

**FOR WHICH FUNCTIONS ARE $f(X_t) - \mathbb{E}f(X_t)$ AND $g(X_t)/\mathbb{E}g(X_t)$
MARTINGALES?**

FRANZISKA KÜHN AND RENÉ L. SCHILLING

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ABSTRACT. Let $X = (X_t)_{t \geq 0}$ be a one-dimensional Lévy process such that each X_t has a C_b^1 -density w.r.t. Lebesgue measure and certain polynomial or exponential moments. We characterize all polynomially bounded functions $f : \mathbb{R} \rightarrow \mathbb{R}$, and exponentially bounded functions $g : \mathbb{R} \rightarrow (0, \infty)$, such that $f(X_t) - \mathbb{E}f(X_t)$, resp. $g(X_t)/\mathbb{E}g(X_t)$, are martingales.

In a series of papers [11, 12, 13] M. Mania and R. Tevzadze studied the interplay of certain Brownian martingales of the form $M_t = f(B_t) - \mathbb{E}f(B_t)$ and $N_t = g(B_t)/\mathbb{E}g(B_t)$, respectively, and Cauchy's functional equation; in particular, they characterized all functions f and g such that M and N are indeed martingales. Their approach is mainly analytic, based on the solution of a certain 'backward' heat equation. In this short note, we provide several alternative, mainly probabilistic, proofs which (i) recover for Brownian motion the results by Mania and Tevzadze, and (ii) extend to the more general class of Lévy processes.

Our main result can be summarized as follows. Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process with a strictly positive transition density:

- if the transition density is of class C_b^1 , if f is polynomially bounded, and if X_t has certain polynomial moments such that $\mathbb{E}f(X_t)$ exists, then the process $M_t = f(X_t) - \mathbb{E}f(X_t)$ is a martingale if, and only if, f is a.e. a second-order polynomial (Theorems 3.1 and 3.2).
- if g is a positive function which grows at most exponentially, and if X_t has exponential moments such that $\mathbb{E}g(X_t)$ exists, then the process $N_t = g(X_t)/\mathbb{E}g(X_t)$ is a martingale if, and only if, $g(x) = ae^{\lambda_1 x} + be^{\lambda_2 x}$ a.e. for suitable coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$ and $a, b \geq 0$. This still holds, if X_t is a one-sided Lévy process (Theorems 3.3 and 3.4)

Note that all of these conditions trivially hold for Brownian motion, i.e. we recover the results by Mania and Tevzadze.

1. LÉVY PROCESSES

A stochastic process $X = (X_t)_{t \geq 0}$ with values in \mathbb{R} is a *Lévy process*, if $t \mapsto X_t$ is right-continuous with finite left-hand limits (càdlàg), and if its increments are independent and stationary random variables, i.e.

$$(X_{t_k} - X_{t_{k-1}})_{k=1, \dots, n} \quad \text{are independent and} \quad X_{t_k} - X_{t_{k-1}} \sim X_{t_k - t_{k-1}} \quad (1)$$

for any choice of $n \in \mathbb{N}$ and $t_0 = 0 \leq t_1 \leq \dots \leq t_n < \infty$. Note that this entails $X_0 \sim \delta_0$, i.e. $X_0 = 0$ a.e. If $\mathcal{F}_t := \sigma(X_s, s \leq t)$ denotes the natural filtration, we can express (1) equivalently in the following form:

$$\mathbb{E} \left(e^{i\xi(X_{t+s} - X_s)} \mid \mathcal{F}_s \right) = e^{-t\psi(\xi)}, \quad s, t \geq 0, \xi \in \mathbb{R}, \quad (2)$$

where $\psi(\xi)$ is the *characteristic exponent* which is given by the Lévy–Khintchine representation

$$\psi(\xi) = -ib\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{y \neq 0} (1 - e^{iy\xi} + iy\xi\mathbf{1}_{(0,1)}(|y|))\nu(dy). \quad (3)$$

The exponent ψ determines (the finite dimensional distributions of) X uniquely, and ψ is uniquely determined by the *Lévy triplet* (b, σ^2, ν) comprising the drift $b \in \mathbb{R}$, the diffusion coefficient $\sigma^2 \geq 0$

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and the Lévy measure ν , i.e. a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{y \neq 0} \min\{y^2, 1\} \nu(dy) < \infty$. The standard reference for Lévy processes is Sato [17], an introduction is given in [18].

Any Lévy process is a Markov process and the transition semigroup is given for any bounded Borel function $u : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_t u(x) = \mathbb{E}^x u(X_t) = \mathbb{E}u(X_t + x); \quad (4)$$

notice the convolution structure of $T_t u = \tilde{\mu}_t * u$, where $\tilde{\mu}_t$ is the law of $-X_t$, which is due to the independent and stationary increment property (1). If u is smooth enough, say if u is in the L. Schwartz space of rapidly decreasing, C^∞ -functions $u \in \mathcal{S}(\mathbb{R})$, then we can use the Fourier transform to determine the infinitesimal generator of $(X_t)_{t \geq 0}$ or $(T_t)_{t \geq 0}$ as follows: write $\widehat{u}(\xi) = (2\pi)^{-1} \int e^{-ix\xi} u(x) dx$ and $\check{u}(x) = \int e^{i\xi x} u(\xi) d\xi$ for the Fourier and the inverse Fourier transform, respectively. Then

$$\widehat{Au} = \lim_{t \rightarrow 0} \frac{1}{t} (\widehat{T_t u} - \widehat{u}) = \lim_{t \rightarrow 0} \frac{1}{t} (e^{-t\psi} \widehat{u} - \widehat{u}) = -\psi \cdot \widehat{u}, \quad (5)$$

i.e. $Au = -\psi(D)u = (-\psi \cdot \widehat{u})^\vee$ is a pseudo-differential operator with symbol $-\psi$. Combining this with the Lévy–Khintchine formula we easily see that

$$Au(x) = bu'(x) + \frac{1}{2} \sigma^2 u''(x) + \int_{y \neq 0} [u(x+y) - u(x) - yu'(x) \mathbf{1}_{(0,1)}(|y|)] \nu(dy). \quad (6)$$

If we consider Lévy processes and their semigroups in the Banach space $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$ of continuous functions vanishing at infinity, which is naturally equipped with the uniform convergence, then the (strong) generator A has the domain

$$\begin{aligned} \mathcal{D}(A) &= \left\{ u \in C_\infty(\mathbb{R}); \|\cdot\|_\infty\text{-}\lim_{t \rightarrow 0} \frac{1}{t} (T_t u - u) \text{ exists} \right\} \\ &= \left\{ u \in C_\infty(\mathbb{R}); \lim_{t \rightarrow 0} \frac{1}{t} (T_t u(x) - u(x)) = g(x) \text{ and } g \in C_\infty(\mathbb{R}) \right\}. \end{aligned}$$

