Transition probabilities of Lévy-type processes: Parametrix construction

Franziska Kühn*

Abstract

We present an existence result for Lévy-type processes which requires only weak regularity assumptions on the symbol $q(x,\xi)$ with respect to the space variable x. Applications range from existence and uniqueness results for Lévy-driven SDEs with Hölder continuous coefficients to existence results for stable-like processes and Lévy-type processes with symbols of variable order. Moreover, we obtain heat kernel estimates for a class of Lévy and Lévy-type processes. The paper includes an extensive list of Lévy(-type) processes satisfying the assumptions of our results.

Keywords: Feller process, existence, heat kernel estimates, Lévy process, jump processes, Lévy-driven stochastic differential equation

MSC 2010: Primary: 60J35. Secondary: 60J25, 60G51, 60H10, 60J75, 35S05.

1 Introduction

The Lévy–Khintchine formula gives a one-to-one correspondence between Lévy processes and continuous negative definite functions (i.e. characteristic exponents). For any continuous negative definite function $\psi : \mathbb{R}^d \to \mathbb{C}$ with $\psi(0) = 0$, i.e. any function of the form

$$\psi(\xi) = \psi(0) - ib \cdot \xi + \frac{1}{2}\xi \cdot Q\xi + \int_{y \neq 0}(1 - e^{i\xi \cdot y} + i\xi \cdot y \mathbb{1}_{(0,1)}(|y|))\, \nu(dy), \qquad \xi \in \mathbb{R}^d$$

for some $b \in \mathbb{R}^d$, a positive semidefinite symmetric matrix $Q \in \mathbb{R}^{d \times d}$ and a measure ν on $\mathbb{R}^d \backslash \{0\}$ such that $\int_{\mathbb{R}^d \backslash \{0\}} |y|^2 \wedge 1\, \nu(dy) < \infty$, there exists a Lévy process with characteristic exponent ψ. Over the past years, there has been an increasing interest in so-called Lévy-type processes. This is a class of Markov processes which behave locally like a Lévy process, but the (analogue of the) Lévy triplet depends on the current position of the process in the state space. If the smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the generator of a Lévy-type process $(X_t)_{t \geq 0}$, then the process can be characterized via its symbol q,

$$q(x,\xi) = q(x,0) - ib(x) \cdot \xi + \frac{1}{2}\xi \cdot Q(x)\xi + \int_{y \neq 0}(1 - e^{i\xi \cdot \xi} + i\xi \cdot y \mathbb{1}_{(0,1)}(|y|))\, \nu(x,dy), \quad x,\xi \in \mathbb{R}^d, \quad (1)$$

which is a continuous negative definite function for each fixed $x \in \mathbb{R}^d$. Typical examples are processes with variable index of stability (this corresponds to $q(x,\xi) = |\xi|^{\alpha(x)}$) and solutions of Lévy-driven SDEs, see Table 3 on page 15 for further examples.

It is natural to ask whether for a given function q of the form (1), i.e. family $(q(x,\xi))_{x \in \mathbb{R}^d}$ of continuous negative definite functions, there exists a Lévy-type process with symbol q. The answer is, in general, *no* (see e.g. [7, Example 2.26] for counterexamples), and therefore it is important to find sufficient conditions on the symbol q or the characteristics $(b(x), Q(x), \nu(x,dy))_{x \in \mathbb{R}^d}$ which ensure the existence of a Lévy-type process with a given symbol q. Many existence results in the literature (see [7] for an overview) are rather restrictive in the sense that they either

---

*Institut für Mathematische Stochastik, Fachrichtung Mathematik, Technische Universität Dresden, 01062 Dresden, Germany, `franziska.kuehn1@tu-dresden.de`

assume that q is of a particular form (typically "stable-like" or "lower-order perturbation") or they require strong assumptions on the regularity of the symbol q with respect to the space variable x (typically smoothness).

In this paper, we present a new existence result for Lévy-type processes which requires only mild regularity assumptions on the symbol q with respect to x. Applications range from variable order subordination and Lévy-type processes with symbols of variable order to existence and uniqueness results for solutions to Lévy-driven stochastic differential equations (SDEs) with globally Hölder continuous coefficients. The proof of the main result, Theorem 4.1, relies on a parametrix construction; this is an analytical method which has become an increasingly popular tool in probability theory to prove the existence of certain stochastic processes and to establish heat kernel estimates, e.g. processes with variable order of differentiation (Kolokoltsov [14, 15] and Chen & Zhang [9]), gradient perturbations of Lévy generators (Bogdan & Jakubowski [6] and Jakubowski & Szczypkowski [12]) and solutions of SDEs with Hölder continuous coefficients (Knopova & Kulik [16, 22] and Huang [11]). Using the parametrix construction, we can not only prove the existence of Lévy-type processes with symbols from certain classes, but also get additional information on the process such as heat kernel estimates, well-posedness of the associated martingale problem or the richness of the domain of the generator, cf. Theorem 4.2.

As a by-product of the parametrix construction, we obtain heat kernel estimates for a class of Lévy processes; the estimates are crucial to prove the convergence of the parametrix expansion. Estimates for the transition density of Lévy processes have attracted a lot of attention, for example heat kernel estimates for unimodal [5], rotationally invariant [8], tempered stable [27] Lévy processes or Lévy processes with exponential moments [17, 28], to mention but a few. In contrast to many of these results, we state our assumptions on the Lévy process in terms of the characteristic exponent ψ and not in terms of the Lévy triplet (b, Q, ν). Since there are few Lévy processes for which both the characteristic exponent and the Lévy triplet can be calculated explicitly, both approaches (i.e. via Lévy triplet or via characteristic exponent) have their own justification. In fact, for most of the examples which we present in Section 3 the Lévy triplet is unknown, and therefore it is very hard to verify conditions on the Lévy triplet. Our result applies, for instance, to relativistic stable, Lamperti stable, normal tempered stable, truncated Lévy processes, cf. Example 3.4, and a class of subordinators; see Table 1 and Table 2 in Section 3 for further examples.

This paper is organized as follows. In Section 2 we introduce the basic definitions and some notation. Section 3 is devoted to heat kernel estimates for Lévy processes. In Section 4 we present our main result, the existence result for Lévy-type processes, and discuss several applications, including a new existence and uniqueness result for Lévy-driven SDEs with Hölder continuous coefficients (Corollary 4.7), an existence result for stable-like processes (Example 4.6) and Feller processes with symbols of variable order (Corollary 4.8).

The results presented in this paper are essentially taken from my PhD thesis [19]. The aim of this paper is to give a brief summary of the most important results and make them accessible to a larger audience; in particular, we do not include proofs since they are very technical and lengthy, and we refer to [19, 20] for full proofs.

2   Preliminaries

We consider the Euclidean space $\mathbb{R}^d$ endowed with the canonical scalar product $x \cdot y = \sum_{j=1}^d x_j y_j$ and the Borel-σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The continuous bounded functions are denoted by $C_b(\mathbb{R}^d)$, and $C_\infty(\mathbb{R}^d)$ is the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity. Superscripts $k \in \mathbb{N}$ are used to denote the order of differentiability, e.g. $f \in C_\infty^k(\mathbb{R}^d)$ means that f and its derivatives up to order k are $C_\infty(\mathbb{R}^d)$-functions. A function $\ell : (0, \infty) \to (0, \infty)$ is *slowly varying (at infinity)* if

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \qquad \text{for all} \ \ \lambda > 0.$$

We say that a function $f : \mathbb{R}^d \to \mathbb{R}^k$ is (globally) Hölder continuous if there exist constants $\varrho \in (0, 1]$, $C > 0$ such that $|f(x) - f(y)| \le C|x - y|^\varrho$ for all $x, y \in \mathbb{R}^d$.

A d-dimensional Markov process $(\Omega, \mathcal{A}, \mathbb{P}^x, x \in \mathbb{R}^d, X_t, t \geq 0)$ with càdlàg (right-continuous with left-hand limits) sample paths is called a *Lévy-type process* if the associated semigroup $(P_t)_{t \geq 0}$ defined by

$$P_t f(x) := \mathbb{E}^x f(X_t), \quad x \in \mathbb{R}^d, \ f \in \mathcal{B}_b(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R}; f \text{ bounded, Borel measurable}\}$$

has the *Feller property* and $(P_t)_{t \geq 0}$ is *strongly continuous at $t = 0$*, i.e. $P_t f \in C_\infty(\mathbb{R}^d)$ for all $C_\infty(\mathbb{R}^d)$ and $\|P_t f - f\|_\infty \xrightarrow{t \to 0} 0$ for any $f \in C_\infty(\mathbb{R}^d)$. Lévy-type processes are also known as *Feller processes*; we use both terms synonymously. A semigroup $(P_t)_{t \geq 0}$ has the *strong Feller property* if $P_t f \in C_b(\mathbb{R}^d)$ for any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $t \geq 0$. If the smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the generator $(L, \mathcal{D}(L))$, then we speak of a *rich* Lévy-type process. A result due to von Waldenfels and Courrège, cf. [7, Theorem 2.21], states that the generator L of a rich Lévy-type process is, when restricted to $C_c^\infty(\mathbb{R}^d)$, a pseudo-differential operator with negative definite symbol:

$$L f(x) = -\int_{\mathbb{R}^d} e^{i\,x \cdot \xi} q(x, \xi) \hat{f}(\xi) \, d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d), \ x \in \mathbb{R}^d,$$

where $\hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) \, dx$ denotes the Fourier transform of f and

$$q(x, \xi) = q(x, 0) - ib(x) \cdot \xi + \frac{1}{2} \xi \cdot Q(x) \xi + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{iy \cdot \xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|)\right) \nu(x, dy). \quad (2)$$

We call q the *symbol* of the rich Lévy-type process $(X_t)_{t \geq 0}$ and $-q$ the symbol of the pseudo-differential operator. For each fixed $x \in \mathbb{R}^d$, $(b(x), Q(x), \nu(x, dy))$ is a Lévy triplet, i.e. $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix and $\nu(x, dy)$ a σ-finite measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ satisfying $\int_{y \neq 0} \min\{|y|^2, 1\} \nu(x, dy) < \infty$. A set $\mathcal{D} \subseteq C_\infty(\mathbb{R}^d)$ is a *core* for the generator $(L, \mathcal{D}(L))$ if $\overline{(L, \mathcal{D})}^{\|\cdot\|_\infty} = (L, \mathcal{D}(L))$. Our standard reference for Lévy-type processes is the monograph [7].

