Existence and estimates of moments for Lévy-type processes

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Abstract

In this paper, we establish the existence of moments and moment estimates for Lévy-type processes. We discuss whether the existence of moments is a time dependent distributional property, give sufficient conditions for the existence of moments and prove estimates of fractional moments. Our results apply in particular to SDEs and stable-like processes.

Keywords: Lévy-type processes, existence of moments, generalized moments, fractional moments.

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1 Introduction

For a Lévy process $(X_t)_{t\geq 0}$ and a submultiplicative function $f \geq 0$ it is known

(i) ... that the existence of the generalized moment $\mathbb{E}f(X_t)$ does not depend on time, i. e. $\mathbb{E}f(X_{t_0}) < \infty$ for some $t_0 > 0$ implies $\mathbb{E}f(X_t) < \infty$ for all $t \geq 0$, see e. g. [16, Theorem 25.18].

(ii) ... that the existence of moments can be characterized in terms of the Lévy triplet, see e. g. [16, Theorem 25.3].

(iii) ... what the small-time asymptotics of fractional moments $\mathbb{E}(|X_t|^\alpha)$, $\alpha > 0$, looks like, cf. [5] and [13].

The first two problems are of fundamental interest; the asymptotics of fractional moments has turned out to be of importance in various parts of probability theory, e. g. to obtain Harnack inequalities [5] or to prove the existence of densities for solutions of stochastic differential equations [7]. Up to now, there is very little known about the answers for the larger class of Lévy-type processes which includes, in particular, stable-like processes, affine processes and solutions of (Lévy-driven) stochastic differential equations. The aim of this work is to extend results which are known for Lévy processes from the Lévy case to Lévy-type processes.

In the last years, heat kernel estimates for Lévy(-type) processes have attracted a lot of attention. Let us point out that the results obtained here have several applications in this area. In a future work[1], we will show that any rich Lévy-type process $(X_t)_{t\geq 0}$ with triplet $(b(x), Q(x), N(x, dy))$ satisfies the integrated heat kernel estimate

$$\frac{\mathbb{P}^x(|X_t - x| \geq R)}{t} \xrightarrow{t\to 0} N(x, \{y \in \mathbb{R}^d; |y| \geq R\}) \tag{1}$$

for all $R > 0$ such that $N(x, \{y \in \mathbb{R}^d; |y| = R\}) = 0$. Combining this with the statements from Section 4 gives the small-time asymptotics of $t^{-1}\mathbb{E}^x f(X_t)$ for a large class of functions f; the functions need not to be bounded or differentiable. The corresponding results for Lévy

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[1] Update: The paper [12] is available now.

processes have been discussed by Jacod [10] and Figueroa-López [6]. As suggested in [6], this gives the possibility to extend the generator of the process to a larger class of functions. Moreover, following a similar approach as Fournier and Printems [7], the estimates of the fractional moments show the existence of (L^2-)densities for Lévy-type processes with Hölder-continuous symbols.

The structure of this paper is as follows. In Section 2, we introduce basic definitions and notation. The problems mentioned above will be answered in Sections 3–5; starting with the question whether the existence of moments is a time dependent distributional property in Section 3, we give sufficient conditions for the existence of moments in Section 4 and finally present estimates of fractional moments in Section 5. In each of these sections, we give a brief overview on known results, state some generalizations and illustrate them with examples.

2 Basic definitions and notation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For a random variable X on $(\Omega, \mathcal{A}, \mathbb{P})$ we denote by $\mathbb{P}_X$ the distribution of X with respect to $\mathbb{P}$. We say that two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ are *comparable* and write $f \asymp g$ if there exists a constant $c > 0$ such that $c^{-1} f(x) \le g(x) \le c f(x)$ for all $x \in \mathbb{R}^d$. Moreover, we denote by $\mathcal{B}_b(\mathbb{R}^d)$ the space of all bounded Borel-measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ and by $C_c^2(\mathbb{R}^d)$ the space of functions with compact support which are twice continuously differentiable. For $x \in \mathbb{R}^d$ and $r > 0$ we set $B(x,r) := \{y \in \mathbb{R}^d ; |y-x| < r\}$ and $B[x,r] := \{y \in \mathbb{R}^d ; |y-x| \le r\}$. The j-th unit vector in $\mathbb{R}^d$ is denoted by e_j and $x \cdot y = \sum_{j=1}^n x_j y_j$ is the Euclidean scalar product. For a function $u : \mathbb{R}^d \to \mathbb{R}$ we denote by $\partial_{x_j}^k u(x)$ the k-th order partial derivative with respect to x_j and by $\nabla^2 u$ the Hessian matrix. The *Fourier transform* of an integrable function $u : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\hat{u}(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i x \cdot \xi} u(x)\, dx, \qquad \xi \in \mathbb{R}^d.$$

We call a stochastic process $(L_t)_{t \ge 0}$ a (d-dimensional) *Lévy process* if $L_0 = 0$ almost surely, $(L_t)_{t \ge 0}$ has stationary and independent increments and $t \mapsto L_t(\omega)$ is càdlàg for almost all $\omega \in \Omega$. It is well-known, cf. [16], that $(L_t)_{t \ge 0}$ can be uniquely characterized via its *characteristic exponent*,

$$\psi(\xi) = -i\, b \cdot \xi + \frac{1}{2} \xi \cdot Q\xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i\, y \cdot \xi} + i\, y \cdot \xi \mathbb{1}_{(0,1]}(|y|))\, \nu(dy), \qquad \xi \in \mathbb{R}^d;$$

here, $b \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix and ν is a measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ such that $\int_{\mathbb{R}^d \setminus \{0\}} (|y|^2 \wedge 1)\, \nu(dy) < \infty$. The triplet (b, Q, ν) is called *Lévy triplet*. Our standard reference for Lévy processes is the monograph by Sato [16]. A stochastic process $(X_t)_{t \ge 0}$ is said to be a (rich) *Lévy-type process* (or (rich) *Feller process*) if $(X_t)_{t \ge 0}$ is a Markov process whose associated semigroup is Feller on the space of continuous functions vanishing at infinity and the domain of the generator contains the compactly supported smooth functions $C_c^\infty(\mathbb{R}^d)$; for further details we refer the reader to [3]. A theorem due to Courrège and Waldenfels, cf. [3, Corollary 2.23], states that the generator A restricted to $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator of the form

$$Au(x) = -\int_{\mathbb{R}^d} e^{i\, x \cdot \xi} q(x, \xi) \hat{u}(\xi)\, d\xi, \qquad u \in C_c^\infty(\mathbb{R}^d),$$

where

$$q(x, \xi) = q(x, 0) - i\, b(x) \cdot \xi + \frac{1}{2} \xi \cdot Q(x) \xi + \int_{\mathbb{R}^d} (1 - e^{i\, y \cdot \xi} + i\, y \cdot \xi \mathbb{1}_{(0,1]}(|y|))\, N(x, dy) \qquad (2)$$

is the *symbol*. For each fixed $x \in \mathbb{R}^d$, $(b(x), Q(x), N(x, dy))$ is a Lévy triplet. Throughout this work, we will assume that $q(x, 0) = 0$. Using well-known results from Fourier analysis, it is not difficult to see that

$$Au(x) = b(x) \cdot \nabla u(x) + \frac{1}{2} \operatorname{tr}(Q(x) \cdot \nabla^2 u(x)) + \int_{\mathbb{R}^d \setminus \{0\}} (u(x+y) - u(x) - \nabla u(x) \cdot y \mathbb{1}_{(0,1]}(|y|))\, N(x, dy)$$

for any $u \in C_c^\infty(\mathbb{R}^d)$, see e. g. [3, Theorem 2.21]. We write $(X_t)_{t\geq 0} \sim (b(x), Q(x), N(x,dy))$ to indicate that $(X_t)_{t\geq 0}$ is a Lévy-type process with triplet $(b(x), Q(x), N(x,dy))$. The symbol of a Lévy-type process is locally bounded, cf. [3, Theorem 2.27(d)]. A Lévy-type process has *bounded coefficients* if $|q(x,\xi)| \leq C(1+|\xi|^2)$ for some constant $C>0$ which does not depend on $x \in \mathbb{R}^d$. By [19, Lemma 6.2], the following statements are equivalent for any compact set $K \subseteq \mathbb{R}^d$:

(i) $\sup_{x\in K} \sup_{|\xi|\leq 1} |q(x,\xi)| < \infty$,

(ii) $\sup_{x\in K} |q(x,\xi)| \leq C_K(1+|\xi|^2)$ for all $\xi \in \mathbb{R}^d$,

(iii) $\sup_{x\in K}(|b(x)|+|Q(x)|+\int_{\mathbb{R}^d\setminus\{0\}}(|y|^2\wedge 1)\, N(x,dy)) < \infty$; here $|\cdot|$ denotes an arbitrary vector norm and matrix norm, respectively.

If $(X_t)_{t\geq 0}$ has bounded coefficients, then the statements are also equivalent for $K = \mathbb{R}^d$. We will use the following result frequently; it is compiled from [4, Theorem 3.13]. We remind the reader that a *Cauchy process* is a Lévy process with characteristic exponent $\psi(\xi) = |\xi|$.

2.1 Theorem *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with triplet $(b(x), Q(x), N(x,dy))$. There exist a Markov extension $(\Omega^\circ, \mathcal{A}^\circ, \mathcal{F}_t^\circ, \mathbb{P}^{\circ,x})$, a Brownian motion $(W_t^\circ)_{t\geq 0}$ and a Cauchy process $(L_t^\circ)_{t\geq 0}$ with jump measure N° on $(\Omega^\circ, \mathcal{A}^\circ, \mathcal{F}_t^\circ, \mathbb{P}^{\circ,x})$ such that*

$$X_t - X_0 = X_t^1 + X_t^2$$

with

$$\begin{aligned} X_t^1 &:= \int_0^t b(X_{s-})\, ds + \int_0^t \sigma(X_{s-})\, dW_s^\circ + \int_0^t \int_{|k|\leq 1} k(X_{s-},z)\, (N^\circ(dz,ds) - \nu^\circ(dz)\, ds) \\ X_t^2 &:= \int_0^t \int_{|k|>1} k(X_{s-},z)\, N^\circ(dz,ds) \end{aligned}$$

for measurable functions $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $k : \mathbb{R}^d \times (\mathbb{R}\setminus\{0\}) \to \mathbb{R}^d$ satisfying

$$N(x,B) = \int_{\mathbb{R}\setminus\{0\}} \mathbb{1}_B(k(x,z))\, \nu^\circ(dz), \qquad B \in \mathcal{B}(\mathbb{R}^d\setminus\{0\}),\ x \in \mathbb{R}^d, \tag{3}$$

and $Q(x) = \sigma(x)\sigma(x)^T$; here $\nu^\circ(dz) = (2\pi)^{-1} z^{-2}\, dz$ denotes the Lévy measure of a (one-dimensional) Cauchy process.