We can use this representation to extend A to twice differentiable functions $u \in C_b^2(\mathbb{R})$, and beyond. Assume that $\int_{|y| \geq 1} |y|^n \nu(dy) < \infty$, then it follows from Taylor's formula and (6) that $Au(x)$ exists if $u''(x)$ grows at most like $|x|^{n-2}$ as $|x| \rightarrow \infty$. In fact, it is possible to show that for $g(x) = |x|^n$

$$\forall t : \mathbb{E}g(X_t) < \infty \iff \int_{|y| \geq 1} g(y) \nu(dy) < \infty \iff \forall \phi \in C_c^\infty(\mathbb{R}) : A^* \phi \in L^1(g(x) dx), \quad (7)$$

cf. e.g. [3, Theorem 3]. Here, A^* is the (formal) adjoint of the operator A . Since $\widehat{A\phi} = -\psi \widehat{\phi}$, it is not hard to see that A^* is the generator of the Lévy process $(-X_t)_{t \geq 0}$ and that it is a pseudo-differential operator with the conjugate-complex symbol $-\overline{\psi(\xi)}$. Using (7) we can extend, in a weak sense, the generator A to the space of polynomially bounded Borel functions $B(\mathbb{R})$ via

$$\langle Af, \phi \rangle := \langle f, A^* \phi \rangle, \quad \phi \in C_c^\infty(\mathbb{R}).$$

We will call this the *weakly extended generator*. Write $g(x) = \text{bp-}\lim_{t \rightarrow 0} \frac{1}{t} (T_t f(x) - f(x))$ if the limit exists pointwise for each x and $\sup_x \frac{1}{t} |T_t f(x) - f(x)| < \infty$ ('bp' stands for 'boundedly pointwise'). If $f \in B(\mathbb{R})$ is polynomially bounded, then $g = Af$ for the weakly extended generator. Indeed, denote by $(T_t^*)_{t \geq 0}$ the semigroup associated with $(-X_t)_{t \geq 0}$. Applying Dynkin's formula, and then Fubini's theorem, yields that

$$\begin{aligned} \left\langle \frac{T_t f - f}{t}, \phi \right\rangle &= \left\langle f, \frac{T_t^* \phi - \phi}{t} \right\rangle = \frac{1}{t} \int_0^t \langle f, T_s^* A^* \phi \rangle ds = \frac{1}{t} \int_0^t \langle T_s f, A^* \phi \rangle ds \\ &= \int_{\mathbb{R}} \frac{1}{t} \int_0^t \mathbb{E}f(x + X_s) ds A^* \phi(x) dx. \end{aligned} \quad (8)$$

Note that Fubini's theorem is applicable because of (7) and the polynomial boundedness of f ; the latter implies that

$$\left| \frac{1}{t} \int_0^t \mathbb{E}f(x + X_s) ds \right| \leq \sup_{s \leq t} |\mathbb{E}f(x + X_s)| \leq C(1 + |x|)^n \sup_{s \leq t} \mathbb{E}((1 + |X_s|)^n) < \infty,$$

see e.g. [3, Theorem 3]. Because of the existence of the bp-limit, the mapping $s \mapsto \mathbb{E}f(x + X_s) = T_s f(x)$ is continuous at $s = 0$, and so

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathbb{E}f(x + X_s) ds = f(x).$$

Using the previous estimate and (7), we can let $t \rightarrow 0$ in (8) using dominated convergence and get $\langle g, \phi \rangle = \langle f, A^* \phi \rangle$.

2. SOME AUXILIARY RESULTS

In this section we collect a few results which will be used in the proof of the main results.

2.1. Lemma. *Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process with characteristic exponent ψ such that $\mathbb{E}[|X_t|^n] < \infty$ for some $n \in \mathbb{N}$. If $p : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial of degree $\deg(p) = n$, then $q(t) := \mathbb{E}[p(X_t)]$ is a polynomial of degree $\deg(q) \leq n$.*

Proof. It is enough to show the assertion for $p(x) = x^n$. Since X_t has n th moments, we can use the characteristic function to see

$$q(t) = \mathbb{E}[X_t^n] = \frac{1}{i^n} \partial_\xi^n \mathbb{E}[e^{i\xi X_t}] \Big|_{\xi=0} = \frac{1}{i^n} \partial_\xi^n e^{-t\psi(\xi)} \Big|_{\xi=0}.$$

An application of Faa di Bruno's formula yields

$$\partial_\xi^n e^{-t\psi(\xi)} = \sum_{k=1}^n (-t)^k e^{-t\psi(\xi)} \sum_{\substack{i_1=0 \\ \vdots \\ i_n=0 \\ i_1 + i_2 + \dots + i_n = k \\ i_1 + 2i_2 + \dots + ni_n = n}} \dots \sum_{i_n=0}^k \frac{n!}{i_1! \cdot i_2! \cdot \dots \cdot i_n!} \prod_{j=1}^n \left(\frac{\partial^j \psi(\xi)}{j!} \right)^{i_j}.$$

Since $\psi(0) = 0$, we conclude that $\deg(q) \leq n$. \square

The key technique in the proof of the main result is the use of Liouville-type results for the generator of certain Lévy processes. The following theorem is due to one of us [9, Thm. 1]. We state here only the one-dimensional case, but it holds in any dimension.

2.2. Theorem (Liouville). *Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process such that each X_t has a transition density $p_t \in C_b^1(\mathbb{R})$ and $\mathbb{E}[|X_t|^{n+\epsilon}] < \infty$ for some $n \in \mathbb{N}$ and $\epsilon > 0$. Denote by A the infinitesimal generator of $(X_t)_{t \geq 0}$. If $u : \mathbb{R} \rightarrow \mathbb{R}$ is polynomially bounded such that $|u(x)| \leq c(1+|x|^n)$ and a weak solution to the equation $Au = 0$, then u is a polynomial of degree $\deg(u) \leq n$.*

If the characteristic exponent ψ of the Lévy process $(X_t)_{t \geq 0}$ is smooth, then a standard result shows $\mathbb{E}[X_t^{2k}] < \infty$ for all $k \in \mathbb{N}$, cf. [20, §2.12, Thm. 1, p. 334] or [4, Lemma 4], and so u can be of arbitrary polynomial growth. We give a slightly different version of Liouville's theorem and present an analytical proof. Recall that $\mathcal{S}(\mathbb{R})$ are the rapidly decreasing, smooth L. Schwartz functions and their dual space $\mathcal{S}'(\mathbb{R})$ are the tempered distributions.