A *Lévy process* $(L_t)_{t \geq 0}$ is a rich Feller process whose symbol q does not depend on x. This is equivalent to saying that $(L_t)_{t \geq 0}$ has stationary and independent increments and càdlàg sample paths, cf. [7, Theorem 2.6]. The symbol $q = q(\xi)$ (also called *characteristic exponent*) and the Lévy process $(L_t)_{t \geq 0}$ are related through the Lévy–Khintchine formula:

$$\mathbb{E}^x e^{i\xi \cdot (L_t - x)} = e^{-tq(\xi)} \qquad \text{for all} \ \ t \geq 0, \ x, \xi \in \mathbb{R}^d.$$

A Lévy process with non-decreasing sample paths is a *subordinator* and can be characterized by its Laplace exponent, cf. [25]. We refer to Sato [23] for a detailed discussion of Lévy processes and to Schilling [13] for an introduction to Lévy and Lévy-type processes.

3 Heat kernel estimates for Lévy processes

In this section, we present transition density estimates for a class of Lévy processes. The main results are Theorem 3.1 (heat kernel estimates for rotationally invariant Lévy processes in dimension $d \geq 1$), Theorem 3.2 (heat kernel estimates for one-dimensional Lévy processes which are not necessarily symmetric) and Corollary 3.5 (heat kernel estimates for subordinators). In contrast to many results in the literature, we state our assumptions on the Lévy process $(L_t)_{t \geq 0}$ in terms of the characteristic exponent and not in terms of the Lévy triplet.

3.1 Theorem *Let $(L_t)_{t \geq 0}$ be a d-dimensional Lévy process and suppose that its characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ satisfies* (L1)-(L3).

(L1) *ψ is rotationally invariant, i.e. there exists $\Psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(\xi) = \Psi(|\xi|)$, $\xi \in \mathbb{R}^d$. If $m > 0$: Ψ is even, i.e. $\Psi(r) = \Psi(-r)$ for all $r \geq 0$.*

(L2) *There exists $\theta \in (0, \frac{\pi}{2})$ and $m \geq 0$ such that Ψ has a holomorphic extension to*

$$\Omega := \Omega(m, \theta) := \{z \in \mathbb{C}; |\operatorname{Im} z| < m\} \cup \{z \in \mathbb{C} \setminus \{0\}; \arg z \in (-\theta, \theta) \cup (\pi - \theta, \pi + \theta)\}. \quad (3)$$
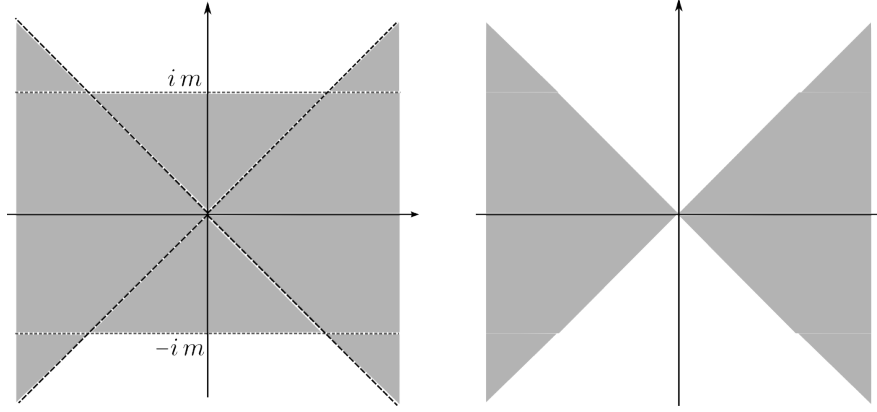
Figure 1: The domain $\Omega = \Omega(m, \vartheta)$ for $m > 0$ (left) and $m = 0$ (right).

(L3) *There exist constants $c_1, c_2 > 0$ and $\gamma_0, \gamma_\infty \in (0, 2]$ and a slowly varying (at infinity) increasing function $\ell : (0, \infty) \to (0, \infty)$ such that*

$$\mathrm{Re}\,\Psi(z) \geq \frac{c_1}{\ell(|z|)}|z|^{\gamma_\infty} \qquad \text{for all}\ \ z \in \Omega, |z| \geq 1 \tag{4}$$

and

$$|\Psi(z)| \leq c_2 \ell(|z|)\left(|z|^{\gamma_0}\mathbb{1}_{\{|z| \leq 1\}} + |z|^{\gamma_\infty}\mathbb{1}_{\{|z| > 1\}}\right) \qquad \text{for all}\ \ z \in \Omega. \tag{5}$$

Then the transition density

$$p_t(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}e^{-t\psi(\xi)}\,d\xi$$

exists, is infinitely often differentiable and satisfies the estimates

$$|p_t(x)| \leq C S(x, t)$$
$$\left|\frac{\partial}{\partial t}p_t(x)\right| \leq C t^{-1} S(x, t)$$

for any $x \in \mathbb{R}^d$, $t \in (0, T]$ where $C = C(T) > 0$ is an absolute constant and

$$S(x, t) := S_m(x, t) := \exp\left(-\frac{m}{4}|x|\right)\left(1 + \ell(ct^{-1/\gamma_\infty})\right)\begin{cases} t^{-d/\gamma_\infty}, & |x| \leq t^{1/\gamma_\infty} \wedge 1, \\ t/|x|^{d+\gamma_\infty}, & t^{1/\gamma_\infty} < |x| \leq 1, \\ t/|x|^{d+\gamma_\infty \wedge \gamma_0}, & |x| > 1 \end{cases}$$

for some absolute constant $c = c(T) > 0$. Moreover, for any multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ and any $T > 0$ there exists a constant $c > 0$ such that

$$\left|\frac{\partial^\beta}{\partial x^\beta}p_t(x)\right| \leq c t^{-(|\beta_1|+\ldots+|\beta_d|)/\gamma_\infty}S(x, t) \qquad \text{for all}\ \ x \in \mathbb{R}^d, t \in (0, T].$$

In dimension $d = 1$ we do not need any symmetry assumption like rotational invariance in (L1).

3.2 Theorem *Let $(L_t)_{t \geq 0}$ be a one-dimensional Lévy process with characteristic exponent ψ. If $\Psi(\xi) := \psi(\xi)$ satisfies (L2) and (L3), then the results of Theorem 3.1 remain valid.*

To prove Theorem 3.1 and Theorem 3.2 we apply Cauchy's theorem to shift the contour of integration; for details see [20, Section 4.1] or [19, Section 4.1]. Using this idea, we can also obtain heat kernel estimates for subordinators, cf. Corollary 3.5. Let us give some remarks on both Theorem 3.1 and Theorem 3.2.

4

3.3 Remarks (i). Condition (L3) implies that ψ satisfies the *sector condition*, i.e. there exists a constant $C_1 > 0$ such that

$$|\operatorname{Im}\psi(\xi)| \le C_1 |\operatorname{Re}\psi(\xi)| \qquad \text{for all } \xi \in \mathbb{R}^d.$$

Moreover, it follows from (L3) that there exists a constant $C_2 > 0$ such that

$$\operatorname{Re}\Psi(z) \ge -C_2 > -\infty \qquad \text{for all } z \in \Omega.$$

Note that the existence of such a constant is not trivial; although any continuous negative definite function $\psi : \mathbb{R}^d \to \mathbb{C}$ satisfies $\operatorname{Re}\psi(\xi) \ge 0$ for all $\xi \in \mathbb{R}^d$, the inequality does, in general, not need to be true for *complex z*.

(ii). The constant $m/4$ in the definition of the function S introduced in Theorem 3.1 can be replaced by $m(1-\delta)$ for any $\delta \in (0,1)$. Since it is well-known, cf. [29, Theorem 5.26], that the characteristic function of L_t

$$\mathbb{E}e^{i\xi L_t} = e^{-t\psi(\xi)}, \qquad t > 0, \xi \in \mathbb{R}^d,$$

is analytic on the strip $\{z \in \mathbb{C}; |\operatorname{Im}z| < m\}$ if, and only if, $\mathbb{E}e^{m(1-\delta)|L_t|} < \infty$ for any $\delta \in (0,1)$, the exponential decay $\exp(-|x|m(1-\delta))$ is, in general, the best we can expect.

(iii). Let ψ be a continuous negative definite function with Lévy triplet $(b, 0, \nu)$. If ψ satisfies the sector condition, then it is possible to give sufficient conditions in terms of fractional moments of $\nu|_{B(0,1)}$ and $\nu|_{B(0,1)^c}$ which ensure that ψ satisfies the growth condition (5) for *real z*, cf. Blumenthal & Getoor [4] and Schilling [24]. This is, however, no longer possible for the holomorphic extension. For instance if we consider $\psi(\xi) := 1 - \cos\xi$, then the associated Lévy measure $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ has arbitrary moments, but the (unique) holomorphic extension $\Psi(z) = 1 - \cos z$ does not satisfy (5).

If the characteristic exponent ψ is given in closed form, it is usually easy to check whether the assumptions of Theorem 3.1 are satisfied.

3.4 Example Let $(L_t)_{t\ge0}$ be a d-dimensional Lévy process with one of the following characteristic exponents.

(i). (isotropic stable) $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}^d$, $\alpha \in (0,2]$,

(ii). (relativistic stable) $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha$, $\xi \in \mathbb{R}^d$, $\varrho > 0$, $\alpha \in (0,2)$,

(iii). (Lamperti stable) $\psi(\xi) = (|\xi|^2 + \varrho)_\alpha - (\varrho)_\alpha$, $\xi \in \mathbb{R}^d$, $\varrho > 0$, $\alpha \in (0,1)$, where $(r)_\alpha := \Gamma(r+\alpha)/\Gamma(r)$ denotes the Pochhammer symbol,

(iv). (TLP: truncated Lévy process) $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2}\cos(\alpha\arctan(\varrho^{-1}|\xi|)) - \varrho^\alpha$, $\xi \in \mathbb{R}^d$, $\alpha \in (0,2)$, $\varrho > 0$,

(v). $\psi(\xi) = (|\xi|^\beta - 1)/(|\xi|^\alpha - 1)$, $\xi \in \mathbb{R}^d$, $0 < \alpha < \beta \le 1$,

(vi). (NTS: normal tempered stable, $d = 1$) $\psi(\xi) = (\kappa^2 + (\xi - ib)^2)^{\alpha/2} - (\kappa^2 - b^2)^{\alpha/2}$, $\xi \in \mathbb{R}$, $\alpha \in (0,2)$, $b > 0$, $\kappa > b$.

