t)_{t\geq 0}.$$

(ii) $\Longrightarrow$ (i): similar reasoning, use Lemma 3.2 (simpler than first part). □

Note that Corollary 3.4 and Corollary 3.5 do not require any regularity assumptions on b and σ.

3.6 Corollary Assume that the coefficients $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ of the SDE (3.1) are continuous and bounded. Then there exists for any initial distribution μ a (non-explosive) weak solution to (3.1).

Proof. By Corollary 3.4 it suffices to show that there exists a solution to the $(L, C_c^2(\mathbb{R}))$-martingale problem with initial distribution μ. To this end, we apply Theorem 2.6. Check assumptions:

- $C_c^2(\mathbb{R})$ is dense in $C_\infty(\mathbb{R})$ and $Lf \in C_\infty(\mathbb{R})$ for any $f \in C_c^2(\mathbb{R})$,
- L satisfies the positive maximum principle, cf. Section 2.1
- Choose $(f_n)_{n\in\mathbb{N}} \subseteq C_c^2(\mathbb{R})$ such that $0 \leq f_n \leq 1$, $\sup_n \|f_n\|_{C^2} < \infty$ and

$$f_n(x) = \begin{cases} 1, & |x| \leq n, \\ 0, & |x| \geq n+1 \end{cases}$$

Then $\lim_{n\to\infty} f_n = 1$,

$$\lim_{n\to\infty} f_n'(x) = 0, \lim_{n\to\infty} f_n''(x) = 0 \implies \lim_{n\to\infty} Lf_n = 0$$

and

$$\sup_{n\geq 1} (\|f_n\|_\infty + \|Lf_n\|_\infty) \leq 1 + (\|b\|_\infty + \|\sigma\|_\infty) \sup_n \|f_n\|_{C_b^2} < \infty$$

□

3.7 Remark The above results can be extended to SDEs of the form

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dB_t + g(X_{t-})\,dJ_t \tag{3.3}$$

where $(J_t)_{t\geq 0}$ is a jump Lévy process. Need to replace L by

$$\begin{aligned} Lf(x) &= b(x)\nabla f(x) + \frac{1}{2}\mathrm{tr}(a(x)\nabla^2 f(x)) \\ &= + \int_{y\neq 0} (f(x+g(x)y) - f(x) - \nabla f(x)\cdot(g(x)y)\mathbb{1}_{(0,1)}(|y|))\,\nu(dy) \end{aligned}$$

where ν is the Lévy measure of $(J_t)_{t\geq 0}$. See [10] for details. Corollary 3.4,3.5 remain valid (without 'continuous'), cf. [10], and similar to Corollary 3.6 we get a general existence result for weak solutions to (3.3), see e.g. [7, Theorem 3.4(i)].

4 Markov processes

4.1 Semigroups

Throughout, $(\Omega, \mathcal{F})$ is a measurable space.

4.1 Definition A Markov process is a tuple $(X_t, \mathcal{F}_t, \mathbb{P}^x)$ consisting of

- a filtration $(\mathcal{F}_t)_{t\geq 0}$,
- an adapted $\mathbb{R}^d$-valued càdlàg process $(X_t, \mathcal{F}_t)_{t\geq 0}$,
- a family of probability measures $\mathbb{P}^x$, $x \in \mathbb{R}^d$

such that the Markov property

$$\mathbb{E}^x(u(X_t) \mid \mathcal{F}_s) = \mathbb{E}^{X_s} u(X_{t-s}) \quad \mathbb{P}^x\text{-a.s.} \tag{4.1}$$

holds for any $s \leq t$, $x \in \mathbb{R}^d$ and $u \in \mathcal{B}_b(\mathbb{R}^d)$. (Implicitly: everything is measurable.) We call

$$P_t u(x) := \mathbb{E}^x u(X_t), \qquad t \geq 0, x \in \mathbb{R}^d, u \in \mathcal{B}_b(\mathbb{R}^d)$$

the semigroup associated with $(X_t)_{t\geq 0}$.

