A probabilistic proof of Schoenberg's theorem

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Abstract

Assume that $g(|\xi|^2)$, $\xi \in \mathbb{R}^k$, is for every dimension $k \in \mathbb{N}$ the characteristic function of an infinitely divisible random variable X^k. By a classical result of Schoenberg $f := -\log g$ is a Bernstein function. We give a simple probabilistic proof of this result starting from the observation that $X^k = X_1^k$ can be embedded into a Lévy process $(X_t^k)_{t\geq 0}$ and that Schoenberg's theorem says that $(X_t^k)_{t\geq 0}$ is subordinate to a Brownian motion. A key ingredient of our proof are concrete formulae which connect the transition densities, resp., Lévy measures of subordinated Brownian motions across different dimensions. As a by-product of our proof we obtain a gradient estimate for the transition semigroup of a subordinated Brownian motion.

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1. Introduction

I.J. Schoenberg proved in 1938 [26] the following

> **Theorem A:** *If $\mathbb{R}^k \ni (x_1, \dots, x_k) \mapsto g(x_1^2 + \dots + x_k^2)$ is positive definite (in the sense of Bochner) for any dimension $k \in \mathbb{N}$, then $g(r)$, $r > 0$, is a completely monotone function.*

Recall that $g : (0, \infty) \to (0, \infty)$ is *completely monotone* (notation: $g \in \mathcal{CM}$), if

$$g \in C^\infty \quad \text{and} \quad (-1)^n g^{(n)}(r) \geq 0 \quad \text{for all} \quad r > 0,\ n \in \mathbb{N}_0. \tag{1}$$

Schoenberg used Theorem A to determine all positive definite functions in a Hilbert space; this was part of his programme to characterize all metrics ρ in $\mathbb{R}^k$ such that $(\mathbb{R}^k, \rho)$ can be isometrically embedded into a Hilbert space, cf. [27]. A necessary and sufficient condition turns out to be that $\mathbb{R}^k \ni x \mapsto e^{-t\rho^2(x,0)}$ is positive definite; in

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other words: all such metrics are of the form $\rho(x,y)=\sqrt{\psi(x-y)}$ where ψ is the (non-negative!) characteristic exponent of a symmetric Lévy process. This allows us to re-cast Schoenberg's theorem in the form of

> **Theorem B:** *If* $\mathbb{R}^k \ni (x_1,\dots,x_k) \mapsto \exp\left[-tf(x_1^2+\cdots+x_k^2)\right]$ *is positive definite (in the sense of Bochner) for all* $t>0$ *and any dimension* $k\in\mathbb{N}$*, then* $g(r)=e^{-tf(r)}$*,* $r>0$*, is a completely monotone function.*

In probabilistic terms, this means that $g(x_1^2+\cdots+x_k^2)$ is an infinitely divisible characteristic function and $f:(0,\infty)\to(0,\infty)$ is a *Bernstein function*, i.e.

$$f\in C^\infty,\quad f\geq 0\quad\text{and}\quad (-1)^{n-1}f^{(n)}(r)\geq 0\quad\text{for all}\quad r>0,\ n\in\mathbb{N}. \tag{2}$$

Both theorems have attracted a lot of attention and there are several proofs highlighting various (hidden) aspects of Schoenberg's result. Let us briefly describe some of the developments. Modern (streamlined) versions of the classical proof of Theorem A can be found in Donoghue [5, p. 205] and Steerneman & van Perlo-ten Kleij [28], where the presentation of the convergence argument as $k\to\infty$ is simplified; following Bochner [3, p. 99], Theorem B is e.g. proved in [24, Theorem 13.14].