The characteristic exponents (i)-(v) and (vi) satisfy the assumptions of Theorem 3.1 and Theorem 3.2, respectively, with

(i). $\gamma_0 = \gamma_\infty = \alpha$, $m = 0$, $\ell = 1$,

(ii). $\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m = \varrho$, $\ell = 1$,

(iii). $\gamma_0 = 2$, $\gamma_\infty = 2\alpha$, $m = \sqrt{\varrho}$, $\ell = 1$,

(iv). $\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m = \varrho$, $\ell = 1$,

(v). $\gamma_0 = \alpha$, $\gamma_\infty = \beta - \alpha$, $m = 0$, $\ell = 1$,

(vi). $\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m = \kappa - b$, $\ell = 1$.

Consequently, Theorem 3.1, respectively, Theorem 3.2, provide heat kernel estimates for the transition density p, its time derivative and derivatives with respect to the space variable x. For the particular case of isotropic stable Lévy processes, we recover well-known (sharp) estimates for the heat kernel, cf. [2, 3].

The following list of examples satisfying the assumptions of Theorem 3.1 and Theorem 3.2, respectively, is taken from [20, Table 5.2].

	Name	char. Exponent	Parameter	Dim.	Heat Kernel Estimate (Thm. 3.1)
1	isotropic α-stable	$\lvert\xi\rvert^\alpha$	$\alpha \in (0,2]$	$d \geq 1$	$\gamma_0 = \gamma_\infty = \alpha$, $m = 0$, $\ell = 1$
2	relativistic stable	$(\lvert\xi\rvert^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha$	$\alpha \in (0,2)$, $\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m \in (0,\varrho)$, $\ell = 1$
3	normal tempered stable (NTS)	$(\kappa^2 + (\xi - ib)^2)^{\alpha/2} - (\kappa^2 - b^2)^{\alpha/2}$	$\alpha \in (0,2)$, $\kappa \in (0,\infty)$, $b < \kappa$	$d = 1$	$\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m \in (0,\kappa-b)$, $\ell = 1$
4		$\dfrac{\lvert\xi\rvert^2}{\sqrt{\lvert\xi\rvert^2 + \varrho}}$	$\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = 2$, $\gamma_\infty = 1$, $m \in (0,\sqrt{\varrho})$, $\ell = 1$
5		$\dfrac{\lvert\xi\rvert^2}{(\lvert\xi\rvert^2 + \varrho)^\alpha}$	$\alpha \in (0,2)$, $\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = 2$, $\gamma_\infty = 2 - \alpha$, $m \in (0,\sqrt{\varrho})$, $\ell = 1$
6		$\dfrac{\lvert\xi\rvert^\beta - 1}{\lvert\xi\rvert^\alpha - 1} - 1$ (extended by continuity at $\xi = 1$)	$0 < \alpha < \beta < 1$	$d \geq 1$	$\gamma_0 = \alpha$, $\gamma_\infty = \beta - \alpha$, $m = 0$, $\ell = 1$
7		$-\dfrac{\lvert\xi\rvert^\alpha - 1}{\lvert\xi\rvert^{\alpha-2} - 1}$ (extended by continuity at $\xi = 1$)	$\alpha \in (0,2)$	$d \geq 1$	$\gamma_0 = 2 - \alpha$, $\gamma_\infty = 2$, $m = 0$, $\ell = 1$
8		$\dfrac{\lvert\xi\rvert^\alpha - 1}{\lvert\xi\rvert^{\alpha-2} - 1} - 1$ (extended by continuity at $\xi = 1$)	$\alpha \in (2,4]$	$d \geq 1$	$\gamma_0 = \alpha - 2$, $\gamma_\infty = 2$, $m = 0$, $\ell = 1$
9		$\lvert\xi\rvert^2 \dfrac{\lvert\xi\rvert^\alpha - \varrho^\alpha}{\lvert\xi\rvert^2 - \varrho^2}$ (extended by continuity at $\xi = \varrho$)	$\alpha \in (0,2)$, $\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m = 0$, $\ell = 1$
10		$(\lvert\xi\rvert^{-\alpha} + \lvert\xi\rvert^{-\beta})^{-1}$ (extended by continuity at $\xi = 0$)	$\alpha, \beta \in (0,2]$	$d \geq 1$	$\gamma_0 = \alpha \vee \beta$, $\gamma_\infty = \alpha \wedge \beta$, $m = 0$, $\ell = 1$
11		$\lvert\xi\rvert(1 - e^{-2\varrho\lvert\xi\rvert})$	$\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = 2$, $\gamma_\infty = 1$, $m = 0$, $\ell = 1$
12		$\lvert\xi\rvert(1 + e^{-2\varrho\lvert\xi\rvert})$	$\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = \gamma_\infty = 1$, $m = 0$, $\ell = 1$
13		$\varrho\lvert\xi\rvert^2(\lvert\xi\rvert^2 + 1)\log(1 + \lvert\xi\rvert^{-2})$	$\varrho \in (0,\infty)$	$d \geq 1$	$\gamma_0 = \gamma_\infty = 2$, $m = 0$, $\ell(r) = \log(r \vee e)$

Table 1: Examples of Lévy processes satisfying the assumptions of Theorem 3.1 and Theorem 3.2, respectively

.

	Name	char. Exponent	Parameter	Dim.	Heat Kernel Estimate (Thm. 3.1)
14		$\varrho\dfrac{|\xi|^2(|\xi|^2+1)}{(|\xi|^2+2)\log(|\xi|^2+2)}$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=\gamma_\infty=2,\ m\in(0,\sqrt{2}),\ \ell(r)=\log(r\vee e)$
15		$|\xi|\arctan(\varrho|\xi|)$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=\gamma_\infty=1,\ m=0,\ \ell=1$
16	truncated Lévy process (TLP)	$(|\xi|^2+\varrho^2)^{\alpha/2}\cos\left(\alpha\arctan\frac{|\xi|}{\varrho}\right)-\varrho^\alpha$	$\alpha\in(0,2),\ \varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=\alpha,\ m\in(0,\varrho),\ \ell=1$
17		$\varrho|\xi|\dfrac{\cosh^2(\sqrt{2}|\xi|)}{\sinh(2\sqrt{2}|\xi|)}$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=\gamma_\infty=1,\ m=0,\ \ell=1$
18		$\varrho|\xi|\dfrac{\sinh^2(\sqrt{2}|\xi|)}{\sinh(2\sqrt{2}|\xi|)}$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=\gamma_\infty=1,\ m=0,\ \ell=1$
19		$\varrho|\xi|\coth((2|\xi|)^{-1})-\varrho|\xi|^2$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=1,\ \gamma_\infty=2,\ m=0,\ \ell=1$
20		$\varrho\log(\sinh(\sqrt{2}|\xi|))-\varrho\log(\sqrt{2}|\xi|)$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=1,\ m=0,\ \ell=1$
21	isotropic Meixner	$\varrho\log(\cosh(\sqrt{2}|\xi|))$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=1,\ m=0,\ \ell=1$
22		$|\xi|\log(1+\varrho\tanh(b|\xi|))$	$b\in(0,\infty),\ \varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=\gamma_\infty=1,\ m=0,\ \ell=1$
23		$\dfrac{\Gamma(\varrho|\xi|^2+1/2)}{\Gamma(\varrho|\xi|^2)}$	$\varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=1,\ m\in(0,1/\sqrt{2\varrho}),\ \ell=1$
24		$|\xi|^2\dfrac{\Gamma(\alpha|\xi|^2+1-\alpha)}{\Gamma(\alpha|\xi|^2+1)}$	$\alpha\in(0,1)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=2-\alpha,\ \ell=1,\ m\in(0,\sqrt{\alpha^{-1}-1})$
25		$\dfrac{\Gamma(\alpha|\xi|^2+1)}{\Gamma(\alpha|\xi|^2+1-\alpha)}-\dfrac{1}{\Gamma(1-\alpha)}$	$\alpha\in(0,1)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=\alpha,\ m\in(0,\sqrt{\alpha^{-1}}),\ \ell=1$
26	Lamperti stable	$\dfrac{\Gamma(|\xi|^2+\alpha+\varrho)}{\Gamma(|\xi|^2+\varrho)}-\dfrac{\Gamma(\alpha+\varrho)}{\Gamma(\varrho)}$	$\alpha\in(0,1),\ \varrho\in(0,\infty)$	$d\geq 1$	$\gamma_0=2,\ \gamma_\infty=2\alpha,\ m\in(0,\sqrt{\varrho+\alpha}),\ \ell=1$

Table 1: (cont.)

If $(L_t)_{t\geq 0}$ is a subordinator, i. e. a Lévy process with non-decreasing sample paths, then we can relax the assumptions of Theorem 3.2; the reason is that we know that the support of L_t is contained in $[0, \infty)$ and therefore we have to establish upper bounds for $p_t(x)$ only for $x \geq 0$. Note that the characteristic exponent ψ of a subordinator with Laplace exponent f is given by $\psi(\xi) = f(-i\xi)$, $\xi \in \mathbb{R}$.