3 Existence of moments - time independence

In this section we adress the question whether the existence of moments is a time dependent distributional property in the class of Lévy-type processes. Given a Lévy-type process $(X_t)_{t\geq 0}$ and a measurable function $f : \mathbb{R}^d \to [0,\infty)$, then under which additional assumptions on $(X_t)_{t\geq 0}$ and f does the equivalence

$$\mathbb{E}^x f(X_t) < \infty \text{ for some } t > 0 \iff \mathbb{E}^x f(X_t) < \infty \text{ for all } t > 0 \tag{4}$$

hold true? It is well-known that (4) holds for any Lévy process $(X_t)_{t\geq 0}$ if f is a locally bounded function which is submultiplicative (i. e. there exists $c > 0$ such that $f(x+y) \leq cf(x)f(y)$ for all $x,y \in \mathbb{R}^d$), see [16, Theorem 25.3]. Analogous results for Lévy-type processes seem to be unknown. First we discuss whether moments exist backward in time, i. e. whether

$$\mathbb{E}^x f(X_t) < \infty \text{ for some } t > 0 \iff \mathbb{E}^x f(X_s) < \infty \text{ for all } s \leq t. \tag{5}$$

The following theorem is the main result of this section.

3.1 Theorem *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with bounded coefficients and $f : \mathbb{R}^d \to (0,\infty)$ measurable.*

(i) *Suppose there exists a bounded measurable function* $g : \mathbb{R}^d \to [0, \infty)$*, such that* $\inf_{|y|\le r} g(y) > 0$ *for* $r > 0$ *sufficiently small and*

$$\inf_{y\in\mathbb{R}^d} \frac{f(z+y)}{f(y)} \ge g(z) \tag{6}$$

for all $z \in \mathbb{R}^d$*. Then*

$$\mathbb{E}^x f(X_t) < \infty \iff \sup_{s\le t} \mathbb{E}^x f(X_s) < \infty. \tag{7}$$

(ii) (6)*, hence* (7)*, holds if one of the following conditions is satisfied.*

(a) f *is submultiplicative and locally bounded.*

(b) $\log f$ *is Hölder continuous.*

(c) f *is Hölder continuous and* $\inf_{x\in\mathbb{R}^d} f(x) > 0$.

(d) f *is differentiable and* $\sup_{y\in\mathbb{R}^d} \sup_{|z|\le r} \frac{|\nabla f(y+z)|}{f(y)} < \infty$ *for* $r > 0$ *sufficiently small.*

(e) f *is differentiable,* $\inf_{y\in\mathbb{R}^d} f(y) > 0$*,* $\sup_{y\in\mathbb{R}^d} \frac{|\nabla f(y)|}{f(y)} < \infty$ *and* ∇f *is uniformly continuous.*

For the proof of Theorem 3.1 we need two auxiliary results.

3.2 Lemma (Maximal inequality) *Let* $(X_t)_{t\ge 0}$ *be a Lévy-type process with symbol* q *and denote by* $\tau_r^x := \inf\{t > 0; X_t \notin B[x,r]\}$ *the exit time from the closed ball* $B[x,r] = \{y \in \mathbb{R}^d; |y-x| < r\}$*. Then there exists* $C > 0$ *such that*

$$\mathbb{P}^x\left(\sup_{s\le\sigma} |X_s - x| > r\right) \le C\mathbb{E}^x\left(\int_{[0,\sigma\wedge\tau_r^x)} \sup_{|\xi|\le r^{-1}} |q(X_s,\xi)|\, ds\right) \tag{8}$$

for all stopping times σ *and* $r > 0$*. In particular,*

$$\mathbb{P}^x\left(\sup_{s\le\sigma} |X_s - x| > r\right) \le C\mathbb{E}^x(\sigma) \sup_{|y-x|\le r} \sup_{|\xi|\le r^{-1}} |q(y,\xi)|. \tag{9}$$

Let us remark that (9) is already known for $\sigma := t$, see [3, Theorem 5.1] for a proof.

Proof. By the truncation inequality, see e. g. [15, (Proof of) Lemma 1.6.2], we have

$$\begin{aligned}\mathbb{P}^x\left(\sup_{s\le\sigma} |X_s - x| > r\right) &\le \mathbb{P}^x(\tau_r^x \le \sigma) \le \mathbb{P}^x(|X_{\sigma\wedge\tau_r^x} - x| \ge r) \\ &\le 7r^d \int_{[-r^{-1},r^{-1}]^d} \mathrm{Re}(1 - \mathbb{E}^x e^{i\xi(X_{\sigma\wedge\tau_r^x}-x)})\, d\xi.\end{aligned}$$

An application of Dynkin's formula yields

$$\mathbb{P}^x\left(\sup_{s\le t} |X_s - x| > r\right) \le 7r^d \int_{[-r^{-1},r^{-1}]^d} \mathrm{Re}\,\mathbb{E}^x\left(\int_{[0,\sigma\wedge\tau_r^x)} q(X_s,\xi) e^{i\xi(X_s-x)}\, ds\right) d\xi.$$

Now (8) follows from the triangle inequality and Fubini's theorem; (9) is a direct consequence of (8). $\square$

3.3 Lemma *Let* $(X_t)_{t\ge 0}$ *be a Lévy-type process with bounded coefficients and* $g \in \mathcal{B}_b(\mathbb{R}^d)$*,* $g \ge 0$*, such that* $\inf_{y\in B[0,r]} g(y) > 0$ *for* $r > 0$ *sufficiently small. Then*

$$\exists \alpha > 0, \delta > 0\ \forall x \in \mathbb{R}^d,\ t \in (0,\delta]: \quad \mathbb{E}^x g(X_t - x) \ge \alpha.$$

Proof. Denote by $\tau_r^x := \inf\{t > 0; X_t \notin B[x,r]\}$ the exit time from $B[x,r]$. Obviously,

$$\begin{aligned}\mathbb{E}^x g(X_t - x) &= \mathbb{E}^x\big(g(X_t - x)\mathbb{1}_{\{\tau_r^x > t\}} + g(X_t - x)\mathbb{1}_{\{\tau_r^x \le t\}}\big) \\ &\ge \inf_{|y-x|\le r} g(y-x)(1 - \mathbb{P}^x(\tau_r^x \le t)) - \|g\|_\infty \mathbb{P}^x(\tau_r^x \le t) \\ &\ge \inf_{|y|\le r} g(y) - 2\|g\|_\infty \mathbb{P}^x(\tau_r^x \le t).\end{aligned}$$

By (the proof of) the maximal inequality and boundedness of the coefficients of the symbol, we have

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}^x(\tau_r^x \le t) \le Ct\left(1+\frac{1}{r^2}\right)$$

for some constant $C>0$ which does not depend on t, r. The claim follows by choosing $r>0$ and $\delta>0$ sufficiently small. □

Proof of Theorem 3.1. (i) Obviously, it suffices to prove "⇒". By Lemma 3.3, there exist $\delta>0$, $\alpha\in(0,1)$ such that $\mathbb{E}^y g(X_r-y)\ge\alpha$ for all $y\in\mathbb{R}^d$ and $r\in(0,\delta]$. Using the Markov property, we get

$$\begin{aligned}\mathbb{E}^x f(X_t) &= \mathbb{E}^x\left(\mathbb{E}^{X_s} f(X_{t-s})\right)\\ &= \int_\Omega\int_{\mathbb{R}^d} \frac{f((z-y)+y)}{f(y)} f(y)\,\mathbb{P}^y_{X_{t-s}}(dz)\Big|_{y=X_s}\,d\mathbb{P}^x\\ &\ge \int_\Omega\int_{\mathbb{R}^d} f(y)g(z-y)\,\mathbb{P}^y_{X_{t-s}}(dz)\Big|_{y=X_s}\,d\mathbb{P}^x\\ &\ge \alpha\mathbb{E}^x f(X_s)\end{aligned}$$

for all $s\in[t-\delta,t]$. Iterating this procedure gives $\mathbb{E}^x f(X_t)\ge\alpha^n\mathbb{E}^x f(X_s)$ for any $s\in[t-n\delta,t]$. Choosing $n\in\mathbb{N}$ sufficiently large proves $\sup_{s\le t}\mathbb{E}^x f(X_s)\le\alpha^{-n}\mathbb{E}^x f(X_t)$.

(ii) We have to check that there exists a suitable function g satisfying (6).

(a) Since $f(y)\le cf(y+z)f(-z)$, we have

$$\inf_{y\in\mathbb{R}^d}\frac{f(z+y)}{f(y)} \ge \frac{1}{c}\frac{1}{f(-z)} \ge \min\left\{1, \frac{1}{c}\frac{1}{f(-z)}\right\} =: g(z), \qquad z\in\mathbb{R}^d.$$

Moreover, as f is locally bounded, $\inf_{y\in B[0,r]} g(y)>0$ for r sufficiently small.

(b) $|\log f(z)-\log f(y)|\le c|z-y|^\gamma$ implies

$$\frac{f(z+y)}{f(y)} = \exp\left(\log f(z+y)-\log f(y)\right) \ge \exp\left(-c|z|^\gamma\right) =: g(z), \qquad z\in\mathbb{R}^d.$$

(c) As $f>c>0$, Hölder continuity of f implies Hölder continuity of $\log f$, and the claim follows from (b).

(d) By the gradient theorem,

$$|f(y+z)-f(y)| = \left|\int_0^1 \nabla f(y+tz)\cdot z\,dt\right| \le |z|\sup_{|z|\le r}|\nabla f(y+z)|$$

for all $|z|\le r$ and $y\in\mathbb{R}^d$. Applying the Cauchy–Schwarz inequality gives

$$\frac{f(z+y)}{f(y)} \ge \min\left\{1, 1-|z|\sup_{y\in\mathbb{R}^d}\sup_{|z|\le r}\frac{|\nabla f(y+z)|}{f(y)}\right\} =: g(z).$$

(e) This is an immediate consequence of (ii)(d). □

The proof of Theorem 3.1 actually shows that, under the assumptions of Theorem 3.1(i),

$$\sup_{x\in K}\mathbb{E}^x f(X_t-x)<\infty \implies \sup_{x\in K}\sup_{s\le t}\mathbb{E}^x f(X_s-x)<\infty$$

for any set $K\subseteq\mathbb{R}^d$. Next we show that the moments also exist forward in time provided that $\mathbb{E}^x f(X_t-x)$ is bounded in x and f is submultiplicative.