4.2 Proposition Let $(P_t)_{t\geq 0}$ be the semigroup associated with a Markov process $(X_t)_{t\geq 0}$.

(i). $P_{t+s} = P_t P_s$ for all $s \leq t$ (semigroup property)

(ii). $0 \leq P_t u \leq 1$ for any $0 \leq u \leq 1$ (sub Markov property)

(iii). $P_t 1 = 1$ (conservative)

Proof. (i). Fix $u \in \mathcal{B}_b(\mathbb{R}^d)$. By the Markov property (4.1) and the tower property of conditional expectation, we have

$$\begin{aligned} P_{t+s}u(x) = \mathbb{E}^x u(X_{t+s}) = \mathbb{E}^x\Big[\mathbb{E}^x(u(X_{t+s}) \mid \mathcal{F}_t)\Big] &= \mathbb{E}^x\Big[\mathbb{E}^{X_t} u(X_s)\Big] \\ &= P_t P_s u(x). \end{aligned}$$

(ii). Clear from $P_t u(x) = \mathbb{E}^x u(X_t)$

(iii). $P_t 1(x) = \mathbb{E}^x 1_{X_t} = 1$ (since we assume $X_t \in \mathbb{R}^d$ for all $t \geq 0$).

□

4.3 Definition A Markov process $(X_t)_{t\geq 0}$ is called a *Feller process* if the associated semigroup $(P_t)_{t\geq 0}$ is a *Feller semigroup*, i.e.

(i). $P_t f \in C_\infty(\mathbb{R}^d)$ for any $t \geq 0$, $f \in C_\infty(\mathbb{R}^d)$ (Feller property)

(ii). $\|P_t f - f\|_\infty \to 0$ as $t \to 0$ for any $f \in C_\infty(\mathbb{R}^d)$ (strong continuity at $t = 0$).

4.4 Example Any Lévy process, i.e. any process with stationary and independent increments, is a Feller process. Examples: Brownian motion, compound Poisson process, ...

4.5 Example Let $(B_t)_{t\geq 0}$ be a Brownian motion and let b, σ be bounded continuous mappings. If the SDE

$$X_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \qquad X_0 = x$$

has a unique weak solution for any $x \in \mathbb{R}^d$, then the solution $(X_t)_{t\geq 0}$ is a Feller process.

Remark (i). The assumption on boundedness of b, σ can be relaxed, see e.g. [1, Theorem 3.6].

(ii). Example 4.5 remains valid if we replace B by a Lévy process, see [7] for a detailed discussion.

4.2 Infinitesimal generator

Aim: Characterize semigroup $(P_t)_{t\geq 0}$ by *one* operator. Idea:

$$P_{t+s} = P_t P_s$$

is "operator version" of Cauchy–Abel equation

$$\phi(t+s) = \phi(t)\phi(s), \qquad s,t \geq 0, \qquad \phi \text{ cts} \tag{4.2}$$

Any solution to (4.2) is of the form $\phi(t) = \phi(0)e^{at}$ for some constant $a > 0$. Hence, we expect

$$P_t = e^{tA} \qquad\qquad A = \frac{d}{dt} P_t \Big|_{t=0} \tag{4.3}$$

for some (unbounded) operator A. How to make sense of (4.3)?

4.6 Definition Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$. Then the operator defined by

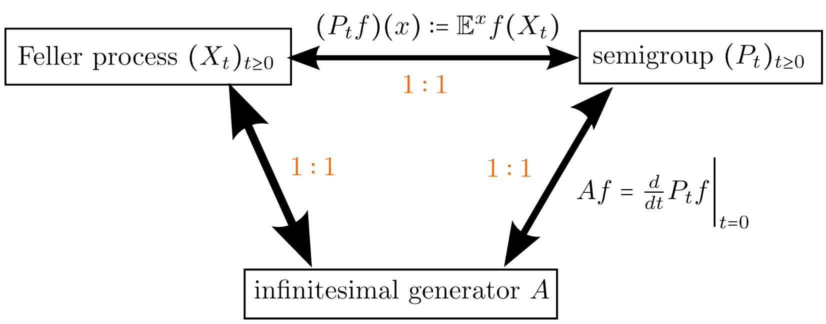
$$\mathcal{D}(A) := \left\{ u \in C_\infty(\mathbb{R}^d); \exists g \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0} \left\| \frac{P_t u - u}{t} - g \right\|_\infty = 0 \right\}$$

$$Au := \lim_{t\to 0} \frac{P_t u - u}{t}, \qquad u \in \mathcal{D}(A)$$

is called infinitesimal generator of $(X_t)_{t\geq 0}$ (and $(P_t)_{t\geq 0}$).