3.5 Corollary *Let $(S_t)_{t\geq 0}$ be a subordinator with Laplace exponent f satisfying* (S1), (S2).

(S1) *There exist $\theta \in (0, \frac{\pi}{2})$ and $m \geq 0$ such that f has a holomorphic extension F to*

$$\Upsilon := \Upsilon(m, \theta) := \{z \in \mathbb{C}; -m < \operatorname{Re} z < 0\} \cup \{z \in \mathbb{C}\backslash\{0\}; \arg z \in (\pi/2, \pi - \theta) \cup (-\pi + \theta, -\pi/2)\}.$$

(S2) *There exist constants $c_1, c_2 > 0$ and $\gamma_0, \gamma_\infty \in (0, 2]$ and a slowly varying (at infinity) increasing function $\ell : (0, \infty) \to (0, \infty)$ such that*

$$\operatorname{Re} F(z) \geq \frac{c_1}{\ell(|z|)}|z|^{\gamma_\infty} \qquad \text{for all } z \in \Upsilon, |z| \geq 1$$

and

$$|F(z)| \leq c_2 \ell(|z|)\left(|z|^{\gamma_0}\mathbb{1}_{\{|z|\leq 1\}} + |z|^{\gamma_\infty}\mathbb{1}_{\{|z|>1\}}\right) \qquad \text{for all } z \in \Upsilon.$$

Then S_t has a density p_t with respect to Lebesgue measure,

$$p_t(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\cdot\xi}e^{-tf(-i\xi)}\,d\xi.$$

The density is infinitely often differentiable and satisfies the estimates

$$|p_t(x)| \leq C\mathbb{1}_{[0,\infty)}(x)S(x,t)$$

$$\left|\frac{\partial}{\partial t}p_t(x)\right| \leq C\mathbb{1}_{[0,\infty)}(x)t^{-1}S(x,t)$$

for any $x \in \mathbb{R}$, $t \in (0, T]$ where $C = C(T) > 0$ is an absolute constant and

$$S(x,t) := S_m(x,t) := \exp\left(-\frac{m}{4}|x|\right)\left(1 + \ell(ct^{-1/\gamma_\infty})\right)\begin{cases} t^{-1/\gamma_\infty}, & |x| \leq t^{1/\gamma_\infty} \wedge 1, \\ t/|x|^{1+\gamma_\infty}, & t^{1/\gamma_\infty} < |x| \leq 1, \\ t/|x|^{1+\gamma_\infty\wedge\gamma_0}, & |x| > 1 \end{cases}$$

for some absolute constant $c = c(T) > 0$. Moreover, for any $k \in \mathbb{N}$ and $T > 0$ there exists a constant $c > 0$ such that

$$\left|\frac{\partial^k}{\partial x^k}p_t(x)\right| \leq c\mathbb{1}_{[0,\infty)}(x)t^{-k/\gamma_\infty}S(x,t) \qquad \text{for all } x \in \mathbb{R}, t \in (0, T].$$

Corollary 3.5 follows from the proof of Theorem 3.2, see [19, pp. 131] or [20, Section 4.9]. In Table 2 we have collected examples of Laplace exponents satisfying the assumptions of Corollary 3.5.

	Laplace Exponent	Parameter	Heat Kernel Estimate (Cor. 3.5)
1	λ^{α}	$\alpha \in (0,1]$	$\gamma_0 = \gamma_\infty = \alpha$, $m = 0$, $\ell = 1$
2	$(\lambda + \varrho)^{\alpha} - \varrho^{\alpha}$	$\alpha \in (0,1)$, $\varrho \in (0,\infty)$	$\gamma_0 = 2$, $\gamma_\infty = \alpha$, $m \in (0,\varrho)$, $\ell = 1$
3	$\dfrac{\lambda}{\sqrt{\lambda + \varrho}}$	$\varrho \in (0,\infty)$	$\gamma_0 = 1$, $\gamma_\infty = 1/2$, $m \in (0,\varrho)$, $\ell = 1$
4	$\dfrac{\lambda}{(\lambda + \varrho)^{\alpha}}$	$\alpha \in (0,1)$, $\varrho \in (0,\infty)$	$\gamma_0 = 1$, $\gamma_\infty = 1 - \alpha$, $m \in (0,\sqrt{\varrho})$, $\ell = 1$
5	$\dfrac{\lambda^{\beta} - 1}{\lambda^{\alpha} - 1} - 1$ (extended by continuity at $\lambda = 1$)	$0 < \alpha < \beta < 1$	$\gamma_0 = \alpha$, $\gamma_\infty = \beta - \alpha$, $m = 0$, $\ell = 1$
6	$-\dfrac{\lambda^{\alpha} - 1}{\lambda^{\alpha-1} - 1}$ (extended by continuity at $\lambda = 1$)	$\alpha \in (0,1)$	$\gamma_0 = 1 - \alpha$, $\gamma_\infty = 1$, $m = 0$, $\ell = 1$
7	$\dfrac{\lambda^{\alpha} - 1}{\lambda^{\alpha-1} - 1} - 1$ (extended by continuity at $\lambda = 1$)	$\alpha \in (1,2]$	$\gamma_0 = \alpha - 1$, $\gamma_\infty = 1$, $m = 0$, $\ell = 1$
8	$\lambda \dfrac{\lambda^{\alpha} - \varrho^{\alpha}}{\lambda - \varrho}$ (extended by continuity at $\lambda = \varrho$)	$\alpha \in (0,1)$, $\varrho \in (0,\infty)$	$\gamma_0 = 1$, $\gamma_\infty = \alpha$, $m = 0$, $\ell = 1$
9	$(\lambda^{-\alpha} + \lambda^{-\beta})^{-1}$ (extended by continuity at $\lambda = 0$)	$\alpha, \beta \in (0,1]$	$\gamma_0 = \alpha \vee \beta$, $\gamma_\infty = \alpha \wedge \beta$, $m = 0$, $\ell = 1$
10	$\sqrt{\lambda}(1 - e^{-2\varrho\sqrt{\lambda}})$	$\varrho \in (0,\infty)$	$\gamma_0 = 1$, $\gamma_\infty = 1/2$, $m = 0$, $\ell = 1$
11	$\sqrt{\lambda}(1 + e^{-2\varrho\sqrt{\lambda}})$	$\varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1/2$, $m = 0$, $\ell = 1$
12	$\varrho\lambda(\lambda + 1)\log(1 + 1/\lambda)$	$\varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1$, $m = 0$, $\ell(r) = \log(r \vee e)$

Table 2: Examples of Laplace exponents satisfying the assumptions of Corollary 3.5

.

	Laplace Exponent	Parameter	Heat Kernel Estimate (Cor. 3.5)
13	$\varrho \dfrac{\lambda(\lambda+1)}{(\lambda+2)\log(\lambda+2)}$	$\varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1,\ m \in (0,2),\ \ell(r) = \log(r \vee e)$
14	$\sqrt{\lambda}\arctan(\varrho\sqrt{\lambda})$	$\varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1/2,\ m = 0,\ \ell = 1$
15	$(\lambda+\varrho)^\alpha \cos\left(\alpha\arctan\sqrt{\tfrac{\lambda}{\varrho}}\right) - \varrho^\alpha$	$\alpha \in (0,1),\ \varrho \in (0,\infty)$	$\gamma_0 = 1,\ \gamma_\infty = \alpha,\ m \in (0,\varrho),\ \ell = 1$
16	$\varrho\sqrt{\lambda}\dfrac{\cosh^2(\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})}$	$\varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1/2,\ m = 0,\ \ell = 1$
17	$\varrho\sqrt{\lambda}\dfrac{\sinh^2(\sqrt{2\lambda})}{\sinh(2\sqrt{2\lambda})}$	$\varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1/2,\ m = 0,\ \ell = 1$
18	$\varrho\sqrt{\lambda}\coth((2\sqrt{\lambda})^{-1}) - \varrho\lambda$	$\varrho \in (0,\infty)$	$\gamma_0 = 1/2,\ \gamma_\infty = 1,\ m = 0,\ \ell = 1$
19	$\varrho\log(\sinh(\sqrt{2\lambda})) - \varrho\log(\sqrt{2\lambda})$	$\varrho \in (0,\infty)$	$\gamma_0 = 1,\ \gamma_\infty = 1/2,\ m = 0,\ \ell = 1$
20	$\varrho\log(\cosh(\sqrt{2\lambda}))$	$\varrho \in (0,\infty)$	$\gamma_0 = 1,\ \gamma_\infty = 1/2,\ m = 0,\ \ell = 1$
21	$\sqrt{\lambda}\log(1 + \varrho\tanh(b\sqrt{\lambda}))$	$b \in (0,\infty),\ \varrho \in (0,\infty)$	$\gamma_0 = \gamma_\infty = 1/2,\ m = 0,\ \ell = 1$
22	$\dfrac{\Gamma(\varrho\lambda + 1/2)}{\Gamma(\varrho\lambda)}$	$\varrho \in (0,\infty)$	$\gamma_0 = 1,\ \gamma_\infty = 1/2,\ m \in (0,1/(2\varrho)),\ \ell = 1$
23	$\lambda\dfrac{\Gamma(\alpha\lambda + 1 - \alpha)}{\Gamma(\alpha\lambda + 1)}$	$\alpha \in (0,1)$	$\gamma_0 = 1,\ \gamma_\infty = 1 - \alpha,\ \ell = 1,\ m \in (0,\alpha^{-1} - 1)$
24	$\dfrac{\Gamma(\alpha\lambda + 1)}{\Gamma(\alpha\lambda + 1 - \alpha)} - \dfrac{1}{\Gamma(1 - \alpha)}$	$\alpha \in (0,1)$	$\gamma_0 = 1,\ \gamma_\infty = \alpha,\ m \in (0,\alpha^{-1}),\ \ell = 1$
25	$\dfrac{\Gamma(\lambda + \alpha + \varrho)}{\Gamma(\lambda + \varrho)} - \dfrac{\Gamma(\alpha + \varrho)}{\Gamma(\varrho)}$	$\alpha \in (0,1),\ \varrho \in (0,\infty)$	$\gamma_0 = 1,\ \gamma_\infty = \alpha,\ m \in (0,\varrho + \alpha),\ \ell = 1$

Table 2: (cont.)

4 Existence result for Lévy-type processes

Let $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be defined by

$$q(x, \xi) = \psi_{\alpha(x)}(\xi), \qquad x, \xi \in \mathbb{R}^d$$

for a family $(\psi_\beta)_{\beta \in I}$ of continuous negative definite functions, a Hölder continuous mapping $\alpha : \mathbb{R}^d \to I$ and a set of parameters $I \subseteq \mathbb{R}^n$. Our main result, Theorem 4.1, gives a sufficient condition on $(\psi_\beta)_{\beta \in I}$ for the existence of a rich Lévy-type process with symbol q.