3.4 Corollary *Let $(X_t)_{t\ge0}$ be a Lévy-type process with bounded coefficients and $f:\mathbb{R}^d\to(0,\infty)$ a locally bounded measurable submultiplicative function. Then*

$$\exists t>0: \sup_{x\in\mathbb{R}^d}\mathbb{E}^x f(X_t-x)<\infty \implies \forall s\ge0: \sup_{r\le s}\sup_{x\in\mathbb{R}^d}\mathbb{E}^x f(X_r-x)<\infty.$$

Proof. Fix $t>0$ such that $\sup_{x\in\mathbb{R}^d}\mathbb{E}^x f(X_t-x)<\infty$. It follows from Theorem 3.1 that $M_1:=1\vee\sup_{x\in\mathbb{R}^d}\sup_{s\leq t}\mathbb{E}^x f(X_s-x)<\infty$. Using the Markov property and the submultiplicativity of f, we find

$$\mathbb{E}^x f(X_r-x)=\mathbb{E}^x\left(\mathbb{E}^y f(X_{r-t}-x)\Big|_{y=X_t}\right)\leq c\mathbb{E}^x\left(\mathbb{E}^y f(X_{r-t}-y)f(y-x)\Big|_{y=X_s}\right)\leq cM_1^2$$

for all $r\in[t,2t]$ and $x\in\mathbb{R}^d$. Hence, $M_2:=1\vee\sup_{r\leq 2t}\sup_{x\in\mathbb{R}^d}\mathbb{E}^x f(X_r-x)<\infty$. By iteration, we obtain $M_k:=1\vee\sup_{r\leq kt}\sup_{x\in\mathbb{R}^d}\mathbb{E}^x f(X_r-x)<\infty$ for all $k\in\mathbb{N}$ and

$$\sup_{x\in\mathbb{R}^d}\sup_{r\leq(k+1)t}\mathbb{E}^x f(X_r-x)\leq cM_k^2<\infty. \qquad\square$$

Remark If f is not submultiplicative, then Corollary 3.4 does, in general, not hold true. For a counterexample in the Lévy case see e. g. [16, Remark 25.9].

4 Existence of moments - sufficient conditions

In this part, we present sufficient conditions for the existence of moments for Lévy-type processes. Let us recall the corresponding well-known result for Lévy processes (cf. [16, Theorem 25.3]): For a Lévy process $(X_t)_{t\geq 0}$ with Lévy triplet (b,Q,ν), we have

$$\mathbb{E}^x f(X_t)<\infty \text{ for some (all) } t>0 \iff \int_{|y|\geq 1} f(y)\,\nu(dy)<\infty$$

for any locally bounded measurable submultiplicative function $f:\mathbb{R}^d\to(0,\infty)$. In [3, Theorem 5.11] it was observed that for $f(y):=\exp(\zeta y)$, $\zeta\in\mathbb{R}^d$, the implication

$$\sup_{x\in\mathbb{R}^d}\int_{|y|\geq 1} f(y)\,N(x,dy)<\infty \implies \forall x\in\mathbb{R}^d, t\geq 0: \mathbb{E}^x f(X_t)<\infty \tag{10}$$

still holds true for any Lévy-type process $(X_t)_{t\geq 0}$ with bounded coefficients. In Theorem 4.1 we extend this result and show (10) for any function $f\geq 0$ which is comparable to a submultiplicative C^2-function. In the second part of this section, we discuss the connection between differentiability of the symbol and existence of moments.

4.1 Theorem *Let $(X_t)_{t\geq 0}\sim(b(x),Q(x),N(x,dy))$ be a Lévy type process and $K\subseteq\mathbb{R}^d$ a compact set. Let $f:\mathbb{R}^d\to[0,\infty)$ be a measurable function and $g\in C^2$ submultiplicative such that $g\geq 0$ and $f\asymp g$. Then for any $t>0$*

$$\sup_{x\in K}\int_{|y|\geq 1} f(y)\,N(x,dy)<\infty \implies \sup_{s\leq t}\sup_{x\in K}\mathbb{E}^x f(X_{s\wedge\tau_K}-x)<\infty$$

and

$$\mathbb{E}^x f(X_{t\wedge\tau_K})\leq Cf(x)\exp\left(C(M_1+M_2)t\right) \tag{11}$$

where $\tau_K:=\inf\{t>0; X_t\notin K\}$ denotes the exit time from the set K, $C=C(K)>0$ is a constant (which does not depend on $(X_t)_{t\geq 0}$ and t) and

$$M_1:=\sup_{x\in K}\left(|b(x)|+|Q(x)|+\int_{\mathbb{R}^d\setminus\{0\}}(|y|^2\wedge 1)N(x,dy)\right)<\infty \qquad M_2:=\sup_{x\in K}\int_{|y|\geq 1} f(y)\,N(x,dy)<\infty.$$

If $(X_t)_{t\geq 0}$ has bounded coefficients, then the claim holds for $K=\mathbb{R}^d$.

Proof. To keep notation simple, we only give the proof for $d=1$. We can assume without loss of generality that $f\in C^2$ is submultiplicative (otherwise replace f by g). Let $(\Omega^\circ,\mathcal{A}^\circ,\mathcal{F}_t^\circ,\mathbb{P}^{\circ,x})$, $(W_t^\circ)_{t\geq 0}$, $(L_t^\circ)_{t\geq 0}$, N° and k,σ be as in Theorem 2.1. For fixed $R>0$ define an $\mathcal{F}_t^\circ$-stopping time by

$$\tau_R^x:=\inf\{t>0; \max\{|X_t^1|,|X_t^2|\}\geq R\}$$

and set $\tau := \tau_K \wedge \tau_R^x$. By the submultiplicativity of f, we have

$$f(X_t - X_0) = f(X_t^1 + X_t^2) \leq c f(X_t^1) f(X_t^2)$$

for some constant $c > 0$. Since a submultiplicative function growths at most exponentially, cf. [16, Lemma 25.5], there exist constants $a, b > 0$ such that

$$f(X_t - X_0) \leq a \exp\Big(b(\sqrt{(X_t^1)^2 + 1} - 1)\Big) f(X_t^2) =: h(X_t^1) f(X_t^2).$$

Moreover, a straightforward calculation shows

$$|h'(x)| + |h''(x)| \leq C_1 h(x), \qquad x \in \mathbb{R}, \tag{12}$$

for some constant $C_1 > 0$. By Itô's formula and optional stopping,

$$\begin{aligned}
&\mathbb{E}^{\circ,x}(h(X_{t\wedge\tau}^1) f(X_{t\wedge\tau}^2)) - a f(0) \\
&\quad= \mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} h'(X_{s-}^1) f(X_{s-}^2) b(X_{s-})\, ds\Big) + \frac{1}{2}\mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} h''(X_{s-}^1) f(X_{s-}^2)\sigma^2(X_{s-})\, ds\Big) \\
&\quad+ \mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} \int_{|k|\leq 1} f(X_{s-}^2)(h(X_{s-}^1 + k(X_{s-}, y)) - h(X_{s-}^1) - h'(X_{s-}^1) k(X_{s-}, y))\, \nu^\circ(dy)\, ds\Big) \\
&\quad+ \mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} \int_{|k|>1} h(X_{s-}^1)(f(X_{s-}^2 + k(X_{s-}, y)) - f(X_{s-}^2))\, \nu^\circ(dy)\, ds\Big) \\
&\quad=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Recall that ν° denotes the Lévy measure of the Cauchy process $(L_t^\circ)_{t\geq 0}$. We estimate the terms separately. By (12) and the definition of M_1, it follows easily that

$$|I_1| + |I_2| \leq C_1 M_1 \mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} h(X_{s-}^1) f(X_{s-}^2)\, ds\Big).$$

For I_4 we note that by the submultiplicativity of f and (3),

$$\begin{aligned}
|I_4| &\leq c\mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)}\int_{|k|>1} h(X_{s-}^1) f(X_{s-}^2)(1 + f(k(X_{s-}, y)))\, \nu^\circ(dy)\, ds\Big) \\
&\leq c(M_1 + M_2)\mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} h(X_{s-}^1) f(X_{s-}^2)\, ds\Big).
\end{aligned}$$

It remains to estimate I_3. By Taylor's formula, we have

$$|h(x+z) - h(x) - h'(x)z| \leq \frac{1}{2}|h''(\xi)| z^2$$

for some intermediate value $\xi = \xi(x,z) \in (x, x+z)$. Since there exists $C_2 > 0$ such that $|h''(\xi)| \leq C_2 h(x)$ for all $|z| \leq 1$ and $x \in \mathbb{R}$, we get

$$|I_3| \leq C_2 M_1 \mathbb{E}^{\circ,x}\Big(\int_{[0,t\wedge\tau)} h(X_{s-}^1) f(X_{s-}^2)\, ds\Big).$$

Combining all estimates shows that $\varphi(t) := \mathbb{E}^{\circ,x}(h(X_{t\wedge\tau}^1) f(X_{t\wedge\tau}^2) \mathbb{1}_{\{t<\tau\}})$ satisfies

$$\varphi(t) \leq \mathbb{E}^{\circ,x}\left(h(X_{t\wedge\tau}^1) f(X_{t\wedge\tau}^2)\right) \leq a f(0) + C_3 \int_0^t \varphi(s)\, ds$$

for some constant $C_3 = C_3(M_1, M_2, f)$. Now it follows from Gronwall's inequality, see e. g. [17, Theorem A.43], that $\varphi(t) \leq a f(0) e^{C_3 t}$. Finally, using Fatou's lemma, we can let $R \to \infty$ and obtain

$$\mathbb{E}^x f(X_{t\wedge\tau_K} - x) \leq \mathbb{E}^{\circ,x}(h(X_{t\wedge\tau_K}^1) f(X_{t\wedge\tau_K}^2)) \leq a f(0) e^{C_3 t}.$$

This proves $\sup_{x\in K}\sup_{s\leq t}\mathbb{E}^x f(X_{s\wedge\tau_K} - x) < \infty$; (11) follows from $f(X_t) \leq c f(X_t - x) f(x)$ and the previous inequality. $\square$

Remark The proof of Theorem 4.1 shows that the statement holds true for any function f such that there exist $g_1 \in C^2$ submultiplicative, $g_2 \in C^2$ subadditive, $g_1 \geq 0$, $\inf_{x\in\mathbb{R}^d} g_2(x) > 0$ and $f \asymp g := g_1 \cdot g_2$.

4.2 Example (i) Let $(X_t)_{t\geq 0}$ be a Lévy-type process with uniformly bounded jumps, i. e. there exist $R_1, R_2 > 0$ such that $\operatorname{supp} N(x,\cdot) \subseteq \{y \in \mathbb{R}^d; R_1 \leq |y| \leq R_2\}$ for all $x \in \mathbb{R}^d$. Then we have

$$\sup_{x\in\mathbb{R}^d} \sup_{s\leq t} \mathbb{E}^x f(X_s - x) < \infty \qquad \text{for all } t \geq 0$$

for any measurable function $f \geq 0$ which is comparable to a submultiplicative C^2-function (e. g. $f(x) = |x|^\alpha \vee 1$, $\alpha > 0$, $f(x) = \exp(|x|^\beta)$, $\beta \in (0,1]$, $f(x) = \log(|x| \vee e)$, ...)