Using Bochner's characterization of positive definite functions and the solution of Hausdorff's moment problem, Ressel [21] proves Theorem A in a completely different way. This approach is generalized to semigroups in Berg *et al.* [1, Chapter 5]. Independent of Ressel, Kahane [8] uses essentially the same argument to prove both Theorem A and B. Combining Bochner's theorem with the characterization of completely monotone functions by iterated differences[1], Wendland [29, Theorem 7.13] gives a short proof of Theorem A which is inspired by earlier work by Kuelbs [13] and Wells & Williams [30, Chapter II]. Let us point out that the essential step in these proofs, [29, p. 94, last 4 lines] (also [13, Lemma 2.1], [30, Theorem 7.2]) is already present in Harzallah's proof that Bernstein functions operate on negative definite functions [6, Lemma 6], see also Jacob [7, Lemma 3.9.22] who works out this detail.

The methods to prove Theorems A and B are closely related to the so-called *Schoenberg's problem* in the geometry of Banach spaces:

> *Determine all values* $\alpha\geq 0$ *such that* $\exp(-\|x\|_{\ell^p}^\alpha)$ *is positive definite on* $\mathbb{R}^k$ *with* $k\geq 2$ *and* $p\geq 1$*.*

For $1\leq p\leq 2$ this is discussed by Bretagnolle *et al.* [4] (who establish the connection with the embeddability of normed linear spaces into L^p); Zastavnyi [33] has the definitive solution.

Our approach to prove Theorem B uses elements of the Fourier approach from the original proof of Schoenberg's theorem, but the rather awkward limiting argument, sending the dimension $k\to\infty$, is now replaced by a "dimension walk" argument which was pioneered by Matheron who calls it the *montée et descente en clavier isotrope* [17, pp. 31–37], see also the unpublished manuscript [18].

[1] In the end, this characterization relies on a deep application of the Krein–Milman theorem, cf. [24, Theorem 4.8].

2. Preliminaries

A function $u:\mathbb{R}^k\to\mathbb{R}$ is called *rotationally invariant* if $u(x)$ depends only on $|x|$, i.e. if $u(x)=U(|x|)$ for some function $U:[0,\infty)\to\mathbb{R}$. In abuse of notation we write $u(r)=U(r)$ for $r\geq 0$. For an integrable function $u:\mathbb{R}^k\to\mathbb{R}$ we denote by

$$\begin{aligned}\mathcal{F}_k u(\xi) &:= \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} e^{-ix\cdot\xi}u(x)\,dx, \quad && \xi\in\mathbb{R}^k,\\ \mathcal{F}_k^{-1}u(\xi) &:= \int_{\mathbb{R}^k} e^{ix\cdot\xi}u(x)\,dx, && \xi\in\mathbb{R}^k,\end{aligned} \tag{3}$$

the *Fourier transform* and *inverse Fourier transform* of u, respectively. If u is rotationally invariant, then both $\mathcal{F}_k u$ and $\mathcal{F}_k^{-1}u$ are rotationally invariant and

$$\mathcal{F}_k u(r) = \frac{1}{(2\pi)^k}\mathcal{F}_k^{-1}u(r) = \frac{1}{(2\pi)^{k/2}r^{k/2-1}}\int_{(0,\infty)} u(s)s^{k/2}J_{k/2-1}(sr)\,ds \tag{4}$$

where J_ν denotes the Bessel function of the first kind, see e.g. [23, Example 19.4] or [29, Theorem 5.26] for a proof. Using (4) and some identities for Bessel functions [20, (10.6.2)] (see also the proof of Theorem 3.3), it is not hard to see that

$$\mathcal{F}_{k+2}u(r) = -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}\mathcal{F}_k u(r) \tag{5}$$

for any rotationally invariant function such that $u(|\cdot|)\in L^1(\mathbb{R}^k,dx)\cap L^1(\mathbb{R}^{k+2},dx)$; this observation is due to Matheron [17, (1.4.9)].