2.3. Lemma. *Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process with characteristic exponent ψ . Moreover, assume that $\psi \in C^\infty(\mathbb{R})$ is smooth and $\xi = 0$ is the only zero of $\psi(\xi)$. Then $\mathbb{E}[|X_t|^n] < \infty$ for all $n \in \mathbb{N}$, and any polynomially bounded measurable function $u : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $Au = c$ weakly is a polynomial.*

It is a classic result that X_t has a [possibly degenerate] lattice distribution if, and only if, ψ has a zero $\xi_0 \neq 0$. In this case, ψ is a [degenerate, i.e. constant] periodic function, cf. Sato [17, Section 24] or [3, Theorem 10]. The following proof works almost literally for any dimension $d \geq 1$, but we need only the case $d = 1$.

Proof. By assumption, the characteristic function $\mathbb{E}e^{i\xi X_t} = e^{-t\psi(\xi)}$ is of class $C^{2k}(\mathbb{R})$ for any $k \in \mathbb{N}$, and so $\mathbb{E}[X_t^{2k}] < \infty$. It is not hard to see with the help of the Lévy–Khintchine formula, cf. [4, Lemma 4], that a smooth Lévy exponent ψ and all of its derivatives $\partial^n \psi$ are polynomially bounded. In particular, $\overline{\psi} \cdot \phi$ is for every Schwartz function $\phi \in \mathcal{S}(\mathbb{R})$ again a Schwartz function.

Since u is polynomially bounded, the inverse Fourier transform \tilde{u} is well-defined as a tempered distribution. For any $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$\langle Au, \phi \rangle = \langle u, A^* \phi \rangle = -\langle u, \overline{\psi}(D)\phi \rangle = -\langle \tilde{u}, \overline{\psi}\widehat{\phi} \rangle.$$

In the last equality we use Plancherel's theorem for the canonical dual pairing of $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$; here we need that $\overline{\psi} \cdot \phi \in \mathcal{S}(\mathbb{R})$. On the other hand, $Au = c$ weakly gives

$$-\langle \check{u}, \overline{\psi\phi} \rangle = \langle Au, \phi \rangle = \int_{\mathbb{R}} c\phi(x) dx = 2\pi c\widehat{\phi}(0) = \langle 2\pi c\delta_0, \widehat{\phi} \rangle.$$

Since this holds for all $\phi \in \mathcal{S}(\mathbb{R})$, and since the Fourier transform is a bijection on $\mathcal{S}(\mathbb{R})$, we conclude from this that $\check{u}\overline{\psi} = -2\pi c\delta_0$. Now we can use the fact that $\xi = 0$ is the only zero of ψ . Therefore, \check{u} has support $\text{supp } u \subset \{0\}$, and a standard structural result for distributions supported in $\{0\}$, see e.g. Rudin [16, Thm. 6.25], shows that

$$\check{u} = \sum_{k=0}^N c_k \partial^k \delta_0$$

for some $N \in \mathbb{N}$. From this we get by Fourier inversion that $u(x) = (2\pi)^{-1} \sum_{k=0}^N c_k (-i)^k x^k$. \square

The following theorem shows that the polynomial u appearing in Theorem 2.2 or Lemma 2.3 has at most degree 2.

2.4. Lemma. *Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process such that $\mathbb{E}[|X_t|^{n_0}] < \infty$ for some $n_0 \in \mathbb{N}$. Assume that $(X_t)_{t \geq 0}$ is non-trivial, i.e. $X_t \not\equiv 0$. If p is a polynomial of degree $\deg(p) \leq n_0$ such that $Ap \equiv \text{const.}$, then $\deg(p) \leq 2$.*

Proof. Let (b, σ^2, ν) be the Lévy triplet of the characteristic exponent ψ . Since $\mathbb{E}[|X_t|^{n_0}] < \infty$ entails that $\int_{|y| \geq 1} |y|^{n_0} \nu(dy) < \infty$, see e.g. [3, Thm. 3], we can use the integro-differential form of the generator

$$Au(x) = bu'(x) + \frac{1}{2}\sigma^2 u''(x) + \int_{y \neq 0} [u(x+y) - u(x) - yu'(x)\mathbb{1}_{(0,1)}(|y|)] \nu(dy)$$

to extend A to all polynomials of degree less or equal than n_0 . For $n \leq n_0$ we get

$$\begin{aligned} Ax^n &= bnx^{n-1} + \frac{1}{2}\sigma^2 n(n-1)x^{n-2} + \sum_{k=2}^n \binom{n}{k} x^{n-k} \int_{|y| < 1} y^k \nu(dy) + \sum_{k=1}^n \binom{n}{k} x^{n-k} \int_{|y| \geq 1} y^k \nu(dy) \\ &= (b + M_1)nx^{n-1} + \binom{n}{2} (\sigma^2 + m_2 + M_2)x^{n-2} + \sum_{k=3}^n \binom{n}{k} (m_k + M_k)x^{n-k}, \end{aligned}$$

where we use the shorthand notation $m_k = \int_{|y| < 1} y^k \nu(dy)$ and $M_k = \int_{|y| \geq 1} y^k \nu(dy)$.

Now let $p(x) = a_n x^n + \dots + a_1 x + a_0$ be a polynomial such that $Ap(x) \equiv c$. Without loss of generality we may assume that $a_n = 1$. Then

$$Ap(x) = (b + M_1)nx^{n-1} + \binom{n}{2} (\sigma^2 + m_2 + M_2)x^{n-2} + \sum_{k=0}^{n-3} c_k x^k$$

for some coefficients $c_k \in \mathbb{R}$. If $\deg(p) = n \geq 3$, then it follows from $Ap(x) \equiv c$ by a comparison of coefficients that $\sigma^2 + m_2 + M_2 = 0$, implying $\sigma^2 = 0$ and $\nu = 0$, and $0 = b + M_1 = b$. Hence, $X_t \equiv 0$ is the trivial Lévy process. In other words, if $(X_t)_{t \geq 0}$ is non-trivial, then any polynomial p with $Ap(x) \equiv c$ satisfies $\deg(p) \leq 2$. \square

Denote by $\Delta_y f(x) := f(x+y) - f(x)$ the difference with step y and by $\Delta_y^{n+1} f := \Delta_y(\Delta_y^n f)$ the iterated differences. The next lemma can be reduced to the form $\Delta_y^n f = 0$ and then be derived from what is known in the literature as *Fréchet's functional equation* or *Cauchy's generalized functional equation*, cf. [1, pp. 129–130] for an overview on the literature (but without proofs). For continuous solutions f , the earliest proof is due to Anghelutza [2], a proof via approximation can be found in Butzer & Kozakiewicz [5]. The following short and elementary argument seems to be new.