(ii) Let $(X_t)_{t\geq 0}$ be a stable-like process, that is a Lévy-type process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for some function $\alpha : \mathbb{R}^d \to (0,2)$; for the existence of such processes see [9]. If we set $\alpha_l := \inf_{x\in\mathbb{R}^d} \alpha(x)$, then, by Theorem 4.1,

$$\sup_{x\in\mathbb{R}^d} \sup_{s\leq t} \mathbb{E}^x(|X_s - x|^\alpha) < \infty \qquad \text{for all } \alpha \in [0, \alpha_l).$$

Now we turn to the question whether regularity of the symbol is related to the existence of moments. It is a classical result that for the characteristic function $\chi(\xi) := \mathbb{E}e^{i\xi X}$ of a random variable X,

$$\chi \text{ is } 2n \text{ times differentiable at } \xi = 0 \iff \mathbb{E}(X^{2n}) < \infty$$

for all $n \in \mathbb{N}$. In particular for a Lévy process $(X_t)_{t\geq 0}$ with characteristic exponent ψ it follows easily from the Lévy-Khintchine formula that

$$\psi \text{ is } 2n \text{ times differentiable at } \xi = 0 \implies \forall t \geq 0 : \mathbb{E}(|X_t|^{2n}) < \infty.$$

Theorem 4.4 below shows that this result can be extended to Lévy-type processes. For the proof we use the following statement which is of independent interest. To keep notation simple we state the result only in dimension $d = 1$; it can be easily extended to higher dimensions by considering $q_j(x,\eta) := q(x, \eta e_j)$, $\eta \in \mathbb{R}$, for $j \in \{1, \dots, d\}$. Here, e_j denotes the j-th unit vector in $\mathbb{R}^d$.

4.3 Lemma *Let $(q(x,\xi))_{x\in\mathbb{R}}$ be a family of negative definite functions with Lévy-Khintchine representation* (2) *and assume that $q(x,0) = 0$ for all $x \in \mathbb{R}$. Let $n \in \mathbb{N}$ and $K \subseteq \mathbb{R}$ be a compact set. Then the following statements are equivalent.*

(i) $q(x,\cdot)$ is $2n$ times differentiable for all $x \in K$, $\xi \in \mathbb{R}$ and $\sup_{x\in K} \sup_{\xi\in\mathbb{R}} |\partial_\xi^{2n} q(x,\xi)| < \infty$.

(ii) $q(x,\cdot)$ is $2n$ times differentiable at $\xi = 0$ for all $x \in K$ and $\sup_{x\in K} |\partial_\xi^{2n} q(x,0)| < \infty$.

(iii) $\sup_{x\in K} \int_{\mathbb{R}\setminus\{0\}} y^{2n} \, N(x,dy) < \infty$.

In this case,

$$\frac{\partial^k}{\partial\xi^k} q(x,\xi) = \begin{cases} -ib(x) + Q(x)\xi + i\int_{\mathbb{R}\setminus\{0\}} (\mathbb{1}_{(0,1]}(|y|)) - e^{iy\xi}) y \, N(x,dy), & k = 1, \\ Q(x) + \int_{\mathbb{R}\setminus\{0\}} y^2 e^{iy\xi} \, N(x,dy), & k = 2, \\ i^{k+2} \int_{\mathbb{R}\setminus\{0\}} y^k e^{iy\xi} \, N(x,dy), & k \in \{3, \dots, 2n\}. \end{cases} \tag{13}$$

If q has bounded coefficients, then (i)-(iii) are equivalent for $K = \mathbb{R}$.

Proof. Obviously, (i) $\Rightarrow$ (ii), so it suffices to prove (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). We prove the claim by induction.

$n = 1$: Suppose that (ii) holds true. Using the classical identities

$$\frac{1}{2} = \lim_{y\to 0} \frac{1 - \cos(y)}{y^2} \qquad \text{and} \qquad \lim_{h\to 0} \frac{\phi(2h) - 2\phi(0) + \phi(-2h)}{4h^2} = \phi''(0) \tag{14}$$

for ϕ twice differentiable at 0, we find by Fatou's lemma

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} y^2\, N(x,dy) &= 2\int_{\mathbb{R}\setminus\{0\}} y^2 \lim_{h\to 0}\frac{1-\cos(2hy)}{(2hy)^2}\, N(x,dy)\\ &\leq \liminf_{h\to 0}\frac{1}{2h^2}\int_{\mathbb{R}}(1-\cos(2hy))\, N(x,dy)\\ &= 2\liminf_{h\to 0}\left(\frac{q(x,2h)+q(x,-2h)}{4h^2}-Q(x)\right)\\ &= 2\frac{\partial^2}{\partial\xi^2}q(x,0)-2Q(x).\end{aligned}$$

Since Q is locally bounded, cf. [3, Theorem 2.27], (iii) follows. On the other hand, if (iii) holds, then it is obvious from the Lévy-Khintchine representation that $q(x,\cdot)$ is twice differentiable and that (13) holds for $k=1,2$.

$n-1\to n$: Suppose that (ii) holds for $n\geq 2$. Then, by the induction hypothesis, we get as in the first part of the proof

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} y^{2n}\, N(x,dy) &\leq \liminf_{h\to 0}\frac{1}{2h^2}\int_{\mathbb{R}\setminus\{0\}} y^{2(n-1)}(1-\cos(2hy))\, N(x,dy)\\ &= 2(-1)^{n-1}\liminf_{h\to 0}\frac{1}{4h^2}\left(\frac{\partial^{2n-2}}{\partial\xi^{2n-2}}q(x,2h)-2\frac{\partial^{2n-2}}{\partial\xi^{2n-2}}q(x,0)+\frac{\partial^{2n-2}}{\partial\xi^{2n-2}}q(x,-2h)\right)\\ &= 2(-1)^{n-1}\frac{\partial^{2n}}{\partial\xi^{2n}}q(x,0).\end{aligned}$$

This shows (iii). If (iii) holds, then we can use again the Lévy-Khintchine representation to conclude that $q(x,\cdot)$ is $2n$ times differentiable, $\sup_{x\in K}\sup_{\xi\in\mathbb{R}}|q^{(2n)}(x,\xi)|<\infty$ and that (13) holds. □

Using Lemma 4.3, we obtain the following statement.

4.4 Theorem *Let $(X_t)_{t\geq 0}=(X_t^{(1)},\dots,X_t^{(d)})_{t\geq 0}\sim(b(x),Q(x),N(x,dy))$ be a Lévy-type process with symbol q and let $K\subseteq\mathbb{R}^d$ be compact. Suppose that $\mathbb{R}\ni\xi\mapsto q_j(x,\xi):=q(x,\xi e_j)$ is $2n$ times differentiable at $\xi=0$ for all $x\in\mathbb{R}^d$ and*

$$\left|\frac{\partial^k}{\partial\xi^k}q_j(x,0)\right|\leq c_k(1+|x_j|^k),\qquad k=1,\dots,2n, \tag{15}$$

for some constants $c_k>0$. Then there exist $C_1,C_2>0$ such that

$$\sup_{x\in K}\sup_{s\leq t}\mathbb{E}^x((X_s^{(j)}-x_j)^{2n})\leq C_1te^{C_2t}\qquad \textit{for all } t\geq 0.$$

Proof. We show the result only for $d=1$; for $d>1$ replace h by $h\cdot e_j$. Throughout this proof, we denote by L the operator

$$Lf(x):=b(x)f'(x)+\frac{1}{2}Q(x)f''(x)+\int_{\mathbb{R}\setminus\{0\}}(f(x+y)-f(x)-f'(x)y\mathbb{1}_{(0,1]}(|y|))\, N(x,dy),\quad x\in\mathbb{R},$$

which is well-defined for any $f\in C_b^2(\mathbb{R})$. We remind the reader that any function $f\in C_c^2(\mathbb{R})$ is contained in the domain of the generator A of $(X_t)_{t\geq 0}$ and that $Af=Lf$.

We prove the claim by induction and start with $n=1$. By Lemma 4.3, $\sup_{x\in K}\int_{\mathbb{R}\setminus\{0\}}y^2\, N(x,dy)<\infty$. Set $f_{h,x}(z):=e^{i(z-x)h}-1$ for fixed $h,x\in\mathbb{R}$. Using Taylor's formula and the identity

$$\begin{aligned}&Lf_{h,x}(z)+Lf_{-h,x}(z)\\ &\quad= -2h\sin((z-x)h)b(z)-2\cos(h(z-x))h^2Q(z)\\ &\quad+2\int_{\mathbb{R}\setminus\{0\}}(\cos((z+y-x)h)-1)-(\cos((z-x)h)-1)+yh\mathbb{1}_{(0,1]}(|y|))\sin(h(z-x))\, N(z,dy),\end{aligned}$$

it follows easily that $\sup_{|h|\le 1}(Lf_{h,x}(z)+Lf_{-h,x}(z))/h^2$ is locally bounded (in z). For fixed $R>0$ set $\tau := \tau_R^x := \inf\{t>0; X_t \notin B(x,R)\}$ and $\varphi(t) := \mathbb{E}^x(|X_{t\wedge\tau}-x|^2 \mathbb{1}_{\{t<\tau\}})$. By (14),

$$\begin{aligned}\varphi(t) \le \mathbb{E}^x(|X_{t\wedge\tau}-x|^2) &= 2\int_\Omega |X_{t\wedge\tau}-x|^2 \lim_{h\to 0}\frac{1-\cos(2h(X_{t\wedge\tau}-x))}{4h^2(X_{t\wedge\tau}-x)^2}\,d\mathbb{P}\\ &\le \liminf_{h\to 0}\frac{1}{4h^2}\left(-\mathbb{E}^x e^{i\,2h(X_{t\wedge\tau}-x)}+2-\mathbb{E}^x e^{-i\,2h(X_{t\wedge\tau}-x)}\right).\end{aligned}$$

Pick a cut-off function $\chi\in C_c^2(\mathbb{R})$ such that $\mathbb{1}_{B(0,1)}\le\chi\le\mathbb{1}_{B(0,2)}$. Applying Dynkin's formula to the truncated functions $y\mapsto(-e^{-i2h(y-x)}+1)\chi(y/n)\in C_c^2(\mathbb{R})$ and $y\mapsto(-e^{i2h(y-x)+1}+1)\chi(y/n)\in C_c^2(\mathbb{R})$ and letting $n\to\infty$ using the dominated convergence theorem, we find

$$\varphi(t)\le\liminf_{h\to 0}\mathbb{E}^x\left(\int_{[0,t\wedge\tau)}\frac{Lf_{2h,x}(X_s)+Lf_{-2h,x}(X_s)}{4h^2}\,ds\right).$$

By the above considerations, we may apply the dominated convergence theorem and obtain using (14)

$$\varphi(t)\le\mathbb{E}^x\left(\int_{[0,t\wedge\tau)}\frac{\partial^2}{\partial h^2}Lf_{h,x}(X_s)\bigg|_{h=0}\,ds\right)=\mathbb{E}^x\left(\int_{[0,t\wedge\tau)}Lg_x(X_s)\,ds\right)$$

where $g_x(z):=(z-x)^2$. The growth assumptions (15) for $k=1,2$ imply, by (13), that

$$\left|b(z)+\int_{|y|\ge 1}y\,N(z,dy)\right|\le c_1(1+|z|)\quad\text{and}\quad Q(z)+\int_{\mathbb{R}\setminus\{0\}}y^2\,N(z,dy)\le c_2(1+z^2)$$

for all $z\in\mathbb{R}$. Therefore it is not difficult to see from the definition of L that there exist constants $C_1,C_2>0$ (which depend (continuously) on x, but not on R) such that φ satisfies the integral inequality

$$\varphi(t)\le C_1t+C_2\int_0^t\varphi(s)\,ds\qquad\text{for all } t\ge 0.$$

By the Gronwall inequality, cf. [17, Theorem A.43], we get $\varphi(t)\le C_1t\exp(C_2t)$. Since the constants C_1,C_2 do not depend on R, the claim follows from Fatou's lemma.