4.1 Theorem *Let $I \subseteq \mathbb{R}^n$ be open and convex and $m \geq 0$. Let $(\psi_\beta)_{\beta \in I}$ be a family of continuous negative definite functions $\psi_\beta : \mathbb{R}^d \to \mathbb{C}$ with $\psi_\beta(0) = 0$ for all $\beta \in I$. Suppose that there exist $\theta \in (0, \pi/2)$ and constants $c_1, c_2, c_3 > 0$ such that each ψ_β, $\beta \in I$, satisfies (4.1)–(4.1).*

(LTP1) *ψ_β is rotationally invariant for each $\beta \in I$, i.e. there exists $\Psi_\beta : \mathbb{R} \to \mathbb{R}$ such that $\psi_\beta(\xi) = \Psi_\beta(|\xi|)$, $\xi \in \mathbb{R}^d$. If $m > 0$: $\Psi_\beta(r) = \Psi_\beta(-r)$ for all $r \geq 0$.*

(LTP2) *ψ_β has a holomorphic extension to the domain $\Omega = \Omega(m, \theta)$ defined in (3).*

(LTP3) *There exist a measurable mapping $\gamma_0 : I \to (0, 2]$, a Hölder continuous mapping $\gamma_\infty : I \to (0, 2]$ and a slowly varying (at infinity) increasing function $\ell : (0, \infty) \to (0, \infty)$ such that*

$$\operatorname{Re} \Psi_\beta(z) \geq \frac{c_1}{\ell(|z|)} |\operatorname{Re} z|^{\gamma_\infty(\beta)} \qquad \text{for all} \quad z \in \Omega, |z| \geq 1, \beta \in I$$

and

$$|\Psi_\beta(z)| \leq c_2 \ell(|z|) \left(|z|^{\gamma_0(\beta)} \mathbb{1}_{\{|z| \leq 1\}} + |z|^{\gamma_\infty(\beta)} \mathbb{1}_{\{|z| > 1\}} \right) \qquad \text{for all} \quad z \in \Omega, \beta \in I.$$

Moreover, $\gamma_\infty^L := \inf_{\beta \in I} \gamma_\infty(\beta) > 0$, $\gamma_0^L := \inf_{\beta \in I} \gamma_0(\beta) > 0$.

(LTP4) *The partial derivative $\frac{\partial}{\partial \beta_j} \Psi_\beta(r)$ exists for all $r \in \mathbb{R}$ and extends holomorphically to Ω for all $j \in \{1, \ldots, n\}$ and $\beta \in I$. Moreover,*

$$\left| \frac{\partial}{\partial \beta_j} \Psi_\beta(z) \right| \leq c_3 (1 + \ell(|z|)) \left(|z|^{\gamma_0(\beta)} \mathbb{1}_{\{|z| \leq 1\}} + |z|^{\gamma_\infty(\beta)} \mathbb{1}_{\{|z| > 1\}} \right) \qquad \text{for all} \quad z \in \Omega, \beta \in I.$$

Then for any Hölder continuous mapping $\alpha : \mathbb{R}^d \to I$ there exists a rich Lévy-type process $(X_t)_{t \geq 0}$ with symbol

$$q(x, \xi) := \psi_{\alpha(x)}(\xi), \qquad x, \xi \in \mathbb{R}^d.$$

To prove Theorem 4.1 we use the parametrix method, cf. [19, Chapter 4] and [20, Chapter 4]. The idea is to construct the transition density as the fundamental solution of the Cauchy problem for the operator $(\partial_t - L)$ where L equals, when restricted to $C_c^\infty(\mathbb{R}^d)$, the pseudo-differential operator

$$Lf(x) = -\int_{\mathbb{R}^d} q(x, \xi) e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi, \qquad x \in \mathbb{R}^d.$$

The parametrix method gives a candidate for the fundamental solution, and the main part of the proof is to verify that this candidate is indeed a fundamental solution to the Cauchy problem and the transition density of a Feller process.

As a by-product of the parametrix construction, we get the following additional information on $(X_t)_{t \geq 0}$ and its transition semigroup $(P_t)_{t \geq 0}$.

4.2 Theorem *Under the assumptions of Theorem 4.1, the Lévy-type process $(X_t)_{t \geq 0}$ with symbol $q(x, \xi) = \psi_{\alpha(x)}(\xi)$ has the following additional properties:*

(i). *The associated semigroup $(P_t)_{t \geq 0}$ has the strong Feller property, i.e. $P_t f \in C_b(\mathbb{R}^d)$ for any $f \in \mathcal{B}_b(\mathbb{R}^d)$.*

(ii). *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator L and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$,*

$$Lf(x) = b(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q(x) \cdot \nabla^2 f(x)) + \int_{y \neq 0} (f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{B(0,1)}(y)) \, \nu(x, dy)$$

for any $f \in C_\infty^2(\mathbb{R}^d)$; here $(b(x), Q(x), \nu(x, dy))$ denotes the Lévy triplet associated with $q(x, \cdot)$. There exists a constant $C > 0$ such that

$$\|Lf\|_\infty \le C \sum_{0 \le |\alpha| \le 2} \|\partial^\alpha f\|_\infty = C\|f\|_{(2)} \qquad \text{for all} \ \ f \in C_\infty^2(\mathbb{R}^d).$$

Moreover, $P_t f \in \mathcal{D}(L)$ for all $t > 0$ and $f \in C_\infty(\mathbb{R}^d)$.

(iii). The distribution $\mathbb{P}^x(X_t \in \cdot)$ has a density $p(t, x, \cdot)$ with respect to Lebesgue measure for all $t > 0$ and $x \in \mathbb{R}^d$. The mapping $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is continuous and differentiable with respect to t.

(iv). The transition density p is a fundamental solution to the Cauchy problem for the operator $(\partial_t - L)$, i.e. $p(t, \cdot, y)$ converges weakly to δ_x as $t \to 0$, $(0, \infty) \ni t \mapsto p(t, x, y)$ is differentiable, $p(t, \cdot, y) \in \mathcal{D}(L)$ for all $t > 0$, $y \in \mathbb{R}^d$ and

$$(\partial_t - L_x)p(t, x, y) = 0 \qquad \text{for all} \ \ t > 0, x, y \in \mathbb{R}^d.$$

(v). The $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed; its unique (in the sense of finite-dimensional distributions) solution is $(X_t)_{t \ge 0}$.

(vi). Denote by $\varrho \in (0, 1]$ the Hölder exponent of α and choose $\gamma \in (0, 1/\gamma_\infty^U]$ such that $\kappa := \gamma \min\{\varrho, (-d + \gamma_\infty^U) + 1\} > 0$. Define

$$S(x, \beta, t) := \exp\left(-\frac{m}{4}|x|\right) \begin{cases} t^{-d/\gamma_\infty(\beta)}, & |x| \le t^{1/\gamma_\infty(\beta)} \wedge 1, \\ t/|x|^{d+\gamma_\infty(\beta)}, & t^{1/\gamma_\infty(\beta)} \le |x| \le 1, \\ t/|x|^{d+\gamma_0(\beta) \wedge \gamma_\infty(\beta)}, & |x| > 1. \end{cases}$$

For any $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$|p(t, x, y)| \le CS(x - y, \alpha(y), t) + Ct^\kappa \frac{1}{1 + |x - y|^{\gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4}|x - y|\right)$$

$$|\partial_t p(t, x, y)| \le Ct^{-1}S(x - y, \alpha(y), t) + Ct^{-1+\kappa} \frac{1}{1 + |x - y|^{\gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4}|x - y|\right)$$

for all $x, y \in \mathbb{R}^d$ and $t \in (0, T]$.

4.3 Remarks (i). In dimension $d = 1$ we can drop the assumption (4.1) of rotational invariance, see Theorem 4.4 below.

(ii). The constant $m/4$ in the definition of S, cf. Theorem 4.2(vi), may be replaced by $m(1-\delta)$ for any $\delta \in (0, 1)$; see also Remark 3.3.

(iii). In Theorem 4.1 we make separate assumptions on the regularity of $I \ni \beta \mapsto \psi_\beta(\xi)$ (differentiability) and $\mathbb{R}^d \ni x \mapsto \alpha(x) \in I$ (Hölder continuity). Note that this is much weaker than assuming differentiability of $x \mapsto q(x, \xi) = \psi_{\alpha(x)}(\xi)$. For instance, if $\psi_\beta(\xi) := |\xi|^\beta$, then the assumptions of Theorem 4.1 are satisfied for any Hölder continuous function α; in contrast, differentiability of $x \mapsto q(x, \xi) = |\xi|^{\alpha(x)}$ requires differentiability of α.

(iv). The first order approximation of the transition probability p is given by

$$p_0(t, x, y) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(x-y)\cdot\xi} e^{-t\psi_{\alpha(y)}(\xi)} \, d\xi;$$

it is possible to derive upper bounds for $|p(t, x, y) - p_0(t, x, y)|$, cf. [20, Theorem 3.8] for details.

4.4 Theorem (Case $d = 1$) Let $I \subseteq \mathbb{R}^n$ be an open convex set and $m \ge 0$. Suppose that $(\psi_\beta)_{\beta \in I}$ is a family of continuous negative definite functions $\psi_\beta : \mathbb{R} \to \mathbb{C}$, $\psi_\beta(0) = 0$, such that $\Psi_\beta(\xi) := \psi_\beta(\xi)$, $\beta \in I$, satisfies (4.1)-(4.1). Then the statements of Theorem 4.1 and Theorem 4.2 remain valid; in particular, there exists a rich Lévy-type process $(X_t)_{t \ge 0}$ with symbol $q(x, \xi) = \psi_{\alpha(x)}(\xi)$, $x, \xi \in \mathbb{R}$, for any Hölder continuous mapping $\alpha : \mathbb{R} \to I$.

The next theorem shows that in dimension $d = 1$ the transition probability $p(t, x, y)$ is differentiable with respect to x provided that the mappings $I \ni \beta \mapsto \psi_\beta(\xi)$ and $\alpha : \mathbb{R}^d \to I$ are sufficiently smooth.