2.5. Lemma. *Let \mathcal{P}_n be the family of polynomials of degree n . Any continuous solution of the problem*

$$f(x+y) - f(x) \in \mathcal{P}_n, \quad y \in \mathbb{R}, \tag{9}$$

is a polynomial of degree $n+1$.

Proof. Fix $y \neq 0$, assume that $f(x+y) - f(x) = p_y(x)$ where $p_y \in \mathcal{P}_n$; clearly, the coefficients of p_y may depend on the parameter y . It is not hard to see that this problem is solved by some polynomial of degree $n+1$. Indeed, if we expand $p_y(x)$ into a Newton interpolation series,

$$p_y(x) - p_y(0) = \sum_{k=1}^n a_k(y) x^{\binom{k}{y}}, \quad x^{\binom{k}{y}} := x \cdot (x-y) \cdot \dots \cdot (x-(k-1)y),$$

we can reduce the problem to finding polynomials q_k of degree $k+1$ such that $\Delta_y q_k(x) = x^{\binom{k}{y}}$ for $k = 1, \dots, n$. Obviously, $\Delta_y x^{\binom{k+1}{y}} = (k+1)x^{\binom{k}{y}}$, and so $q_y(x) = p_y(0) + \sum_{k=1}^n a_k(y)q_k(x)$ is a polynomial solving the original problem.

If F_1 and F_2 are solutions to $f(x+y) - f(x) = p_y(x)$, then

$$\Delta_y(F_2 - F_1) = \Delta_y F_2 - \Delta_y F_1 = 0,$$

which shows that $(F_2 - F_1)(x) = (F_2 - F_1)(x+y)$, i.e. the difference $F_2 - F_1$ is a y -periodic function. Therefore, the general solution of $\Delta_y f = p_y$ is given by

$$f(x) = q_y(x) + \pi_y(x)$$

where q_y is the particular polynomial solution and $x \mapsto \pi_y(x)$ is a continuous y -periodic function.

By assumption $\Delta_z f(x) = f(x+z) - f(x) \in \mathcal{P}_n$ for any $z \neq y$. We pick z in such a way that $z/y \notin \mathbb{Q}$. Using the representation of f and the periodicity of π_y we conclude that $\pi_y(x+z) - \pi_y(x) \equiv c$.

This leads to the problem $\Delta_z \pi_y(x) = c$, and the discussion above shows that its general solution is of the form $\pi_y(x) = a(y, z)x + b(y, z) + \bar{\pi}_{y,z}(x)$ with a z -periodic continuous function $\bar{\pi}_{y,z}(x)$. Since π_y is y -periodic, $a(y, z) = 0$ and $\bar{\pi}_{y,z}$ is both z - and y -periodic. Because of our assumption $z/y \notin \mathbb{Q}$, the periods of $\bar{\pi}_{y,z}(x)$ form a dense subset of \mathbb{R} ; using the continuity we see that $\pi_{y,z}(x)$, hence $p_y(x)$, is constant. This proves $f = q_y + c \in \mathcal{P}_{n+1}$. \square

Let us close this preparatory section with a simple but extremely useful result.

2.6. Lemma. *Let X be a random variable such that $X \sim p(x) dx$ and $p > 0$ Lebesgue a.e. If u is a Borel measurable function such that $u(X) = c$ almost surely for some constant $c \in \mathbb{R}$, then $u(x) = c$ Lebesgue almost everywhere.*

Proof. Clearly, $\mathbb{E}[(u(X) - c)^2] = 0$. Thus,

$$0 = \int_{\mathbb{R}} (u(x) - c)^2 p(x) dx.$$

Since $p > 0$ a.e., it follows that $u(x) = c$ Lebesgue almost everywhere. \square

3. MAIN RESULTS

We are now ready to state and prove the main results of this note.

3.1. Theorem. *Let $(B_t)_{t \geq 0}$ be a one-dimensional Brownian motion, and f a Borel measurable function which is polynomially bounded. If $f(B_t) - \mathbb{E}[f(B_t)]$ is a martingale, then*

- (i) $f = \tilde{f}$ almost everywhere for a twice differentiable function \tilde{f} satisfying $A\tilde{f} = \text{const}$, where A is the (weakly extended) generator,¹
- (ii) $f(x) = ax^2 + bx + c$ almost everywhere for suitable constants $a, b, c \in \mathbb{R}$.

Proof. Set $\gamma(t) := \mathbb{E}f(B_t)$ and denote by $T_t f(x) := \mathbb{E}^x[f(B_t)] = \mathbb{E}[f(x+B_t)]$ the semigroup generated by the process $(B_t)_{t \geq 0}$. Without loss of generality, we may assume that $f(0) = 0$, hence $\gamma(0) = 0$. We are first going to show that $t \mapsto \gamma(t)$ is a linear function. Since $f(B_t) - \mathbb{E}[f(B_t)]$ is a martingale, we have

$$\mathbb{E}[f(B_{t+s}) - \gamma(t+s) | \mathcal{F}_s] = f(B_s) - \gamma(s)$$

for all $s, t \geq 0$, and so, by the Markov property,

$$f(B_s) - \gamma(s) = \mathbb{E}[f(B_{t+s}) - \gamma(t+s) | \mathcal{F}_s] = \mathbb{E}^{B_s}[f(B_t)] - \gamma(t+s) = (T_t f)(B_s) - \gamma(t+s).$$

We can rearrange this to get

$$(T_t f - f)(B_s) = \gamma(t+s) - \gamma(s) \quad \text{almost surely.}$$

¹ A is, of course, the Laplace operator $\frac{1}{2}\Delta$. In view of the extension to more general Lévy processes, we prefer to use the notation A here.

Since the transition density p_s of B_s is strictly positive, this implies that

$$(T_t f - f)(x) = \gamma(t + s) - \gamma(s) \quad \text{Lebesgue almost everywhere,} \quad (10)$$

see Lemma 2.6. If we choose $s = 0$, then (10) yields

$$(T_t f - f)(x) = \gamma(t) \quad \text{Lebesgue almost everywhere,}$$

and we get that

$$\gamma(t + s) = \gamma(t) + \gamma(s), \quad s, t \geq 0,$$

which means that γ solves the Cauchy functional equation. As f is polynomially bounded, Lemma 2.1 shows that the function γ is bounded on every finite interval. Any solution to the Cauchy functional equation which is bounded on at least one non-degenerate interval is, however, linear, see e.g. Aczél [1, Theorem 1, p. 34], and so $\gamma(t) = \alpha t + \beta$ for some constants $\alpha, \beta \in \mathbb{R}$. From $\gamma(0) = 0$, we conclude that $\beta = 0$.