Now suppose that q satisfies the assumptions of Theorem 4.4 for $n\ge 2$ and that the claim holds true for $n-1$. Then $q(x,\cdot)$ is $2(n-1)$ times differentiable at $\xi=0$ and it follows from the inductional hypothesis and Lemma 4.3 that $\sup_{x\in K}\int_{\mathbb{R}\setminus\{0\}}|y|^{2n-2}\,N(x,dy)<\infty$, $\sup_{x\in K}\mathbb{E}^x(|X_t-x|^{2n-2})<\infty$ and

$$\frac{\partial^{2n-2}}{\partial\xi^{2n-2}}q(x,\xi)=Q(x)\delta_{2,n}+(-1)^{n-1}\int_{\mathbb{R}\setminus\{0\}}y^{2n-2}e^{i\,y\xi}\,N(x,dy).\tag{16}$$

(Here, $\delta_{k,n}$ denotes the Kronecker delta.) For fixed $h,x\in\mathbb{R}$, set $f_{h,x}(z):=(z-x)^{2n-2}(e^{i\,h(z-x)}-1)$. By Taylor's formula and (16), it is not difficult to see that $\sup_{|h|\le 1}(Lf_{h,x}(z)+Lf_{-h,x}(z))/h^2$ is locally bounded (in z). As in the first part, an application of Fatou's lemma and Dynkin's formula yields

$$\begin{aligned}\mathbb{E}^x(|X_{t\wedge\tau}-x|^{2n})&\le\liminf_{h\to 0}\frac{1}{4h^2}\left(\mathbb{E}^x f_{2h,x}(X_{t\wedge\tau})-f_{2h,x}(0)-f_{-2h,x}(0)+\mathbb{E}^x f_{-2h,x}(X_{t\wedge\tau})\right)\\ &=\liminf_{h\to 0}\mathbb{E}^x\left(\int_{[0,t\wedge\tau)}\frac{Lf_{2h,x}(X_s)+Lf_{-2h,x}(X_s)}{4h^2}\,ds\right).\end{aligned}$$

Since $\sup_{|h|\le 1}(Lf_{h,x}(z)+Lf_{-h,x}(z))/h^2$ is locally bounded, it follows from the dominated convergence theorem that

$$\mathbb{E}^x(|X_{t\wedge\tau}-x|^{2n})\le\mathbb{E}^x\left(\int_{[0,t\wedge\tau)}\frac{\partial^2}{\partial h^2}Lf_{h,x}(X_s)\Big|_{h=0}\,ds\right)=\mathbb{E}^x\left(\int_{[0,t\wedge\tau)}Lg_x(X_s)\,ds\right)$$

for $g_x(z):=(z-x)^{2n}$. Using again the growth assumptions and Taylor's formula, we find that $\varphi(t):=\mathbb{E}^x(|X_{t\wedge\tau}-x|^{2n}\mathbb{1}_{\{t<\tau\}})$ satisfies

$$\varphi(t)\le C_1t+C_2\int_0^t\varphi(s)\,ds,\qquad t\ge 0,$$

for $C_1,C_2>0$ (not depending on R). Applying Gronwall's inequality and Fatou's lemma finishes the proof. □

Remark Let $(X_t)_{t\geq 0}$ be a geometric Brownian motion, i. e. a solution to the SDE

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\mu \in \mathbb{R}$, $\sigma > 0$. One can easily verify that

$$\mathbb{E}^x((X_t - x)^2) = x^2(e^{2\mu t}(e^{\sigma^2 t} - 2) + 1).$$

This means that exponential growth for large t and linear growth for small t is the best we can expect; in this sense the estimate in Theorem 4.4 is optimal.

4.5 Example Let $(L_t)_{t\geq 0}$ be a (d-dimensional) Lévy process with characteristic exponent ψ. Suppose that the Lévy-driven SDE

$$dX_t = f(X_{t-})\,dL_t, \qquad X_0 = x, \tag{17}$$

has a unique solution $(X_t)_{t\geq 0}$ which is a Lévy-type process and suppose that its symbol is given by $q(x,\xi) = \psi(f(x)^T\xi)$. If ψ is $2n$-times differentiable at $\xi = 0$, i. e. if $\mathbb{E}(|L_t|^{2n}) < \infty$, then it follows from Theorem 4.4 that $\sup_{s\leq t}\sup_{x\in K}\mathbb{E}^x(|X_s - x|^k) < \infty$ for any compact set $K \subseteq \mathbb{R}^d$ and $k \leq 2n$.

Important classes of examples are the following:

(i) If f is bounded and locally Lipschitz continuous, then the (unique) solution to (17) is a Lévy-type process with symbol $q(x,\xi) = \psi(f(x)^T\xi)$, cf. [19]. The boundedness of f is needed to ensure that $(X_t)_{t\geq 0}$ is a Lévy-type process; see [19, Rem. 3.4] for an example where f is locally Lipschitz continuous, but the solution fails to be a Lévy-type process.

(ii) ($d = 2$) The generalized Ornstein-Uhlenbeck process is the solution to the SDE

$$dX_t = X_{t-}\,dL_t^{(1)} + dL_t^{(2)}, \qquad X_0 = x.$$

In [1, Theorem 3.1], it was shown that $(X_t)_{t\geq 0}$ is a Lévy-type process with symbol $q(x,\xi) = \psi((x,1)^T\xi)$.

5 Fractional moments

This section is devoted to estimates of fractional moments, i. e. we study the small-time and large-time asymptotics of $\mathbb{E}^x(\sup_{s\leq t}|X_s - x|^\alpha)$ for $\alpha > 0$. Depending on α, there are different techniques to prove such estimates; the following ones have recently been used to obtain estimates for Lévy processes:

(i) $\alpha \in (0,1]$: bounded variation technique, cf. [13, Theorem 1].

(ii) $\alpha \geq 1$: martingale technique based on the Burkholder–Davis–Gundy inequality, cf. [13, Theorem 1].

(iii) $\alpha \in (0,2)$: characterization via Blumenthal–Getoor indices, cf. [5, Section 3].

Combining the bounded variation and martingale techniques with Theorem 2.1, we will extend [13, Theorem 1] to Lévy-type processes in the first part of this section (Theorem 5.1, Theorem 5.2). In the second part, we will introduce generalized Blumenthal–Getoor indices and prove extensions of the results presented in [5]; cf. Theorem 5.3, Corollary 5.5 and Theorem 5.6. Let us remark that the small-time estimate

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s - x|^\alpha\right) \leq Ct$$

is the best we can expect; otherwise, the Kolmogorov–Chentsov theorem would imply the existence of a modification with exclusively continuous sample paths.

We start with a combination of the bounded variation and martingale technique. A crucial ingredient to obtain estimates is the Burkholder–Davis–Gundy inequality; for continuous martingales this inequality is standard, but for discontinuous martingales the proof is more involved, see e. g. [14] or [11]. The following theorem generalizes [5, Theorem 3.1] and the corresponding result in [13].

5.1 Theorem *Let $(X_t)_{t\geq 0} \sim (b(x), Q(x), N(x,dy))$ be a Lévy-type process with bounded coefficients and suppose that*

$$M := \sup_{x\in\mathbb{R}^d}\left(\int_{|y|>1}|y|^\alpha\,N(x,dy)+\int_{|y|\leq 1}|y|^\beta\,N(x,dy)\right)<\infty$$

for some $\alpha\in(0,1]$ and $\beta\in[0,2]$.

(i) $\beta\in[1,2]$: Then there exists $C>0$ such that

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right)\leq t^\kappa\sup_{x\in\mathbb{R}^d}|b(x)|^\kappa+Ct^{\kappa/2}\sup_{x\in\mathbb{R}^d}|Q(x)|^{\kappa/2} + Ct^{\kappa/\beta}\left(\sup_{x\in\mathbb{R}^d}\int_{|y|\leq 1}|y|^\beta\,N(x,dy)\right)^{\kappa/\beta}+t^{\kappa/\alpha}\sup_{x\in\mathbb{R}^d}\left(\int_{|y|>1}|y|^\alpha\,N(x,dy)\right)^{\kappa/\alpha}$$

for all $t\geq 0$ and $\kappa\in[0,\alpha]$.

(ii) $\beta\in[\alpha,1]$: Then there exists $C>0$ such that

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right)\leq t^\kappa\sup_{x\in\mathbb{R}^d}\left|b(x)+\int_{|y|\leq 1}y\,N(x,dy)\right|^\kappa+Ct^{\kappa/2}\sup_{x\in\mathbb{R}^d}|Q(x)|^{\kappa/2} + t^{\kappa/\beta}\left(\sup_{x\in\mathbb{R}^d}\int_{|y|\leq 1}|y|^\beta\,N(x,dy)\right)^{\kappa/\beta}+t^{\kappa/\alpha}\sup_{x\in\mathbb{R}^d}\left(\int_{|y|>1}|y|^\alpha\,N(x,dy)\right)^{\kappa/\alpha}$$

for all $t\geq 0$ and $\kappa\in[0,\alpha]$.

(iii) $\beta\in[0,\alpha]$: Then there exists $C>0$ such that

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right)\leq t^\kappa\sup_{x\in\mathbb{R}^d}\left|b(x)+\int_{|y|\leq 1}y\,N(x,dy)\right|^\kappa+Ct^{\kappa/2}\sup_{x\in\mathbb{R}^d}|Q(x)|^{\kappa/2} + t^{\kappa/\alpha}\sup_{x\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d\setminus\{0\}}|y|^\alpha\,N(x,dy)\right)^{\kappa/\alpha}$$

for all $t\geq 0$ and $\kappa\in[0,\alpha]$.