4.5 Theorem *Let $(\psi_\beta)_{\beta \in I}$ be as in Theorem 4.4 and assume additionally that there exists a constant $c_4 > 0$ such that (4.5) holds.*

(LTP5) $\frac{\partial^2}{\partial \beta_j^2} \psi_\beta(\xi)$ *exists for all $\xi \in \mathbb{R}$, $j \in \{1, \ldots, n\}$ and has a holomorphic extension to Ω satisfying*

$$\left| \frac{\partial^2}{\partial \beta_j^2} \psi_\beta(z) \right| \le c_4 (1 + \ell(|z|))(|z|^{\gamma_0(\beta)} \mathbb{1}_{\{|z| \le 1\}} + |z|^{\gamma_\infty(\beta)} \mathbb{1}_{\{|z| > 1\}}), \qquad z \in \Omega, \beta \in I$$

where ℓ denotes the slowly varying function from (4.1).

Let $\alpha : \mathbb{R} \to I$ be such that $\alpha \in C_b^2(\mathbb{R})$. Denote by $(X_t)_{t \ge 0}$ the Lévy-type process from Theorem 4.4 with symbol $q(x, \xi) = \psi_{\alpha(x)}(\xi)$ and transition density p. Then:

(i). *The transition probability $p(t, x, y)$ is continuously differentiable with respect to x for any $t > 0$ and $y \in \mathbb{R}$. For any $T > 0$ there exists a constant $C = C(T) > 0$ such that*

$$\left| \frac{\partial}{\partial x} p(t, x, y) \right| \le C t^{-1/\gamma_\infty^L} \left[S(x - y, \alpha(y), t) + t^\kappa \frac{1}{1 + |x - y|^{d + \gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4} |x - y| \right) \right]$$

for all $t \in (0, T]$ and $x, y \in \mathbb{R}$; see Theorem 4.2(vi) for the definition of κ and S.

(ii). *The semigroup $(P_t)_{t \ge 0}$ asssociated with the Lévy-type process $(X_t)_{t \ge 0}$ satisfies the gradient estimate*

$$\sup_{x \in \mathbb{R}} \left| \frac{\partial}{\partial x} P_t f(x) \right| \le C t^{-1/\gamma_\infty^L} \|f\|_\infty \qquad \text{for all } t \in (0, T], f \in \mathcal{B}_b(\mathbb{R})$$

for some absolute constant $C = C(T) > 0$.

(iii). *Suppose additionally that each $\psi_\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in I$, is even. Then for any $T > 0$ there exist constants $C_1, C_2, C_3 > 0$ such that*

$$p(t, x, y) \ge C_1 t^{-1/\gamma_\infty(\alpha(y))} \left(1 - C_2 t^{-1/\gamma_\infty(\alpha(y))} |x - y| - C_3 t^\kappa \right)^+$$

for all $x, y \in \mathbb{R}$, $t \in (0, T]$.

(iv). *If $\psi_\beta : \mathbb{R} \to \mathbb{R}$ is an even function for all $\beta \in I$, then $(X_t)_{t \ge 0}$ is λ-irreducible, i.e.*

$$\int_{(0, \infty)} \mathbb{P}^x(X_t \in B) \, dt > 0$$

for all $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ with $\lambda(B) > 0$.

The remaining part of this article is devoted to applications of the above results. First, we state an existence result for "stable-like" processes. The most popular examples are isotropic stable-like processes (processes with variable index of stability); this corresponds to symbols of the form $q(x, \xi) = |\xi|^{\alpha(x)}$. Bass [1] proved the well-posedness of the associated martingale problem in dimension $d = 1$ for Dini continuous functions α, and, more recently, Kolokoltsov [15] established the existence of Feller processes with symbol $q(x, \xi) = |\xi|^{\alpha(x)}$ in dimension $d \ge 1$ for Hölder continuous mappings α. Using the results from the first part of this Section, we can derive existence results for many stable-like processes, for instance relativistic stable-like and Lamperti stable-like processes. With the exception of the well-studied isotropic stable case, such existence results were so far only known under much stronger regularity assumptions; e.g. a general existence result by Hoh [10] requires $\alpha \in C^{5d+3}(\mathbb{R}^d)$.

4.6 Example Let $q(x, \xi)$ be one of the following functions.

(i). (isotropic stable-like) $q(x, \xi) = |\xi|^{\alpha(x)}$ where $\alpha : \mathbb{R}^d \to (0, 2]$ is a Hölder continuous mapping such that $\inf_{x \in \mathbb{R}^d} \alpha(x) > 0$.

(ii). (relativistic stable-like) $q(x,\xi) = (|\xi|^2 + \varrho(x)^2)^{\alpha(x)/2} - \varrho(x)^{\alpha(x)}$ for Hölder continuous mappings $\alpha : \mathbb{R}^d \to (0,2)$ and $\varrho : \mathbb{R}^d \to (0,\infty)$ such that

$$\inf_{x \in \mathbb{R}^d} \alpha(x) > 0 \quad \text{and} \quad 0 < \varrho^L := \inf_{x \in \mathbb{R}^d} \varrho(x) \leq \sup_{x \in \mathbb{R}^d} \varrho(x) < \infty.$$

(iii). (Lamperti stable-like) $q(x,\xi) = (|\xi|^2 + \varrho(x))_{\alpha(x)} - (\varrho(x))_{\alpha(x)}$ for Hölder continuous mappings $\alpha : \mathbb{R}^d \to (0,1)$ and $\varrho : \mathbb{R}^d \to (0,\infty)$ such that

$$0 < \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) < 1 \quad \text{and} \quad 0 < \varrho^L := \inf_{x \in \mathbb{R}^d} \varrho(x) \leq \sup_{x \in \mathbb{R}^d} \varrho(x) < \infty;$$

here $(r)_\alpha := \Gamma(r + \alpha)/\Gamma(r)$ denotes the Pochhammer symbol.

(iv). (TLP-like) $q(x,\xi) = (|\xi|^2 + \varrho(x)^2)^{\alpha(x)/2} \cos\left[\alpha(x) \arctan \frac{|\xi|}{\varrho(x)}\right] - \varrho(x)^{\alpha(x)}$ for Hölder continuous mappings $\alpha : \mathbb{R}^d \to (0,1)$ and $\varrho : \mathbb{R}^d \to (0,\infty)$ such that

$$0 < \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) < 1 \quad \text{and} \quad 0 < \varrho^L := \inf_{x \in \mathbb{R}^d} \varrho(x) \leq \sup_{x \in \mathbb{R}^d} \varrho(x) < \infty.$$

By Theorem 4.1, there exists a rich Lévy-type process $(X_t)_{t \geq 0}$ with symbol q. The process $(X_t)_{t \geq 0}$ has the properties listed in Theorem 4.2; the heat kernel estimate 4.2.(vi) holds with

(i). $m = 0$, $\gamma_\infty(\alpha(x)) = \gamma_0(\alpha(x)) = \alpha(x)$

(ii). $m = \varrho^L$, $\gamma_\infty(\alpha(x), \varrho(x)) = \alpha(x)$, $\gamma_0(\alpha(x), \varrho(x)) = 2$,

(iii). $m = \sqrt{\varrho^L}$, $\gamma_\infty(\alpha(x), \varrho(x)) = 2\alpha(x)$, $\gamma_0(\alpha(x), \varrho(x)) = 2$,

(iv). $m = \varrho^L$, $\gamma_\infty(\alpha(x), \varrho(x)) = \alpha(x)$, $\gamma_0(\alpha(x), \varrho(x)) = 2$.

Let us remark that is possible to obtain further information on the richness of the domain of the infinitesimal generator of $(X_t)_{t \geq 0}$, cf. [21, Example 4.11].

Applying Theorem 4.4 we obtain in a similar fashion an existence result for one-dimensional rich Lévy-type processes with symbol

$$q(x,\xi) = (\kappa(x)^2 + (\xi - ib(x))^2)^{\alpha(x)/2} - (\kappa(x)^2 - b(x)^2)^{\alpha(x)/2}, \qquad x, \xi \in \mathbb{R}$$

for Hölder continuous bounded mappings $b : \mathbb{R} \to \mathbb{R}$, $\alpha : \mathbb{R} \to (0,2)$, $\kappa : \mathbb{R} \to (0,\infty)$ such that

$$\alpha^L := \inf_{x \in \mathbb{R}} \alpha(x) > 0, \qquad \kappa^L := \inf_{x \in \mathbb{R}} \kappa(x) > 0, \qquad \kappa^L - \|b\|_\infty > 0;$$

we call such a Lévy-type process an NTS-like process; NTS is short for normal tempered stable. More generally, it is possible to consider symbols of the form

$$q(x,\xi) = f_{\alpha(x)}(|\xi|^2)$$

for a family of Bernstein functions $(f_\beta)_{\beta \in I}$, this leads to, so-called, variable order subordination; see [20, Section 5.1] for a general existence result.