Inserting this into (10) for $s = 0$, we see that

$$T_t f(x) - f(x) = \gamma(t) = \alpha t \quad \text{Lebesgue almost everywhere.}$$

Since $p_t(y) = (2\pi t)^{-1/2} \exp[-\frac{1}{2t}y^2]$ is a smooth kernel, the convolution $x \mapsto T_t f(x)$ is twice continuously differentiable, and so $f = \tilde{f}$ almost everywhere for a twice continuously differentiable function \tilde{f} . Because of the continuity, we then have $T_t \tilde{f}(x) - \tilde{f}(x) = \alpha t$ for all $x \in \mathbb{R}$. In particular, for all $x \in \mathbb{R}$

$$A\tilde{f}(x) = \lim_{t \rightarrow 0} \frac{T_t \tilde{f}(x) - \tilde{f}(x)}{t} = \alpha,$$

and so $A\tilde{f} = \text{const.}$, in the weak sense. We can now apply Lemma 2.3 and 2.4 and see that \tilde{f} is a polynomial of degree $\deg(\tilde{f}) \leq 2$. This finishes the proof. \square

Let us add a further proof which uses only martingale arguments and avoids Cauchy's functional equation. We will present a further proof avoiding Cauchy's functional equation in Theorem 3.2 below, which generalizes Theorem 3.1 to Lévy processes.

Alternative proof of Theorem 3.1. Assume that f is a polynomially bounded function such that $f(B_t) - \mathbb{E}f(B_t)$ is a martingale.

We will first show that we can replace f by a C^2 -function. Pick $s, t \geq 0$ and fix $u > 0$. Set $f_u(x) := \mathbb{E}^x f(B_u) = \mathbb{E}f(B_u + x)$. Using the Markov property and the martingale property, we see

$$\begin{aligned} \mathbb{E}[f_u(B_{t+s}) - \mathbb{E}f_u(B_{t+s}) | \mathcal{F}_t] &= \mathbb{E}[\mathbb{E}^{B_{t+s}} f(B_u) - \mathbb{E}\mathbb{E}^{B_{t+s}} f(B_u) | \mathcal{F}_t] \\ &= \mathbb{E}[f(B_{t+s+u}) - \mathbb{E}f(B_{t+s+u}) | \mathcal{F}_t] \\ &= f(B_t) - \mathbb{E}f(B_t). \end{aligned}$$

If we take $s = 0$, the above calculation shows

$$\mathbb{E}[f_u(B_t) - \mathbb{E}f_u(B_t) | \mathcal{F}_t] = f(B_t) - \mathbb{E}f(B_t) = \mathbb{E}[f_u(B_{t+s}) - \mathbb{E}f_u(B_{t+s}) | \mathcal{F}_t],$$

and since $f_u(B_t)$ is \mathcal{F}_t measurable, we conclude that $M_t := f_u(B_t) - \mathbb{E}f_u(B_t)$ is a martingale.

Noting that $f_u(x) = \mathbb{E}f(B_u + x) = \int_{\mathbb{R}} f(y) p_u(y - x) dy$ with the smooth transition density $p_u(y)$ of B_u , we can use dominated convergence to see that $x \mapsto f_u(x)$ is a C^2 -function. Thus, we can apply Itô's formula to infer that $N_t := f_u(B_t) - \int_0^t Af_u(B_r) dr$ is a local martingale.

Let $(\sigma_n)_{n \in \mathbb{N}}$, $\sigma_n \uparrow \infty$, be a localizing sequence for $(N_t)_{t \geq 0}$. Since $Af_u(x)$ is continuous, the stopping times

$$\tau_n := \sigma_n \wedge \inf \{t \geq 0 : |Af_u(B_t)| \leq n\}$$

are such that $(N_{\tau_n \wedge t})_{t \geq 0}$ is a martingale and $|Af_u(B_s)| \leq n$ for $s \in [0, \tau_n)$. In particular, $t \mapsto \int_0^{t \wedge \tau_n} Af_u(B_r) dr$ is a continuous process which is of bounded variation on compact t -intervals.

The martingale $(N_{t \wedge \tau_n})_{t \geq 0}$ has constant expectation, i.e.

$$\mathbb{E}f_u(B_{t \wedge \tau_n}) = f_u(0) + \mathbb{E} \int_0^{t \wedge \tau_n} Af_u(B_r) dr. \quad (11)$$

By assumption, $|f(x)| \leq C(1 + |x|)^N$ for some $N \in \mathbb{N}$. It is easy to see that f_u is also polynomially bounded and that

$$\begin{aligned} |Af_u(x)| &= \left| \int_{\mathbb{R}} A_x p_u(y-x) f(y) dy \right| \leq \int_{\mathbb{R}} |A p_u(y)| \cdot |f(y+x)| dy \\ &\leq c(1 + |x|)^N \int_{\mathbb{R}} |p_u''(y)| (1 + |y|)^N dy \leq c_u (1 + |x|)^N. \end{aligned}$$

Using that the maximum of Brownian motion has moments of any order, we find that

$$\mathbb{E} \int_0^t |Af_u(B_r)| dr \leq t c_u \mathbb{E} \left[\sup_{r \leq t} (1 + |B_r|)^N \right] < \infty.$$

We can apply dominated convergence in (11) to see that

$$h(t) := \mathbb{E} f_u(B_t) = f_u(0) + \mathbb{E} \int_0^t Af_u(B_r) dr.$$

This shows, in particular, that $t \mapsto h(t)$ is continuous and of bounded variation. Hence,

$$M_{t \wedge \tau_n} - N_{t \wedge \tau_n} = \int_0^{t \wedge \tau_n} Af_u(B_r) dr - \mathbb{E} f_u(B_r) \Big|_{r=t \wedge \tau_n} = \int_0^{t \wedge \tau_n} Af_u(B_r) dr - h(t \wedge \tau_n)$$

is a continuous martingale which is of bounded variation. Such martingales are a.s. constant, cf. [14, Proposition IV.(1.2), p. 120], and so

$$M_{t \wedge \tau_n} - N_{t \wedge \tau_n} = M_{0 \wedge \tau_n} - N_{0 \wedge \tau_n} = -f_u(0).$$

Letting $n \rightarrow \infty$ we see that for all $s < t$

$$\int_s^t Af_u(B_r) dr = \lim_{n \rightarrow \infty} \left[\int_0^{t \wedge \tau_n} Af_u(B_r) dr - \int_0^{s \wedge \tau_n} Af_u(B_r) dr \right] = \lim_{n \rightarrow \infty} [h(t \wedge \tau_n) - h(s \wedge \tau_n)].$$

The right-hand side is the non-random expression $h(t) - h(s)$. Now we divide by $t - s$, and let $s \uparrow t$. This yields

$$Af_u(B_t) = \lim_{s \uparrow t} \frac{1}{t-s} \int_s^t Af_u(B_r) dr = \text{const.}$$

From this point onwards we can argue as in the first proof, and see that $f_u(x) = a_u x^2 + b_u x + c_u$ for suitable constants $a_u, b_u, c_u \in \mathbb{R}$.