Since any Lévy-type process with bounded coefficients satisfies $\sup_{x\in\mathbb{R}^d}\int_{|y|\leq 1}|y|^2\,N(x,dy)<\infty$, Theorem 5.1(i) is applicable with $\beta=2$ whenever $\sup_{x\in\mathbb{R}^d}\int_{|y|>1}|y|^\alpha\,N(x,dy)<\infty$ for some $\alpha\in(0,1]$. Moreover, by the Markov property, Theorem 5.1 gives also bounds for $\mathbb{E}^x\left(\sup_{s\leq t}|X_{s+r}-X_r|^\kappa\right)$ for any fixed $r\geq 0$.

It is well-known that a Lévy process $(X_t)_{t\geq 0}$ with Lévy triplet $(0,0,\nu(dy))$ has sample paths of bounded variation and satisfies $\mathbb{E}(|X_t|^\alpha)<\infty$ if, and only if, $\int_{\mathbb{R}^d\setminus\{0\}}|y|^\alpha\,\nu(dy)<\infty$ for some $\alpha\in(0,1]$, cf. [16, Theorem 21.9, Theorem 25.3]. Theorem 5.1 extends this statement to Lévy-type processes. If $(X_t)_{t\geq 0}\sim(0,0,N(x,dy))$ is a Lévy-type process such that $\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{0\}}|y|^\alpha\,N(x,dy)<\infty$ for some $\alpha\in(0,1]$, then Theorem 5.1 shows that $(X_t)_{t\geq 0}$ has $\mathbb{P}^x$-almost surely a finite (strong) p-variation on compact t-intervals for any $p>\alpha$ and $x\in\mathbb{R}^d$.

Proof of Theorem 5.1. Because of Jensen's inequality, it suffices to prove the claim for $\kappa=\alpha$. By Theorem 2.1, there exist a Markov extension $(\Omega^\circ,\mathcal{A}^\circ,\mathcal{F}_t^\circ,\mathbb{P}^{\circ,x})$, a Brownian motion $(W_t^\circ)_{t\geq 0}$, a Cauchy process $(L_t^\circ)_{t\geq 0}$ with jump measure N° and k,σ such that (3) holds and

$$\begin{aligned}X_t-x&=\int_0^t b(X_{s-})\,ds+\int_0^t\sigma(X_{s-})\,dW_s^\circ+\int_0^t\int_{|k|>1}k(X_{s-},z)\,N^\circ(dz,ds)\\&\quad+\int_0^t\int_{|k|\leq 1}k(X_{s-},z)\,(N^\circ(dz,ds)-\nu^\circ(dz)\,ds)\end{aligned}\tag{18}$$

$\mathbb{P}^{\circ,x}$-almost surely. First, we prove (i). By (18), we have

$$
\begin{aligned}
X_t - x = &\int_0^t b(X_{s-})\,ds + \int_0^t \sigma(X_{s-})\,dW_s^\circ + \sum_{0\le s\le t} k(X_{s-},\Delta L_s^\circ)\mathbb{1}_{\{|k(X_{s-},\Delta L_s^\circ)|>1\}} \\
&+ \int_0^t \int_{|k|\le 1} k(X_{s-},z)\,(N^\circ(dz,ds) - \nu^\circ(dz,ds)).
\end{aligned}
$$

Using the elementary estimate $(u+v)^\alpha \le u^\alpha + v^\alpha$, $u,v\ge 0$, $\alpha\in(0,1]$, yields

$$
\begin{aligned}
\sup_{s\le t}|X_s - x|^\alpha &\le \sup_{s\le t}\left|\int_0^s b(X_{r-})\,dr\right|^\alpha + \sup_{s\le t}\left|\int_0^s \sigma(X_{r-})\,W_r^\circ\right|^\alpha + \sum_{0\le s\le t}|k(X_{s-},\Delta L_s^\circ)|^\alpha \mathbb{1}_{\{|k(X_{s-},\Delta L_s^\circ)|>1\}} \\
&\quad + \sup_{s\le t}\left|\int_0^s\int_{|k|\le 1} k(X_{r-},z)\,(N^\circ(dz,dr)-\nu^\circ(dz,dr))\right|^\alpha \\
&\le \sup_{x\in\mathbb{R}^d}|b(x)|^\alpha t^\alpha + \sup_{s\le t}\left|\int_0^s \sigma(X_{r-})\,dW_r^\circ\right|^\alpha + \int_0^t\int_{|k|>1}|k(X_{s-},z)|^\alpha\,N^\circ(dz,ds) \\
&\quad + \sup_{s\le t}\left|\int_0^s\int_{|k|\le 1} k(X_{r-},z)\,(N^\circ(dz,dr)-\nu^\circ(dz,dr))\right|^\alpha .
\end{aligned}
$$

Integrating both sides and using that, by Jensen's inequality,

$$
\mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\int_0^s \sigma(X_{r-})\,W_r^\circ\right|^\alpha\right) \le \mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\int_0^s \sigma(X_{r-})\,W_r^\circ\right|^2\right)^{\alpha/2}
$$

and

$$
\begin{aligned}
&\mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\int_0^s\int_{|k|\le 1} k(X_{r-},z)\,(N^\circ(dz,dr)-\nu^\circ(dz,dr))\right|^\alpha\right) \\
&\qquad \le \mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\int_0^s\int_{|k|\le 1} k(X_{r-},z)\,(N^\circ(dz,dr)-\nu^\circ(dz,dr))\right|^\beta\right)^{\alpha/\beta},
\end{aligned}
$$

the assertion follows (for $\kappa=\alpha$) from Itô's isometry and the Burkholder–Davis–Gundy inequality [14, Theorem 1].

If $\beta\in[0,1]$, then

$$
\mathbb{E}^{\circ,x}\left(\int_0^t\int_{|k|\le 1}|k(X_{s-},z)|\,\nu^\circ(dz)\,ds\right) = \mathbb{E}^{\circ,x}\left(\int_0^t\int_{|y|\le 1}|y|\,N(X_{s-},dy)\,ds\right) \le Mt < \infty.
$$

Therefore, we can write

$$
\begin{aligned}
X_t - x &= \int_0^t \tilde b(X_{s-})\,ds + \int_0^t \sigma(X_{s-})\,dW_s^\circ + \int_0^t\int_{\mathbb{R}^d\setminus\{0\}} k(X_{s-},z)\,N^\circ(dz,ds) \\
&= \int_0^t \tilde b(X_{s-})\,ds + \int_0^t \sigma(X_{s-})\,dW_s^\circ + \sum_{0\le s\le t} k(X_{s-},\Delta L_s^\circ)
\end{aligned}
$$

where $\tilde b(x) := b(x) + \int_{|y|\le 1} y\,N(x,dy)$. The bounds for drift and diffusion are obtained as in the first part of this proof; it remains to estimate the jump part. If $1\ge\beta\ge\alpha$, then another application of the inequality $(u+v)^\beta\le u^\beta+v^\beta$ and Jensen's inequality yield

$$
\begin{aligned}
\mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\sum_{0\le r\le s} k(X_{r-},\Delta L_r)\mathbb{1}_{\{|k(X_{r-},\Delta L_r)|\le 1\}}\right|^\alpha\right) &\le \mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\sum_{0\le r\le s} k(X_{r-},\Delta L_r)\mathbb{1}_{\{|k(X_{r-},\Delta L_r)|\le 1\}}\right|^\beta\right)^{\alpha/\beta} \\
&\le \mathbb{E}^{\circ,x}\left(\sum_{0\le r\le t}|k(X_{r-},\Delta L_r)|^\beta \mathbb{1}_{\{|k(X_{r-},\Delta L_r)|\le 1\}}\right)^{\alpha/\beta} \\
&= \mathbb{E}^{\circ,x}\left(\int_0^t\int_{|y|\le 1}|y|^\beta\,N(X_{s-},dy)\,ds\right)^{\alpha/\beta}.
\end{aligned}
$$

Estimating the large jumps in exactly the same way (but without applying Jensen's inequality), we get

$$
\mathbb{E}^{\circ,x}\left(\sup_{s\le t}\left|\sum_{0\le r\le s} k(X_{r-},\Delta L_r)\right|^\alpha\right) \le t^{\alpha/\beta}\sup_{x\in\mathbb{R}^d}\left[\int_{|y|\le 1}|y|^\beta\,N(x,dy)\right]^{\alpha/\beta} + t\sup_{x\in\mathbb{R}^d}\int_{|y|>1}|y|^\alpha\,N(x,dy).
$$

Finally, if $\beta\in[0,\alpha]$, then $M<\infty$ for $\beta=\alpha$, and the claim follows from (ii). □

Remark (i) Let $\kappa \in (0,1]$ be such that $\inf_{\theta\in[\kappa,1]} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} |y|^\theta \, N(x,dy) < \infty$ and

$$q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}} (1-e^{iy\xi}) \, N(x,dy).$$

Then an application of Jensen's inequality and Theorem 5.1 show that

$$\mathbb{E}^x \left(\sup_{s\leq t} |X_s - x|^\kappa \right) \leq \inf_{\theta\in[\kappa,1]} \left(t \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} |y|^\theta \, N(x,dy) \right)^{\kappa/\theta}$$

for any Lévy-type process $(X_t)_{t\geq 0}$ with symbol q. This generalizes [5, Theorem 3.2] where the inequality was proved for Lévy processes.

(ii) Let $(X_t)_{t\geq 0}$ be a Lévy-type process and $\alpha \in (0,1]$ such that

$$f(t) := \sup_{s\leq t} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x(|X_s - x|^\alpha) < \infty \qquad \text{for all } t \geq 0. \tag{$\star$}$$

Then $\lim_{t\to\infty} f(t)/t$ exists and is finite.

Indeed: Since $(u+v)^\alpha \leq u^\alpha + v^\alpha$, $u,v \geq 0$, the Markov property gives

$$\mathbb{E}^x(|X_{t+s} - x|^\alpha) \leq \mathbb{E}^x \left[\mathbb{E}^{X_t}(|X_s - X_0|^\alpha) \right] + \mathbb{E}^x(|X_t - x|^\alpha) \leq f(s) + f(t).$$

Consequently, f is subadditive. Applying [8, Theorem 6.6.4] finishes the proof.

Note that, by Theorem 5.1, assumption ($\star$) is, in particular, satisfied if $(X_t)_{t\geq 0}$ has bounded coefficients and α, β are as in Theorem 5.1.