Further examples of families of continuous negative definite functions satisfying (4.1)-(4.1) are listed in Table 3; in Table 3 we use $C^{>0}(I)$ to denote the space of bounded Hölder continuous functions $f : \mathbb{R}^d \to I$ satisfying

$$f^L(x) := \inf_{x \in \mathbb{R}^d} f(x) \in I \quad \text{and} \quad f^U(x) := \sup_{x \in \mathbb{R}^d} f(x) \in I.$$

Name	Symbol	Assumptions	Dim.	Parameters (Thm. 4.1)
isotropic α-stable-like	$\lvert\xi\rvert^{\alpha(x)}$	$\alpha \in C^{>0}((0,2])$	$d \geq 1$	$\gamma_\infty(\alpha(x)) = \gamma_0(\alpha(x)) = \alpha(x),\ m = 0$
relativistic stable-like	$(\lvert\xi\rvert^2 + \varrho^2(x))^{\alpha(x)/2} - \varrho(x)^{\alpha(x)}$	$\alpha \in C^{>0}((0,2])$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x),\varrho(x)) = 2,\ m \in (0,\varrho^L)$ $\gamma_\infty(\alpha(x),\varrho(x)) = \alpha(x)$
NTS-like	$(\kappa(x)^2 + (\xi - ib(x))^2)^{\alpha(x)/2} - (\kappa(x)^2 - b(x)^2)^{\alpha(x)/2}$	$\alpha \in C^{>0}((0,2])$ $\kappa \in C^{>0}((0,\infty))$ $b \in C^{>0}(\mathbb{R})$ $\kappa^L - \lVert b\rVert_\infty > 0$	$d = 1$	$\gamma_0(\alpha(x),b(x),\kappa(x)) = 2$ $\gamma_\infty(\alpha(x),b(x),\kappa(x)) = \alpha(x)$ $m \in (0,\kappa^L - \lVert b\rVert_\infty)$
	$\dfrac{\lvert\xi\rvert^2}{\sqrt{\lvert\xi\rvert^2 + \varrho(x)}}$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2,\ \gamma_\infty(\varrho(x)) = 1$ $m \in (0,\sqrt{\varrho^L})$
	$\dfrac{\lvert\xi\rvert^2}{(\lvert\xi\rvert^2 + \varrho(x))^{\alpha(x)}}$	$\alpha \in C^{>0}((0,2))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x),\varrho(x)) = 2,\ m \in (0,\sqrt{\varrho^L})$ $\gamma_\infty(\alpha(x),\varrho(x)) = 2 - \alpha(x)$
	$\dfrac{\lvert\xi\rvert^{\beta(x)} - 1}{\lvert\xi\rvert^{\alpha(x)} - 1} - 1$ (extended by continuity at $\xi = 1$)	$\alpha, \beta \in C^{>0}((0,1))$ $(\beta - \alpha)^L > 0$	$d \geq 1$	$\gamma_0(\alpha(x),\beta(x)) = \alpha(x),\ m = 0$ $\gamma_\infty(\varrho(x)) = \beta(x) - \alpha(x)$
	$-\dfrac{\lvert\xi\rvert^{\alpha(x)} - 1}{\lvert\xi\rvert^{\alpha(x)-2} - 1}$ (extended by continuity at $\xi = 1$)	$\alpha \in C^{>0}((0,2))$	$d \geq 1$	$\gamma_0(\alpha(x)) = 2 - \alpha(x),\ \gamma_\infty(\alpha(x)) = 2$ $m = 0$
	$\dfrac{\lvert\xi\rvert^{\alpha(x)} - 1}{\lvert\xi\rvert^{\alpha(x)-2} - 1} - 1$ (extended by continuity at $\xi = 1$)	$\alpha \in C^{>0}((2,4])$	$d \geq 1$	$\gamma_0(\alpha(x)) = \alpha(x) - 2,\ \gamma_\infty(\alpha(x)) = 2$ $m = 0$
	$\lvert\xi\rvert^2 \dfrac{\lvert\xi\rvert^{\alpha(x)} - \varrho(x)^{\alpha(x)}}{\lvert\xi\rvert^2 - \varrho(x)^2}$ (extended by continuity at $\xi = \varrho(x)$)	$\alpha \in C^{>0}((0,2))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x),\varrho(x)) = 2,\ m = 0$ $\gamma_\infty(\alpha(x),\varrho(x)) = \alpha(x)$
	$(\lvert\xi\rvert^{-\alpha(x)} + \lvert\xi\rvert^{-\beta(x)})^{-1}$ (extended by continuity at $\xi = 0$)	$\alpha, \beta \in C^{>0}((0,2])$	$d \geq 1$	$\gamma_0(\alpha(x),\beta(x)) = \alpha(x) \vee \beta(x),\ m = 0$ $\gamma_\infty(\alpha(x),\beta(x)) = \alpha(x) \wedge \beta(x)$
	$\lvert\xi\rvert(1 - e^{-2\varrho(x)\lvert\xi\rvert})$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2,\ \gamma_\infty(\varrho(x)) = 1,\ m = 0$
	$\lvert\xi\rvert(1 + e^{-2\varrho(x)\lvert\xi\rvert})$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1,\ m = 0$
	$\varrho(x)\lvert\xi\rvert^2(\lvert\xi\rvert^2 + 1)\log(1 + \lvert\xi\rvert^{-2})$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 2,\ m = 0$

Table 3: Examples of admissible symbols

Name	Symbol	Assumptions	Dim.	Parameters (Thm. 4.1)
	$\varrho(x)\dfrac{\lvert\xi\rvert^2(\lvert\xi\rvert^2+1)}{(\lvert\xi\rvert^2+2)\log(\lvert\xi\rvert^2+2)}$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 2$ $m \in (0,\sqrt{2})$
	$\lvert\xi\rvert\arctan(\varrho(x)\lvert\xi\rvert)$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$, $m = 0$
TLP-like	$(\lvert\xi\rvert^2+\varrho(x)^2)^{\alpha(x)/2}\cos\big(\alpha(x)\arctan\frac{\lvert\xi\rvert}{\varrho(x)}\big) - \varrho(x)^{\alpha(x)}$	$\alpha \in C^{>0}((0,2))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x),\varrho(x)) = 2$, $m \in (0,\varrho^L)$ $\gamma_\infty(\alpha(x),\varrho(x)) = \alpha(x)$
	$\varrho(x)\lvert\xi\rvert\dfrac{\cosh^2(\sqrt{2}\lvert\xi\rvert)}{\sinh(2\sqrt{2}\lvert\xi\rvert)}$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$ $m = 0$
	$\varrho(x)\lvert\xi\rvert\dfrac{\sinh^2(\sqrt{2}\lvert\xi\rvert)}{\sinh(2\sqrt{2}\lvert\xi\rvert)}$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$ $m = 0$
	$\varrho(x)\lvert\xi\rvert\coth((2\lvert\xi\rvert)^{-1}) - \varrho(x)\lvert\xi\rvert^2$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 1$, $\gamma_\infty(\varrho(x)) = 2$, $m = 0$
	$\varrho(x)\log(\sinh(\sqrt{2}\lvert\xi\rvert)) - \varrho(x)\log(\sqrt{2}\lvert\xi\rvert)$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2$, $\gamma_\infty(\varrho(x)) = 1$, $m = 0$
isotropic Meixner-like	$\varrho(x)\log(\cosh(\sqrt{2}\lvert\xi\rvert))$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2$, $\gamma_\infty(\varrho(x)) = 1$, $m = 0$
	$\lvert\xi\rvert\log(1+\varrho(x)\tanh(b(x)\lvert\xi\rvert))$	$b \in C^{>0}((0,\infty))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(b(x),\varrho(x)) = \gamma_\infty(b(x),\varrho(x)) = 1$ $m = 0$
	$\dfrac{\Gamma(\varrho(x)\lvert\xi\rvert^2+1/2)}{\Gamma(\varrho(x)\lvert\xi\rvert^2)}$	$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2$, $\gamma_\infty(\varrho(x)) = 1$ $m \in (0,1/\sqrt{2\varrho^U})$
	$\lvert\xi\rvert^2\dfrac{\Gamma(\alpha(x)\lvert\xi\rvert^2+1-\alpha(x))}{\Gamma(\alpha(x)\lvert\xi\rvert^2+1)}$	$\alpha \in C^{>0}((0,1))$	$d \geq 1$	$\gamma_0(\alpha(x)) = 2$, $\gamma_\infty(\alpha(x)) = 2-\alpha(x)$ $m \in (0,\sqrt{1/\alpha^U-1})$
	$\dfrac{\Gamma(\alpha(x)\lvert\xi\rvert^2+1)}{\Gamma(\alpha(x)\lvert\xi\rvert^2+1-\alpha(x))} - \dfrac{1}{\Gamma(1-\alpha(x))}$	$\alpha \in C^{>0}((0,1))$	$d \geq 1$	$\gamma_0(\alpha(x)) = 2$, $\gamma_\infty(\alpha(x)) = \alpha(x)$ $m \in (0,\sqrt{1/\alpha^U})$
Lamperti stable-like	$\dfrac{\Gamma(\lvert\xi\rvert^2+\alpha(x)+\varrho(x))}{\Gamma(\lvert\xi\rvert^2+\varrho(x))} - \dfrac{\Gamma(\alpha(x)+\varrho(x))}{\Gamma(\varrho(x))}$	$\alpha \in C^{>0}((0,1))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x),\varrho(x)) = 2$, $m \in (0,\sqrt{\varrho^L+\alpha^L})$ $\gamma_\infty(\alpha(x),\varrho(x)) = 2\alpha(x)$

Table 3: (cont.)

Theorem 4.4 allows us to deduce an uniqueness and existence result for solutions of Lévy-driven stochastic differential equations, i. .e. SDEs of the form

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 = x \in \mathbb{R}^d$$

where $(L_t)_{t \geq 0}$ is a n-dimensional Lévy process. *If* the SDE has a unique weak solution, then it is possible to give conditions in terms of σ and the Lévy measure ν of $(L_t)_{t \geq 0}$ which ensure that the solution is a rich Feller process, cf. Kühn [18]. It is, however, in general a non-trivial problem to prove the uniqueness of the solution, see [26] for the particular case that $(L_t)_{t \geq 0}$ has a non-vanishing diffusion part and e. g. [22, 30] for the case that $(L_t)_{t \geq 0}$ is an isotropic stable process.

Using the parametrix construction, we can give sufficient conditions in terms of the characteristic exponent ψ such that the SDE has a unique weak solution which is a rich Feller process.