Let us return to the original problem. We need to show that f_u converges in a suitable sense to f . First we prove that $f_u(B_t)$ converges to $f(B_t)$ in $L^2(\mathbb{P})$. To this end, we note that

$$x \mapsto I(\phi, x) := \int_{\mathbb{R}} |\phi(x+y) - \phi(y)|^2 p_t(y) dy = \mathbb{E}(|\phi(x+B_t) - \phi(B_t)|^2)$$

is continuous for any continuous function ϕ , which grows at most polynomially. Since f is, by assumption, at most of polynomial growth, and since the bounded continuous functions $C_b(\mathbb{R})$ are dense in $L^2(e^{-y^2/4t} dy)$, there is a sequence $(\phi_n)_{n \in \mathbb{N}} \subset C_b(\mathbb{R})$ such that $\phi_n \rightarrow f$ in $L^2(e^{-y^2/4t} dy)$ as $n \rightarrow \infty$. We have

$$\begin{aligned} |\sqrt{I(\phi_n, x)} - \sqrt{I(f, x)}|^2 &\leq |\sqrt{I(\phi_n - f, x)}|^2 \\ &\leq 2\mathbb{E}(|\phi_n(x+B_t) - f(x+B_t)|^2) + 2\mathbb{E}(|\phi_n(B_t) - f(B_t)|^2) \\ &\leq \frac{2}{\sqrt{2\pi t}} \left(1 + e^{x^2/2t}\right) \int_{\mathbb{R}} |\phi_n(y) - f(y)|^2 e^{-y^2/4t} dy, \end{aligned}$$

where we use in the last step the estimate $2|x||y| = 2(\sqrt{2}|x|)(|y|/\sqrt{2}) \leq 2x^2 + y^2/2$ to obtain

$$\exp\left(-\frac{(x-y)^2}{2t}\right) \leq \exp\left(\frac{x^2}{2t}\right) \exp\left(-\frac{y^2}{4t}\right).$$

It follows that $I(\phi_n, x) \rightarrow I(f, x)$ uniformly on compact sets, and so $x \mapsto I(f, x)$ is continuous. Thus,

$$\int_{\mathbb{R}} I(f, x+z) p_u(x) dx \xrightarrow{u \rightarrow 0} I(f, z)$$

for any $z \in \mathbb{R}$. Choosing $z = 0$ and plugging in the definition of I , we get

$$\mathbb{E} \left[\mathbb{E}'(|f(B'_u + B_t) - f(B_t)|^2) \right] = \int_{\mathbb{R}} \mathbb{E}(|f(x+B_t) - f(B_t)|^2) p_u(x) dx \xrightarrow{u \rightarrow 0} 0$$

for an independent copy $(B'_u)_{u \geq 0}$ of $(B_u)_{u \geq 0}$ and with \mathbb{E}' to indicate the expectation acting on B'_u . As

$$\mathbb{E}(|f_u(B_t) - f(B_t)|^2) = \mathbb{E}(|\mathbb{E}' f(B'_u + B_t) - f(B_t)|^2) \leq \mathbb{E}(\mathbb{E}'[|f(B'_u + B_t) - f(B_t)|^2]),$$

we conclude that $f_u(B_t) \rightarrow f(B_t)$ in $L^2(\mathbb{P})$. Since the transition density p_t is bounded away from zero on any compact set, this implies

$$\int_{|x| \leq k} |f_u(x) - f(x)|^2 dx \rightarrow 0$$

for any $k \in \mathbb{N}$. Consequently, $f_u(x) \rightarrow f(x)$ Lebesgue almost everywhere for a suitable subsequence of f_u . By the first part of this proof, f_u is a polynomial of order 2, and so is f (up to a Lebesgue null set). \square

The key ingredients in the first proof of Theorem 3.1 are the following observations:

- B_t is a Lévy process,
- B_t has a transition density which is strictly positive,
- the generator of B_t satisfies the Liouville theorem,

and so the following generalization of Theorem 3.1 is natural. If $(X_t)_{t \geq 0}$ is a Lévy process which has a transition density, then the convolution identity $p_{t+s}(x) = p_t * p_s(x)$ shows that $x \mapsto p_t(x)$ is continuous, i.e. $p_t > 0$ Lebesgue a.e. already means that $p_t > 0$ on \mathbb{R} . Moreover, if $(t, x) \mapsto p_t(x)$ is jointly continuous on $(0, \infty) \times \mathbb{R}$, then $\{x : p_t(x) > 0\}$ is either \mathbb{R} or a half-line, i.e. $\text{supp}(p_t) = \mathbb{R}$ implies $\{p_t > 0\} = \mathbb{R}$; joint continuity is, e.g., implied by $e^{-t\psi(\xi)} \in L^1(d\xi)$ for all $t > 0$, see Sharpe [19] for a discussion, and [8] for further results on the existence and smoothness of Lévy densities.

3.2. Theorem. *Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process which has a transition density p_t for all $t > 0$, such that $p_t > 0$ for some $t > 0$. Assume that*

- either (a) the characteristic exponent ψ is of class C^∞ , and $n \geq 2$ is arbitrary,*
- or (b) $p_t \in C_b^1(\mathbb{R})$, $t > 0$, and $\mathbb{E}|X_1|^{n+\epsilon} < \infty$ for some $\epsilon > 0$ and $n \geq 2$.*

Let $|f(x)| \leq c(1 + |x|^n)$ be a polynomially bounded, measurable function. If $f(X_t) - \mathbb{E}f(X_t)$ is a martingale, then $f = \tilde{f}$ almost everywhere for some continuous function \tilde{f} satisfying $A\tilde{f} = \text{const}$, where A is the (weakly extended) generator. In fact, $\tilde{f}(x) = ax^2 + bx + c$ for suitable constants $a, b, c \in \mathbb{R}$.