The Burkholder–Davis–Gundy inequality yields also estimates of fractional moments for $\alpha \geq 1$:

5.2 Theorem *Let $(X_t)_{t\geq 0} \sim (b(x), Q(x), N(x,dy))$ be a Lévy-type process with bounded coefficients and $\alpha \geq 1$, $\beta \in [1,2]$ such that*

$$M := \sup_{x\in\mathbb{R}^d} \left(\int_{|y|\leq 1} |y|^\beta \, N(x,dy) + \int_{|y|>1} |y|^\alpha \, N(x,dy) \right) < \infty.$$

(i) If $\alpha \in [1,2]$, then there exists $C > 0$ such that

$$\begin{aligned} \mathbb{E}^x \left(\sup_{s\leq t} |X_s - x|^\kappa \right) &\leq C \sup_{x\in\mathbb{R}^d} \left(t^\kappa \left| b(x) + \int_{|y|>1} y \, N(x,dy) \right|^\kappa + t^{\kappa/2} |Q(x)|^{\kappa/2} \right) \\ &\quad + C \sup_{x\in\mathbb{R}^d} \left(t^{\kappa/\beta} \left[\int_{|y|\leq 1} |y|^\beta \, N(x,dy) \right]^{\kappa/\beta} + t^{\kappa/\alpha} \left[\int_{|y|>1} |y|^\alpha \, N(x,dy) \right]^{\kappa/\alpha} \right) \end{aligned}$$

for all $t \geq 0$ and $\kappa \in [0, \alpha \wedge \beta]$.

(ii) If $\alpha > 2$, then there exists $C > 0$ such that

$$\begin{aligned} \mathbb{E}^x \left(\sup_{s\leq t} |X_s - x|^\kappa \right) &\leq C \sup_{x\in\mathbb{R}^d} \left(t^\kappa \left| b(x) + \int_{|y|>1} y \, N(x,dy) \right|^\kappa + t^{\kappa/2} |Q(x)|^{\kappa/2} \right) \\ &\quad + C \sup_{x\in\mathbb{R}^d} \left(t^{\kappa/\alpha} \left[\int_{\mathbb{R}^d\setminus\{0\}} |y|^\alpha \, N(x,dy) \right]^{\kappa/\alpha} + t^{\kappa/2} \left[\int_{\mathbb{R}^d\setminus\{0\}} |y|^2 \, N(x,dy) \right]^{\kappa/2} \right) \end{aligned}$$

for all $t \geq 0$ and $\kappa \in [0, \alpha]$.

(iii) (Wald's identity) Suppose that q is of martingale-type, i. e.

$$q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy\cdot\xi} + iy\cdot\xi) \, N(x,dy),$$

and $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} |y|^\alpha \, N(x,dy) < \infty$ for some $\alpha \in [1,2]$. Then $\mathbb{E}^x(X_\tau) = x$ holds for any stopping time τ such that $\mathbb{E}^x(\tau^{1/\alpha}) < \infty$.

Proof. As in the proof of Theorem 5.1, we fix a Markov extension $(\Omega^\circ, \mathcal{A}^\circ, \mathcal{F}_t^\circ, \mathbb{P}^{\circ,x})$, a Brownian motion $(W_t^\circ)_{t\geq 0}$, a Cauchy process $(L_t^\circ)_{t\geq 0}$ with jump measure N° and k, σ such that (3) and (18) hold. As

$$\mathbb{E}^{\circ,x}\left(\int_0^t \int_{|k|>1} |k(X_{s-},z)|\,\nu^\circ(dz)\,ds\right) = \mathbb{E}^{\circ,x}\left(\int_0^t \int_{|y|>1} |y|\,N(x,dy)\right) \leq Mt,$$

we can write

$$\begin{aligned} X_t - x = &\int_0^t \bar{b}(X_{s-})\,ds + \int_0^t \sigma(X_{s-})\,dW_s^\circ + \int_0^t \int_{|k|>1} k(X_{s-},z)\,(N^\circ(dz,ds) - \nu^\circ(dz)\,ds) \\ &+ \int_0^t \int_{|k|\leq 1} k(X_{s-},z)\,(N^\circ(dz,ds) - \nu^\circ(dz)\,ds) \end{aligned}$$

where $\bar{b}(x) := b(x) + \int_{|y|>1} y\,N(x,dy)$. Note that $\bar{b}$ is well-defined since

$$\int_{|y|>1} |y|\,N(x,dy) \overset{\alpha\geq 1}{\leq} \int_{|y|>1} |y|^\alpha\,N(x,dy) \leq M < \infty.$$

Using the elementary estimate

$$\left(\sum_{i=1}^4 u_i\right)^\alpha \leq 4^{\alpha-1} \sum_{i=1}^4 u_i^\alpha, \qquad u_i \geq 0,$$

(i) and (ii) follow from [14, Theorem 1], (3), Jensen's inequality and Itô's isometry for $\kappa = \alpha$. Again we apply Jensen's inequality to obtain the estimate for $\kappa \in [0,\alpha]$. Wald's identity is a direct consequence of [14, Theorem 2]. □

Remark By Theorem 5.1 and Theorem 5.2,

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} |y|^\alpha\,N(x,dy) < \infty \implies \forall t \leq 1 : \sup_{x\in\mathbb{R}^d} \mathbb{E}^x\left(\sup_{s\leq t} |X_s - x|^\alpha\right) \leq Ct$$

for any Lévy-type process $(X_t)_{t\geq 0} \sim (0,0,N(x,dy))$ and $\alpha > 0$. One can show that at least a partial converse holds true:

$$\forall t \leq 1 : \mathbb{E}^x(|X_t - x|^\alpha) \leq Ct \implies \int_{\mathbb{R}^d\setminus\{0\}} |y|^\alpha\,N(x,dy) < \infty.$$

This follows by combining the integrated heat kernel estimate (1) with Fatou's lemma and the identity

$$\int_{|y|\geq 1} |y|^\alpha\,N(x,dy) = \alpha \int_{[1,\infty)} r^{\alpha-1} N(x,\{y\in\mathbb{R}^d; |y|\geq r\})\,dr,$$

see [12, Proposition 3.10] for details.

In the last part of this section, we describe the asymptotics of fractional moments in terms of the growth of the symbol. To this end, we recall the notion of Blumenthal–Getoor indices. Blumenthal and Getoor [2] introduced various indices for Lévy processes; we will use the following ones: For a Lévy process $(L_t)_{t\geq 0}$ with characteristic exponent ψ and Lévy triplet (b,Q,ν), we call

$$\begin{aligned} \beta_0 &:= \sup\left\{\alpha \geq 0; \lim_{|\xi|\to 0} \frac{|\psi(\xi)|}{|\xi|^\alpha} = 0\right\} = \sup\left\{\alpha \geq 0; \limsup_{|\xi|\to 0} \frac{|\psi(\xi)|}{|\xi|^\alpha} < \infty\right\}, \\ \beta_\infty &:= \inf\left\{\alpha \geq 0; \lim_{|\xi|\to \infty} \frac{|\psi(\xi)|}{|\xi|^\alpha} = 0\right\} = \inf\left\{\alpha \geq 0; \limsup_{|\xi|\to \infty} \frac{|\psi(\xi)|}{|\xi|^\alpha} < \infty\right\} \end{aligned} \tag{19}$$

the *Blumenthal–Getoor index at* 0 *and* ∞, respectively. Then $\beta_0, \beta_\infty \in [0,2]$ and

$$\beta_0 = \sup\left\{\alpha \leq 2; \int_{|y|\geq 1} |y|^\alpha\,\nu(dy) < \infty\right\} = \sup\left\{\alpha \leq 2; \mathbb{E}|L_t|^\alpha < \infty\right\}.$$

For a proof of the first equality see e. g. [18, Proposition 5.4]; the second equality follows from [16, Theorem 25.3]. There are several ways to define so-called generalized Blumenthal–Getoor indices for Lévy-type processes, cf. [18] and [3, Section 5.2]. Following [3], we define for a family $(q(x,\xi))_{x\in\mathbb{R}^d}$ of characteristic exponents the *generalized Blumenthal–Getoor index at* 0 *and* ∞, respectively, as

$$\begin{aligned}\beta_0^x &:= \sup\left\{\alpha \geq 0; \limsup_{|\xi|\to 0} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)| < \infty\right\}, \\ \beta_\infty^x &:= \inf\left\{\alpha \geq 0; \limsup_{|\xi|\to \infty} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)| < \infty\right\}\end{aligned} \tag{20}$$

for $x \in \mathbb{R}^d$. The next theorem is one of our main results.

5.3 Theorem *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with symbol q and let $x \in \mathbb{R}^d$. Suppose that there exist $\alpha, \beta \in (0,2]$, $\gamma < \beta$ and $C > 0$ such that*

$$\begin{aligned}|q(y,\xi)| &\leq C(1+|y|^\gamma)|\xi|^\beta, \qquad \textit{for all } |\xi|\leq 1,\ |y-x|\leq|\xi|^{-1}, \\ |q(y,\xi)| &\leq C(1+|y|^\gamma)|\xi|^\alpha, \qquad \textit{for all } |\xi|\geq 1,\ |y-x|\leq|\xi|^{-1}.\end{aligned}$$

Then

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right) \leq Cf(t)^{\kappa/\gamma} \qquad \textit{for all } t\leq 1,\ \kappa\in[0,\gamma],$$

where $C = C(x,\gamma,\alpha,\beta)$ and $f(t) := t^{\frac{\gamma}{\alpha}\wedge 1}$.

Note that under the assumptions of Theorem 5.3 we can choose for any $\kappa \in (0,\beta)$ some $\gamma < \beta$ such that $\kappa \in (0,\gamma]$; therefore Theorem 5.3 gives moment estimates for all $\kappa \in (0,\beta)$.

Proof of Theorem 5.3. Throughout this proof the constant $C_1 = C_1(\gamma,\alpha,\beta) > 0$ may vary from line to line. Without loss of generality, we may assume that $\gamma \neq \alpha$ and $\kappa \in [0,\gamma]$ (otherwise we choose $\gamma < \beta$ sufficiently large such that these two relations are satisfied). Again, we denote by $\tau^x(r) := \tau_r^x$ the exit time from $B(x,r)$. Fix $R > 0$. By Lemma 3.2,

$$\begin{aligned}\mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\right) &= \int_0^\infty \mathbb{P}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|\geq r^{1/\gamma}\right)dr \\ &\leq \int_0^\infty \min\left\{1,\ C_1\mathbb{E}^x\left(\int_{[0,t\wedge\tau^x(r^{1/\gamma})\wedge\tau_R^x)}\sup_{|\xi|\leq r^{-1/\gamma}}|q(X_s,\xi)|\,ds\right)\right\}dr.\end{aligned}$$

Using the growth assumptions on q, we get

$$\begin{aligned}\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\gamma\right) &\leq \int_0^{t^{\gamma/\alpha}} 1\,dr + C_1\int_{t^{\gamma/\alpha}}^\infty \mathbb{E}^x\left(\int_{[0,t\wedge\tau^x(r^{1/\gamma})\wedge\tau_R^x)}\sup_{|\xi|\leq r^{-1/\gamma}}|q(X_s,\xi)|\,ds\right)dr \\ &\leq t^{\gamma/\alpha} + C_1\left(\int_{t^{\gamma/\alpha}}^1 r^{-\alpha/\gamma}\,dr + \int_1^\infty r^{-\beta/\gamma}\,dr\right)\mathbb{E}^x\left(\int_{[0,t\wedge\tau_R^x)}(1+|X_s|^\gamma)\,ds\right) \\ &\leq f(t) + C_1\int_0^t(1+\mathbb{E}^x(|X_s|^\gamma\mathbb{1}_{\{s<\tau_R^x\}}))\,ds\end{aligned}$$

for all $t \leq 1$. This shows that $\varphi(t) := \mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\mathbb{1}_{\{t<\tau_R^x\}}\right)$ satisfies

$$\varphi(t) \leq \mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\right) \leq C_1 f(t)(1+|x|^\gamma) + C_1\int_0^t\varphi(s)\,ds.$$