Remark (i). In general, it is difficult to determine $\mathcal{D}(A)$. Typically one tries to find "nice" function spaces contained in $\mathcal{D}(A)$, e.g. $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$.

(ii). $\mathcal{D}(A)$ is not "too small" since $\mathcal{D}(A)$ is dense in $C_\infty(\mathbb{R}^d)$, i.e. for any $u \in C_\infty(\mathbb{R}^d)$ there exists $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $\|f_n - u\|_\infty \to 0$.



4.7 Proposition Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$ and infinitesimal generator $(A, \mathcal{D}(A))$. Then

(i). If $u \in \mathcal{D}(A)$ then $P_t u \in \mathcal{D}(A)$ for all $t > 0$ and

$$\frac{d}{dt} P_t u = AP_t u = P_t A u, \qquad t > 0.$$

(ii). For any $u \in \mathcal{D}(A)$ it holds that

$$P_t u - u = \int_0^t P_s A u \, ds = \int_0^t A P_s u \, ds. \tag{4.4}$$

Sketch of proof. (i). Fix $u \in \mathcal{D}(A)$ and $t > 0$. For $s > 0$

$$\left\| \frac{P_s P_t u - P_t u}{s} - P_t A u \right\|_\infty \le \left\| \frac{P_s u - u}{s} - A u \right\|_\infty \xrightarrow{s \to 0} 0.$$

This shows $P_t u \in \mathcal{D}(A)$, $A P_t u = P_t A u$ and $\lim_{h \downarrow 0} \frac{P_{t+h} u - P_t u}{h} = P_t A u$. Similar calculation for $s < 0$ gives $\lim_{h \downarrow 0} \frac{P_{t-h} u - P_t u}{-h} = P_t A u$. Hence, $d/dt P_t u = A P_t u$.

(ii). By (i) and the fundamental theorem of calculus,

$$P_t u - u = \int_0^t \frac{d}{ds} P_s u \, ds = \int_0^t P_s A u \, ds. \quad \square \tag{4.5}$$

Write (4.5) in probabilistic way:

$$\mathbb{E}^x u(X_t) - u(x) = \int_0^t \mathbb{E}^x A u(X_s) \, ds, \qquad u \in \mathcal{D}(A), x \in \mathbb{R}^d, t \ge 0. \tag{4.6}$$

First glimpse of martingale problem!

$$M_t^u := u(X_t) - u(X_0) - \int_0^t A u(X_s) \, ds$$

has constant expectation w.r.t. to $\mathbb{P}^x$.

4.3 From Markov processes to martingale problems

4.8 Proposition Let $(X_t, \mathbb{P}^x)$ be a Feller process with generator $(A, \mathcal{D}(A))$, and let $x \in \mathbb{R}^d$. Then $(X_t)_{t \ge 0}$ is (w.r.t. to $\mathbb{P}^x$) a solution to the $(A, \mathcal{D}(A))$-martingale problem with initial distribution δ_x.

Proof. Fix $u \in \mathcal{D}(A)$. Need to show that $(M_t^u)_{t \ge 0}$ is martingale. By definition,

$$\mathbb{E}^x(M_t^u \mid \mathcal{F}_s) = \mathbb{E}^x\left(u(X_t) - \int_s^t A u(X_r) \, dr \mid \mathcal{F}_s\right) - u(x) - \int_0^s A u(X_r) \, dr \overset{!}{=} M_s^u$$

for $s \le t$, i.e. need to show

$$\mathbb{E}^x\left(u(X_t) - \int_s^t A u(X_r) \, dr \mid \mathcal{F}_s\right) = u(X_s).$$

Use Markov property:

$$\begin{aligned}
\mathbb{E}^x\left(u(X_t) - \int_s^t A u(X_r) \, dr \mid \mathcal{F}_s\right) &= \mathbb{E}^x\left(u(X_t) \mid \mathcal{F}_s\right) - \mathbb{E}^x\left(\int_0^{t-s} A u(X_{s+h}) \, dh \mid \mathcal{F}_s\right) \\
&= \mathbb{E}^{X_s}(u(X_{t-s})) - \int_0^{t-s} \mathbb{E}^{X_s}(A u(X_h)) \, dh \\
&\overset{(4.6), x = X_s}{=} u(X_s).
\end{aligned}$$

$\square$

4.9 Corollary (Dynkin's formula) Let $(X_t, \mathbb{P}^x)$ be a Feller process with generator $(A, \mathcal{D}(A))$. Then

$$\mathbb{E}^x f(X_\tau) - f(x) = \mathbb{E}^x\left(\int_{(0,\tau)} A f(X_s) \, ds\right)$$

for any stopping time τ with $\mathbb{E}^x \tau < \infty$.