Proof. Under the assumption (a), we can literally use the first proof of Theorem 3.1. Notice that the existence of the density p_t and the (trivial) fact that $p_t \in L^1(dx)$ allows us to use the Riemann–Lebesgue lemma to conclude that $\lim_{|\xi| \rightarrow \infty} e^{-t\psi(\xi)} = 0$, hence $\lim_{|\xi| \rightarrow \infty} \psi(\xi) = \infty$ and $\{\psi = 0\} = \{0\}$.

Assume now that (b) holds. As in the first proof of Theorem 3.1, we find that

$$T_t f(x) - f(x) = \gamma(t) \quad \text{a.e.}$$

Since $f \mapsto T_t f$ is a convolution operator, the function $x \mapsto T_t f(x)$ is continuous, and so there is a continuous function \tilde{f} such that $f = \tilde{f}$ almost everywhere and $T_t \tilde{f}(x) - \tilde{f}(x) = \gamma(t)$ for all $x \in \mathbb{R}$. As $T_t \tilde{f}(x+y) - \tilde{f}(x+y) = T_t(\tilde{f}(\cdot + y))(x) - \tilde{f}(\cdot + y)(x) = \gamma(t)$, it follows that the difference $\Delta_y \tilde{f}(x) := \tilde{f}(x+y) - \tilde{f}(x)$ satisfies $T_t(\Delta_y \tilde{f}) - \Delta_y \tilde{f} = 0$. Hence, $A(\Delta_y \tilde{f}) = 0$ weakly, and so the Liouville theorem yields that $\Delta_y \tilde{f}$ is a polynomial. By Lemma 2.5, this implies that \tilde{f} is a polynomial.² Thus, by Lemma 2.1, $\gamma(t) = \mathbb{E}\tilde{f}(X_t)$ is a polynomial. In particular, $\gamma(t)$ is differentiable, which implies

$$A\tilde{f}(x) = \lim_{t \rightarrow 0} \frac{T_t \tilde{f}(x) - \tilde{f}(x)}{t} = \lim_{t \rightarrow 0} \frac{\gamma(t)}{t} = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t} = \gamma'(0), \quad x \in \mathbb{R}.$$

Consequently, Lemma 2.4 shows that \tilde{f} is a polynomial of degree $\deg(\tilde{f}) \leq 2$. \square

We will now turn to the ‘multiplicative’ version of Theorem 3.1.

²It is interesting to note that the particular form of $\gamma(t)$ does not play any role here. To wit: if u is a continuous function such that $T_t u(x) - u(x) = \gamma(t)$ does not depend on x , then u is a polynomial. This allows us to avoid Cauchy’s functional equation, which we used in the first proof of Theorem 3.1.

3.3. Theorem. *Let $(B_t)_{t \geq 0}$ be a one-dimensional Brownian motion, and $g : \mathbb{R} \rightarrow (0, \infty)$ a Borel measurable function which grows at most exponentially. If $g(B_t)/\mathbb{E}g(B_t)$ is a martingale, then $g = \tilde{g}$ almost everywhere for a twice differentiable function \tilde{g} which satisfies*

$$\frac{A\tilde{g}}{\tilde{g}} = \text{const.}$$

where A is the (weakly extended) generator. Moreover, all positive solutions are of the form $g(x) = ae^{cx} + be^{-cx}$ almost everywhere for suitable constants $a, b, c \geq 0$.

Proof. Set $\gamma(t) := \mathbb{E}g(B_t)$ and denote by $T_t g(x) = \mathbb{E}^x g(B_t) = \mathbb{E}g(B_t + x)$ the semigroup generated by the process $(B_t)_{t \geq 0}$. Without loss of generality, we may assume that $g(0) = 1$, hence $\gamma(0) = 1$. From $|g(x)| \leq ce^{\kappa|x|}$ we see that $\sup_{s \leq t} \mathbb{E}|g(B_s)| < \infty$ for any $t > 0$; in particular, $\gamma(t)$ is strictly positive and bounded in some neighbourhood $[0, \epsilon_0]$ of $t = 0$.

We are first going to show that $\gamma(t)$ is of the form $\gamma(t) = e^{\alpha t}$ for some $\alpha \in \mathbb{R}$. We use again the argument from Theorem 3.1. Since $g(B_t)/\gamma(t)$ is a martingale, we have

$$\frac{\mathbb{E}[g(B_{t+s}) | \mathcal{F}_s]}{\gamma(t+s)} = \frac{g(B_s)}{\gamma(s)}$$

for all $s, t > 0$, and so, by the Markov property,

$$\frac{g(B_s)}{\gamma(s)} = \frac{\mathbb{E}[g(B_{t+s}) | \mathcal{F}_s]}{\gamma(t+s)} = \frac{\mathbb{E}^{B_s}[g(B_t)]}{\gamma(t+s)} = \frac{(T_t g)(B_s)}{\gamma(t+s)}.$$

We can rearrange this to get

$$\frac{(T_t g)(B_s)}{g(B_s)} = \frac{\gamma(t+s)}{\gamma(s)} \quad \text{almost surely,}$$

which implies, by the strict positivity of the transition density p_s of B_s ,

$$\frac{T_t g(x)}{g(x)} = \frac{\gamma(t+s)}{\gamma(s)} \quad \text{Lebesgue almost everywhere.} \quad (12)$$

Taking $s = 0$ and $x = 0$, we conclude that

$$\gamma(t+s) = \gamma(t)\gamma(s), \quad s, t \geq 0.$$

This means that $t \mapsto \log \gamma(t)$ satisfies the Cauchy functional equation; moreover, it is bounded on the interval $[0, \epsilon_0]$. Therefore (see the proof of Theorem 3.1), $t \mapsto \log \gamma(t)$ is linear, and we get $\gamma(t) = \gamma(0)e^{\alpha t} = e^{\alpha t}$ for some $\alpha \in \mathbb{R}$. Taking $s = 0$ in (12), we obtain that

$$T_t g(x) = e^{\alpha t} g(x) \quad \text{Lebesgue almost everywhere.}$$

Since $p_t(y) = (2\pi t)^{-1/2} \exp[-\frac{1}{2t}y^2]$ is a smooth kernel, the convolution $x \mapsto T_t g(x)$ is twice continuously differentiable, and so we find that $g = \tilde{g}$ Lebesgue a.e. for a function $\tilde{g} \in C^2(\mathbb{R})$. Finally, we see that

$$A\tilde{g}(x) = \lim_{t \rightarrow 0} \frac{T_t \tilde{g}(x) - \tilde{g}(x)}{t} = \lim_{t \rightarrow 0} \frac{e^{\alpha t} - 1}{t} \tilde{g}(x) = \alpha \tilde{g}(x).$$

Since \tilde{g} is twice differentiable, $A\tilde{g}$ has a classical meaning and all solutions of $A\tilde{g} = -\alpha\tilde{g}$ are solutions of the following second-order linear ODE with constant coefficients: $y'' + \alpha y = 0$ and $y(0) = 1$. Solving this, we see that $y(x) = ae^{cx} + be^{-cx}$ with $c = \sqrt{\alpha} \in \mathbb{C}$ and $a + b = 1$, so all positive solutions are given by $ae^{cx} + be^{-cx}$ with $a, b, c \geq 0$. \square

The next result extends Theorem 3.3 to the class of Lévy processes with finite exponential moments.