Hence, by Gronwall's inequality,

$$\varphi(t) \leq C_1 f(t)(1+|x|^\gamma)\exp(C_1 t).$$

Finally, since the constant C_1 does not depend on R, we can let $R\to\infty$ using Fatou's lemma. For $\kappa\in[0,\gamma]$ apply Jensen's inequality. □

5.4 Example Let $(X_t)_{t\geq 0}$ be a Lévy-type process which is a solution to an SDE of the form

$$dX_t = f(X_{t-})\,dL_t, \qquad X_0 = x,$$

where $(L_t)_{t\geq 0}$ is a Lévy process with characteristic exponent ψ and f a function of sublinear growth, i. e. $|f(x)| \leq C(1+|x|^{1-\epsilon})$ for some $C, \epsilon > 0$. Denote by β_0 and β_∞ the Blumenthal–Getoor indices of ψ at 0 and ∞, cf. (19). Then

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s - x|^\kappa\right) \leq C' t^{\kappa/\beta_\infty \wedge 1} \qquad \text{for all } t \leq 1,\ \kappa \in [0,\beta_0),\ \kappa \neq \beta_\infty.$$

5.5 Corollary *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with symbol q. Assume that*

$$\limsup_{|\xi|\to\infty} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)| < \infty \tag{21}$$

for some $\alpha \in (0,2]$. Then

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s - x|^\kappa\right) \leq \begin{cases} Ct^{\frac{\kappa}{\alpha}\wedge 1}, & \kappa \neq \alpha, \\ Ct|\log t|, & \kappa = \alpha \end{cases}$$

for all $t \leq 1$ and $\kappa \in [0,\beta_0^x)$. Here β_0^x and β_∞^x denote the generalized Blumenthal–Getoor indices at 0 *and* ∞*, respectively, cf.* (20)*.*

Proof. Because of the assumptions on q, the growth conditions in Theorem 5.3 are satisfied for any $\gamma \in (0,\beta)$; in particular we can choose $\gamma = \kappa$. □

Remark (i) Applying Corollary 5.5 to α-stable and tempered α-stable processes shows that the estimates are optimal, cf. [13, p. 431-32].

(ii) By the very definition of the Blumenthal–Getoor index (see (20)), we know that the limit

$$\limsup_{|\xi|\to\infty} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)|$$

is finite (infinite) if $\alpha > \beta_\infty^x$ (if $\alpha < \beta_\infty^x$). Therefore, (21) is violated for any $\alpha \in (0,\beta_\infty^x)$ and automatically satisfied for $\alpha \in (\beta_\infty^x, 2]$. The case $\alpha = \beta_\infty^x$ has to be checked individually.

(iii) Combining Corollary 5.5 and Fatou's lemma shows that

$$\liminf_{t\to 0} \frac{1}{t^{1/\alpha}} \sup_{s\leq t}|X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s.}$$

for any $\alpha > \beta_\infty^x$; see also [3, Theorem 5.16].

(iv) Corollary 5.5 can be proved using a very similar argument as in the proof of Theorem 5.6. The proof then shows in particular that the estimate

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s - x|^\kappa \wedge 1\right) \leq \begin{cases} Ct^{\frac{\kappa}{\alpha}\wedge 1}, & \kappa \neq \alpha, \\ Ct|\log t|, & \kappa = \alpha \end{cases}$$

holds true for any $\kappa > 0$ and $t \leq 1$.

There is an analogous result for the large-time asymptotics; it extends [5, Theorem 3.3].

5.6 Theorem *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with symbol q and $\beta \in (0,2]$ such that*

$$\limsup_{|\xi|\to 0} \frac{1}{|\xi|^\beta} \sup_{|x-y|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(x,\eta)| < \infty,$$

then

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s - x|^\kappa\right) \leq Ct^{\kappa/\beta} \qquad \text{for all } t \geq 1,\ \kappa \in [0,\beta).$$

Proof. An application of the maximal inequality (9) yields

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s-x|^\kappa\right)=\int_0^\infty \mathbb{P}^x\left(\sup_{s\le t}|X_s-x|\ge r^{1/\kappa}\right)dr$$
$$\le\int_0^\infty \min\left\{1, Ct\sup_{|y-x|\le r^{1/\kappa}}\sup_{|\eta|\le r^{-1/\kappa}}|q(y,\eta)|\right\}dr.$$

Hence,

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s-x|^\kappa\right)\le\int_0^{t^{\kappa/\beta}}1\,dr+C't\int_{t^{\kappa/\beta}}^\infty r^{-\beta/\kappa}\,dr=O(t^{\kappa/\beta}).\qquad\square$$

5.7 Example Let $(X_t)_{t\ge0}$ be a stable-like process, i.e. a Lévy-type process with symbol $q(x,\xi)=|\xi|^{\alpha(x)}$ for a (continuous) function $\alpha:\mathbb{R}^d\to(0,2)$. Then $\beta_0^x\ge\alpha_l:=\inf_{y\in\mathbb{R}^d}\alpha(y)$ and $\beta_\infty^x=\alpha(x)$. Hence, by Corollary 5.5, we have for any $\alpha>\alpha(x)$ and $\kappa\in[0,\alpha_l)$,

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s-x|^\kappa\right)\le\begin{cases}Ct^{\frac{\kappa}{\alpha}\wedge1}, & \kappa\ne\alpha,\\ Ct|\log t|, & \kappa=\alpha\end{cases}$$

for all $t\le1$. Moreover, by Theorem 5.6,

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s-x|^\kappa\right)\le Ct^{\kappa/\beta}$$

for all $t\ge1$ and any $\beta<\alpha_l$, $\kappa\in[0,\beta)$.

For the readers' convenience we sum up conditions and results in Table 1.

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assumptions on the symbol	assumptions on the moments		$\mathbb{E}^x\left(\sup_{s\le t}\|X_s - x\|^\kappa\right)$	reference
"bounded variation"-type: $q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-e^{i\,y\xi})\,N(x,dy)$	$\sup_{x\in\mathbb{R}^d}\int_{\|y\|>1}\|y\|^\alpha\,N(x,dy) < \infty$ $\sup_{x\in\mathbb{R}^d}\int_{\|y\|\le 1}\|y\|^\beta\,N(x,dy) < \infty$	$\alpha\in(0,1]$ $\beta\in[0,\alpha]$	$O(t^{\kappa/\alpha})$ for $\kappa\in[0,\alpha]$	Theorem 5.1
"bounded variation"-type: $q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-e^{i\,y\xi})\,N(x,dy)$	$\sup_{x\in\mathbb{R}^d}\int_{\|y\|>1}\|y\|^\alpha\,N(x,dy) < \infty$ $\sup_{x\in\mathbb{R}^d}\int_{\|y\|\le 1}\|y\|^\beta\,N(x,dy) < \infty$	$\alpha\in(0,1]$ $\beta\in[\alpha,1]$	$O(t^{\kappa/\alpha}+t^{\kappa/\beta})$ for $\kappa\in[0,\alpha]$	Theorem 5.1
"pure-jump"-type: $q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-e^{i\,y\xi}+iy\xi\mathbb{1}_{(0,1]}(\|y\|))\,N(x,dy)$	$\sup_{x\in\mathbb{R}^d}\int_{\|y\|>1}\|y\|^\alpha\,N(x,dy) < \infty$ $\sup_{x\in\mathbb{R}^d}\int_{\|y\|\le 1}\|y\|^\beta\,N(x,dy) < \infty$	$\alpha\in(0,1]$ $\beta\in[1,2]$	$O(t^{\kappa/\alpha}+t^{\kappa/\beta})$ for $\kappa\in[0,\alpha]$	Theorem 5.1
"martingale"-type: $q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-e^{i\,y\xi}+iy\xi)\,N(x,dy)$	$\sup_{x\in\mathbb{R}^d}\int_{\|y\|>1}\|y\|^\alpha\,N(x,dy) < \infty$ $\sup_{x\in\mathbb{R}^d}\int_{\|y\|\le 1}\|y\|^\beta\,N(x,dy) < \infty$	$\alpha\in[1,2]$ $\beta\in[1,2]$	$O(t^{\kappa/\alpha}+t^{\kappa/\beta})$ for $\kappa\in[0,\alpha\wedge\beta]$	Theorem 5.2
"martingale"-type: $q(x,\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-e^{i\,y\xi}+iy\xi)\,N(x,dy)$	$\sup_{x\in\mathbb{R}^d}\int_{\|y\|>1}\|y\|^\alpha\,N(x,dy) < \infty$ $\sup_{x\in\mathbb{R}^d}\int_{\|y\|\le 1}\|y\|^2\,N(x,dy) < \infty$	$\alpha>2$	$O(t^{\kappa/\alpha}+t^{\kappa/2})$ for $\kappa\in[0,\alpha]$	Theorem 5.2
$\forall\|\xi\|\le 1: \sup_{\|y-x\|\le\|\xi\|^{-1}}\|q(y,\xi)\|\le C(1+\|x\|^\gamma)\|\xi\|^\beta$ $\forall\|\xi\|\ge 1: \sup_{\|y-x\|\le\|\xi\|^{-1}}\|q(y,\xi)\|\le C(1+\|x\|^\gamma)\|\xi\|^\alpha$		$\gamma<\beta$	$O(t^{\kappa(\alpha^{-1}\wedge\gamma^{-1})})$, $t\to 0$, for $\kappa\in[0,\beta),\kappa\neq\alpha$	Theorem 5.3
$\forall\|\xi\|\ge 1: \sup_{\|y-x\|\le\|\xi\|^{-1}}\sup_{\|\eta\|\le\|\xi\|}\|q(y,\eta)\|\le C\|\xi\|^\alpha$			$O(t^{\kappa/\alpha\wedge 1})$, $t\to 0$, for $\kappa\in[0,\beta_0^x),\kappa\neq\alpha$	Corollary 5.5
$\forall\|\xi\|\le 1: \sup_{\|y-x\|\le\|\xi\|^{-1}}\sup_{\|\eta\|\le\|\xi\|}\|q(y,\eta)\|\le C\|\xi\|^\beta$			$O(t^{\kappa/\beta})$, $t\to\infty$, for $\kappa\in[0,\beta)$	Theorem 5.6

Table 1: Estimates of fractional moments of (pure-jump) Lévy-type processes.

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