4.7 Corollary *Let $(L_t)_{t \geq 0}$ be a one-dimensional Lévy process with characteristic exponent ψ. Suppose that ψ has a holomorphic extension Ψ to $\Omega = \Omega(m^L, \theta)$ for some $m^L \geq 0$, $\theta \in (0, \pi/2)$ which satisfies the following two growth conditions:*

(i). *There exist $\alpha \in (0, 2]$, $\beta \in (1, 2)$ and constants $c_1, c_2 > 0$ such that*

$$\operatorname{Re}\Psi(z) \geq c_1 |\operatorname{Re} z|^\beta \qquad \text{for all } |z| \gg 1, \ z \in \Omega$$

  and

$$|\Psi(z)| \leq c_2\big(|z|^\alpha \mathbb{1}_{\{|z| \leq 1\}} + |z|^\beta \mathbb{1}_{\{|z| > 1\}}\big), \qquad z \in \Omega.$$

(ii). *There exists a constant $c_3 > 0$ such that $|\Psi'(z)| \leq c_3 |z|^{\beta-1}$ for all $z \in \Omega$, $|z| \gg 1$.*

Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be Hölder continuous bounded functions such that

$$0 < \sigma^L := \inf_{x \in \mathbb{R}} |\sigma(x)| \leq \sup_{x \in \mathbb{R}} |\sigma(x)| =: \sigma^U < \infty.$$

Then there exists a unique weak solution to the SDE

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 = x, \tag{6}$$

and the solution is a rich Lévy-type process with symbol $q(x, \xi) = -ib(x)\xi + \psi(\sigma(x)\xi)$. The solution has the following additional properties:

(i). *The transition probability $p : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 4.1 with $\gamma_0(b(x), \sigma(x)) = \min\{\alpha, 1\}$, $\gamma_\infty(b(x), \sigma(x)) = \beta$ and any $m \in (0, m^L/\sigma^U)$.*

(ii). *$C_c^\infty(\mathbb{R})$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t \geq 0}$ and $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*

(iii). *$(X_t)_{t \geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*

(iv). *The associated semigroup has the strong Feller property.*

Corollary 4.7 applies, in particular, to Lévy processes $(L_t)_{t \geq 0}$ with the following characteristic exponents:

(i). (isotropic stable) $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}$, $\alpha \in (1, 2]$,

(ii). (relativistic stable) $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha$, $\xi \in \mathbb{R}$, $\varrho > 0$, $\alpha \in (1, 2)$,

(iii). (Lamperti stable) $\psi(\xi) = (|\xi|^2 + \varrho)_\alpha - (\varrho)_\alpha$, $\xi \in \mathbb{R}$, $\varrho > 0$, $\alpha \in (1/2, 1)$, where $(r)_\alpha := \Gamma(r + \alpha)/\Gamma(r)$ denotes the Pochhammer symbol,

(iv). (truncated Lévy process) $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} \cos(\alpha \arctan(\varrho^{-1}|\xi|)) - \varrho^\alpha$, $\xi \in \mathbb{R}$, $\alpha \in (1, 2)$, $\varrho > 0$,

(v). (normal tempered stable) $\psi(\xi) = (\kappa^2 + (\xi - ib)^2)^{\alpha/2} - (\kappa^2 - b^2)^{\alpha/2}$, $\xi \in \mathbb{R}$, $\alpha \in (1, 2)$, $b > 0$, $|\kappa| > |b|$.

Up to know, this result was only known for the particular case that $(L_t)_{t \geq 0}$ is an isotropic stable process, see Knopova & Kulik [16, 22] and the references therein.

We close this section with an existence result for Lévy-type processes with symbols of variable order.

4.8 Theorem *Let $I \subseteq \mathbb{R}^n$ an open convex set and $\psi_\beta : \mathbb{R}^d \to \mathbb{C}$, $\beta \in I$, be a family of continuous negative definite functions satisfying (4.1)-(4.1) on*

$$\Omega(\vartheta) := \{z \in \mathbb{C} \backslash \{0\}; \arg z \in (-\vartheta, \vartheta) \cup (\pi - \vartheta, \pi + \vartheta)\}$$

for some $\vartheta \in (0, \pi/2)$. Assume, in addition, that

(LTP4') *The partial derivative $\frac{\partial}{\partial \beta_j} \Psi_\beta(r)$ exists for all $r \in \mathbb{R}$ and extends holomorphically to $\Omega(\vartheta)$ for all $j \in \{1, \ldots, n\}$ and $\alpha \in I$. There exist an increasing slowly varying (at ∞) function $\ell : (0, \infty) \to (0, \infty)$ and a constant $c_4 > 0$ such that*

$$\left| \frac{\partial_{\beta_j} \Psi_\beta(z)}{\Psi_\beta(z)} \right| \leq c_4 (1 + \ell(|z|)) \qquad \text{for all } z \in \Omega(\vartheta), \ j = 1, \ldots, n,$$

and

(S) *$(\psi_\beta)_{\beta \in I}$ satisfies the sector condition, i.e. there exists a constant $c > 0$ such that*

$$|\operatorname{Im} \Psi_\beta(z)| \leq c |\operatorname{Re} \Psi_\beta(z)| \qquad \text{for all } z \in \Omega(\vartheta), \ \beta \in I.$$

Then for any two Hölder continuous mappings $\alpha : \mathbb{R}^d \to (0, 1]$ and $\beta : \mathbb{R}^d \to I$ such that $\alpha^L := \inf_{x \in \mathbb{R}^d} \alpha(x) > 0$, there exists a rich Lévy-type process $(X_t)_{t \geq 0}$ with symbol

$$q(x, \xi) := (\psi_{\beta(x)}(\xi))^{\alpha(x)}, \qquad x, \xi \in \mathbb{R}^d.$$

The process $(X_t)_{t \geq 0}$ has the following properties:

(i). *The transition probability $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 4.2 with $\tilde{\gamma}_0(\alpha(x), \beta(x)) := \alpha(x) \gamma_0(\beta(x))$, $\tilde{\gamma}_\infty(\alpha(x), \beta(x)) := \alpha(x) \gamma_\infty(\beta(x))$ and $m = 0$; here $\gamma_0(\beta(x))$ and $\gamma_\infty(\beta(x))$ are the mappings associated with $(\psi_\beta)_{\beta \in I}$ by the growth condition (4.1).*

(ii). *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t \geq 0}$ and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*

(iii). *$(X_t)_{t \geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*

(iv). *The associated semigroup has the strong Feller property.*

In dimension $d = 1$ Theorem 4.8 remains valid if we just assume that $(\psi_\beta)_{\beta \in I}$ satisfies (4.1), (4.1), (4.8) and (S), i.e. we can drop the assumption of rotational invariance.

Acknowledgements I would like to thank René Schilling for helpful comments and suggestions.

References

[1] Bass, R. F.: Uniqueness in Law for Pure Jump Markov Processes. *Probab. Th. Rel. Fields* **79** (1988), 271–287.

[2] Bendikov, A.: Asymptotic formulas for symmetric stable semigroups. *Expo. Math.* **12** (1994), 381–384.

[3] Blumenthal, R., Getoor, R.: Some theorems on stable processes. *Trans. Amer. Math. Soc.* **95** (1960), 263–273.

[4] Blumenthal, R., Getoor, R.: Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.* **10** (1961), 493–516.

[5] Bogdan, K., Grzywny, T., Ryznar, M.: Density and tails of unimodal convolution semi-groups. *J. Funct. Anal.* **266**, 3543–3571.

[6] Bogdan, K., Jakubowski, T.: Estimates of Heat Kernel of Fractional Laplacian Perturbed by Gradient Operators. *Commun. Math. Phys.* **271** (2007), 179–198.

[7] Böttcher, B., Schilling, R. L., Wang, J.: *Lévy-Type Processes: Construction, Approximation and Sample Path Properties.* Springer Lecture Notes in Mathematics vol. **2099**, (vol. III of the "Lévy Matters" subseries). Springer, 2014.

[8] Chen, Z.-Q., Kim, P., Song, R.: Dirichlet Heat Kernel Estimates for Rotationally Symmetric Lévy Processes. *Proc. Lond. Math. Soc.*, DOI: 10.1112/plms/pdt068.

[9] Chen, Z.-Q., Zhang, X.: Heat kernels and analyticity of non-symmetric jump diffusion semigroups. To appear in *Prob. Theory Relat. Fields.*

[10] Hoh, W.: *Pseudo-Differential Operators Generating Markov Processes.* Habilitations-schrift. Universität Bielefeld, Bielefeld 1998.

[11] Huang, L.: Density estimates for SDEs driven by tempered stable processes. Preprint arXiv 1504.04183.

[12] Jakubowski, T., Szczypkowski, K.: Time-dependent gradient perturbations of fractional Laplacian. *J. Evol. Equ.* **10** (2010), 319–339.

[13] Khoshnevisan, D., Schilling, R. L.: *From Lévty-type Processes to Parabolic SPDEs.* Birkhäuser Verlag, 2017.

[14] Kolokoltsov, V. N.: Symmetric stable laws and stable-like jump diffusions. *Proc. Lond. Math. Soc.* **80** (2000), 725–768.

[15] Kolokoltsov, V. N.: *Markov Processes, Semigroups and Generators.* De Gruyter, 2011.

[16] Knopova, V., Kulik, A.: Parametrix construction of the transition probability density of the solution to an SDE driven by α-stable noise. Preprint arXiv 1412.8732.

[17] Knopova, V., Schilling, R. L.: Transition Density Estimates for a Class of Lévy and Lévy-Type Processes. *J. Theor. Probab.* **25** (2012), 144–170.

[18] Kühn, F.: Solutions of Lévy-driven SDEs with unbounded coefficients as Feller processes. Preprint arXiv 1610.02286.

[19] Kühn, F.: Probability and Heat Kernel Estimates for Lévy(-Type) Processes. PhD Thesis, Technische Universität Dresden 2016. http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-214839

[20] Kühn, F.: *Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates.* To appear, Springer. (vol. V of the "Lévy Matters" subseries)

[21] Kühn, F., Schilling, R. L.: On the domain of fractional Laplacians and related generators of Feller processes. Preprint arXiv 1610.08197.

[22] Kulik, A.: On weak uniqueness and distributional properties of a solution to an SDE with α-stable noise. Preprint arXiV 1511.00106.

[23] Sato, K.: *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, 2005.

[24] Schilling, R. L.: Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Th. Rel. Fields* **112** (1998), 565–611.

[25] Schilling, R. L., Song, R., Vondraček, Z.: *Bernstein functions.* De Gruyter, 2012 (2nd edition).

[26] Stroock, D. W.: Diffusion Processes Associated with Lévy Generators. *Z. Wahrschein-lichkeitstheorie verw. Gebiete* **32** (1975), 209–244.

[27] Sztonyk, P.: Estimates of Tempered Stable Densities. *J. Theor. Probab.* **23** (2010), 127–147.

[28] Sztonyk, P.: Estimates of densities for Lévy processes with lower intensity of large jumps. Preprint arXiv 1505.04831. To appear in *Math. Nachr.*

[29] Ushakov, N. G.: *Selected topics on characteristic functions.* De Gruyter, 2011.

[30] Zanzotto, P. A.: On stochastic differential equations driven by a Cauchy process and other stable Lévy motions. *Ann. Probab.* **30** (2002), 802–825.