3.4. Theorem. *Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process which has a transition density p_t for all $t > 0$. Let $g : \mathbb{R} \rightarrow (0, \infty)$ be a Borel measurable function which grows at most exponentially, i.e. $|g(x)| \leq ce^{\kappa|x|}$. Assume that $\mathbb{E}e^{\kappa|X_t|} < \infty$ and*

- either (a) $p_t > 0$ on \mathbb{R} for some $t > 0$
- or (b) $p_t = 0$ on $(-\infty, 0)$ and $p_t > 0$ on $(0, \infty)$ for some $t > 0$.

If $g(B_t)/\mathbb{E}g(B_t)$ is a martingale, then $g = \tilde{g}$ almost everywhere for a continuous function \tilde{g} satisfying

$$\frac{A\tilde{g}(x)}{\tilde{g}(x)} = \alpha,$$

for some constant $\alpha \in \mathbb{R}$, where A denotes the (weakly extended) generator.

The equation $\mathbb{E}\exp(\lambda X_t) = \exp(\alpha t)$ admits at most two (in case (a): at most one) solutions $\lambda = \lambda_1, \lambda_2 \in \mathbb{R}$. If there are solutions, then $g(x) = ae^{\lambda_1 x} + be^{\lambda_2 x}$ almost everywhere for suitable constants $a, b \geq 0$; otherwise there is only the trivial solution $g \equiv 0$.

Proof. Denote by $(T_t)_{t \geq 0}$ the semigroup associated with $(X_t)_{t \geq 0}$, and assume without loss of generality that $g(0) = 1$. Following the reasoning from the proof of Theorem 3.3, we find that there is a constant $\alpha \in \mathbb{R}$ such that $T_t g(x) = e^{\alpha t} g(x)$ for Lebesgue almost all x . Since $g \mapsto T_t g$ is a convolution operator, the function $x \mapsto T_t g(x)$ is continuous, and so $g(x) = \tilde{g}(x)$ Lebesgue a.e. for a continuous function \tilde{g} ; in particular,

$$T_t \tilde{g}(x) = e^{\alpha t} \tilde{g}(x), \quad x \in \mathbb{R}, \quad (13)$$

and $A\tilde{g} = \alpha\tilde{g}$. Pick $t > 0$ such that the transition density p_t satisfies assumption (a) or (b). Then (13) can be seen as a convolution equality $\int \tilde{g}(x+y) \varrho_t(dy) = \tilde{g}(x)$ with the measure $\varrho_t(dy) = e^{-\alpha t} p_t(y) dy$ which has full support in \mathbb{R} under the assumption (a), resp. in $(0, \infty)$ under the assumption (b).

Clearly, the equation $\int_{\mathbb{R}} e^{\lambda x} \varrho_t(dx) = 1$ can be rewritten in the form $M(\lambda) := \mathbb{E}e^{\lambda X_t} = e^{\alpha t}$. Since we have assumed that $\mathbb{E}e^{\kappa|X_t|} < \infty$, it is easy to see that $M''(\lambda) = \mathbb{E}[X_t^2 e^{\lambda X_t}] > 0$ exists and is strictly positive for all $\lambda \in (-\kappa, \kappa)$. Thus, M is strictly convex on $[-\kappa, \kappa]$, and from this it is clear that $M(\lambda) = e^{\alpha t}$ has one or two solutions if $\min\{\mathbb{E}e^{-\kappa X_t}, 1, \mathbb{E}e^{+\kappa X_t}\} \leq e^{\alpha t} \leq \max\{\mathbb{E}e^{-\kappa X_t}, \mathbb{E}e^{+\kappa X_t}\}$.³

If there are solutions $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$, then all positive locally integrable solutions satisfying (13) are of the form

$$\tilde{g}(x) = ae^{\lambda_1 x} + be^{\lambda_2 x}, \quad x \in \mathbb{R},$$

for $a, b \geq 0$; otherwise, (13) has only the trivial solution $\tilde{g} = 0$; see the classical works by Choquet & Deny [6, 7] or the elementary presentation by Ramachandran & Prakasa Rao [15] for the case that ϱ_t has full support in \mathbb{R} and Lau & Rao [10] for the case that ϱ_t is supported in $[0, \infty)$. From the definition of ϱ_t , it is immediate that the condition on λ_i is equivalent to $\mathbb{E}e^{\lambda_i X_t} = e^{\alpha t}$. \square

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³If X_t has exponential moments of all orders, we can take $\kappa \rightarrow \infty$ and we see, letting $\lambda \rightarrow \pm\infty$ in

$$\mathbb{E}e^{\lambda X_t} = \mathbb{E}[e^{\lambda X_t} \mathbf{1}_{\{X_t > 0\}}] + \mathbb{E}[e^{\lambda X_t} \mathbf{1}_{\{X_t < 0\}}],$$

that in the regime (b) the function $M(\lambda)$ is U -shaped with the global minimum $M(0) = 1$ at $\lambda = 0$; thus there are always two solutions if $e^{\alpha t} > 1$, one solution if $e^{\alpha t} = 1$, and no solution otherwise. In the regime (a), $M(\lambda)$ has the limits 0 and ∞ as $\lambda \rightarrow \pm\infty$, and there is always exactly one solution.

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(F. Kühn) TU DRESDEN, FAKULTÄT MATHEMATIK, INSTITUT FÜR MATHEMATISCHE STOCHASTIK, 01062 DRESDEN, GERMANY. E-MAIL: franziska.kuehn1@tu-dresden.de

(R.L. Schilling) TU DRESDEN, FAKULTÄT MATHEMATIK, INSTITUT FÜR MATHEMATISCHE STOCHASTIK, 01062 DRESDEN, GERMANY. E-MAIL: rene.schilling@tu-dresden.de