Proof. Use Proposition 4.8 and optional stopping. $\square$

See [1] for many interesting applications of this formula, e.g. a maximal inequality for Feller processes.

More difficult: uniqueness of solution to martingale problem.

4.10 Theorem Let $(X_t)_{t\geq 0}$ be a Feller process with generator $(A, \mathcal{D}(A))$. Let $\mathcal{D} \subseteq \mathcal{D}(A)$ be a core of $(A, \mathcal{D}(A))$, i.e. for any $f \in \mathcal{D}(A)$ there is $(f_n)_{n\in\mathbb{N}}$ such that $\|f_n - f\|_\infty \to 0$ and $\|Af_n - Af\|_\infty \to 0$. Then the $(A, \mathcal{D})$-martingale problem is well-posed.

4.11 Remark $\mathcal{D} := \mathcal{D}(A)$ is a core for $(A, \mathcal{D}(A))$, and therefore Theorem 4.10 shows, in particular, that the $(A, \mathcal{D}(A))$-martingale problem is well-posed.

Which operators appear as generators of Feller processes? How do generators of Feller processes look like?

4.12 Theorem ([1, Theorem 2.21]) Let $(X_t)_{t\geq 0}$ be a Feller process with generator $(A, \mathcal{D}(A))$. If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ then $A|_{C_c^\infty(\mathbb{R}^d)}$ has a representation of the form

$$Af(x) = b(x)\nabla f(x) + \frac{1}{2}\mathrm{tr}(Q(x)\nabla^2 f(x)) + \int_{y\neq 0}(f(x+y) - f(x) - y \cdot \nabla f(x)\mathbb{1}_{(0,1)}(|y|))\,\nu(x, dy)$$

where $(b(x), Q(x), \nu(x, dy))$, $x \in \mathbb{R}^d$, is the so-called *characteristics* consisting of

- $b(x) \in \mathbb{R}^d$
- $Q(x) \in \mathbb{R}^{d\times d}$ positive semidefinite
- $\nu(x, dy)$ measure satisfying $\int \min\{1, |y|^2\}\,\nu(x, dy) < \infty$.

Interpretation of Theorem 4.12:

$$X_t \approx \int_0^t b(X_{s-})\,ds + \int_0^t \sqrt{Q}(X_{s-})\,dB_s + \text{jump part} \tag{4.7}$$

The jump part comes from the family $(\nu(x, dy))$:

4.13 Proposition ([9]) Let $(X_t)_{t\geq 0}$ be a Feller process with characteristics $(b(x), Q(x), \nu(x, dy))$. Then

$$\lim_{t\to 0} \frac{\mathbb{P}^x(X_t \in x + A)}{t} = \nu(x, A), \qquad A \in \mathcal{B}(\mathbb{R}^d\backslash\{0\}).$$

Interpretation:

$$\lim_{t\to 0} \frac{\mathbb{P}^x(X_t \in x + A)}{t} = \text{(scaled) probability to move very quickly from } X_0 = x \text{ into the set } x + A$$

Observation: diffusion part doesn't play a role, i.e. diffusion is moving too slowly. Only possibility: jumps.

$$\nu(x, A) = \text{likelihood that process jumps from } X_0 = x \text{ to } x + A$$

More generally

$$\nu(X_{s-}, A) = \text{likelihood that process jumps current position } X_{s-} \text{ to } X_{s-} + A$$

For rigorous statement of (4.7) see [1, Proposition 3.10] or [3].

4.14 Remark Many open questions on existence of Feller processes and associated martingale problems, e.g.

- Under which assumptions on $(b(x), Q(x), \nu(x, dy))$ does there exist a Feller process with characteristics $(b(x), Q(x), \nu(x, dy))$?
- Under which assumptions is $C_c^\infty(\mathbb{R}^d)$ a core for the generator? (important for well-posedness, cf. Theorem 4.12)

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