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space. A random variable $X:\Omega\to\mathbb{R}^k$ is called *unimodal isotropic* if $\mathbb{P}(X\in dx)=c\delta_0(dx)+p(|x|)\,dx$ for some non-increasing $p:(0,\infty)\to[0,\infty)$ and $c\in[0,1]$. A family of random variables $X_t:\Omega\to\mathbb{R}^k$ is called (*k-dimensional*) *Lévy process* if $X_0=0$ a.s., $(X_t)_{t\geq 0}$ has independent and stationary increments and $t\mapsto X_t(\omega)$ is for almost all $\omega\in\Omega$ right-continuous with finite left limits. Our standard reference for Lévy processes is the monograph by Sato [22]. For an introduction to Lévy processes we also recommend [11]. We will often use the superscript to indicate the dimension, i.e. we write $(X_t^k)_{t\geq 0}$ for a k-dimensional Lévy process. It is well known, cf. [22], that $(X_t)_{t\geq 0}$ can be uniquely characterized via its *characteristic exponent*,

$$\psi(\xi) = -i\,b\cdot\xi+\frac{1}{2}\xi\cdot Q\xi+\int_{\mathbb{R}^k\setminus\{0\}}\left(1-e^{iy\cdot\xi}+i\,y\cdot\xi\mathbf{1}_{(0,1)}(|y|)\right)\nu(dy), \quad \xi\in\mathbb{R}^k;$$

the *Lévy triplet* (b,Q,ν) consists of the *drift* $b\in\mathbb{R}^k$, a positive semi-definite symmetric matrix $Q\in\mathbb{R}^{k\times k}$ and the *Lévy measure* ν on $(\mathbb{R}^k\setminus\{0\},\mathcal{B}(\mathbb{R}^k\setminus\{0\}))$ satisfying $\int_{\mathbb{R}^k\setminus\{0\}}\min\{1,|y|^2\}\,\nu(dy)<\infty$. We say that ψ satisfies the *Hartman–Wintner condition* if

$$\lim_{|\xi|\to\infty}\frac{\operatorname{Re}\psi(|\xi|)}{\log|\xi|}=\infty. \tag{HW}$$

It is shown in [12] that the Hartman–Wintner condition is equivalent to the existence of a smooth transition density p_t for all $t>0$. A function ψ is *continuous negative definite* (in the sense of Schoenberg) if, and only if, it is the characteristic exponent of a Lévy process. The domain of the *generator* $A = A_k$ of a k-dimensional Lévy process contains the compactly supported smooth functions $C_c^\infty(\mathbb{R}^k)$ and

$$A_k u(x) = -\int_{\mathbb{R}^k} e^{ix\cdot\xi}\psi(\xi)\mathcal{F}_k u(\xi)\,d\xi = -\mathcal{F}_k^{-1}(\psi\cdot\mathcal{F}_k u)(x), \quad u\in C_c^\infty(\mathbb{R}^k),\ x\in\mathbb{R}^k, \tag{6}$$

cf. [11, Theorem 6.8]. If ψ is a rotationally invariant characteristic exponent of a k-dimensional Lévy process and $u(x) = u(|x|)$ a rotationally invariant function with compact support, then we write in accordance with (4)

$$A_k u(r) := -\mathcal{F}_k^{-1}(\psi\cdot\mathcal{F}_k u)(r), \quad r\geq 0. \tag{7}$$

The *jump measure* N of $(X_t)_{t\geq 0}$ is given by

$$N_t(B) := \#\{s\in[0,t]; \Delta X_s := X_s - X_{s-}\in B\}, \quad B\in\mathcal{B}(\mathbb{R}^k\setminus\{0\}),\ t\geq 0. \tag{8}$$

For any fixed Borel set $B\in\mathcal{B}(\mathbb{R}^k\setminus\{0\})$ the process $(N_t(B))_{t\geq 0}$ is a Poisson process with intensity $\nu(B)$, cf. [11, Lemma 9.4].

A one-dimensional Lévy process $(S_t)_{t\geq 0}$ is called a *subordinator* if $(S_t)_{t\geq 0}$ has non-decreasing sample paths. A subordinator is uniquely characterized by its Laplace transform $\mathbb{E}e^{-uS_t} = e^{-tf(u)}$, $u\geq 0$; the characteristic (Laplace) exponent f is a *Bernstein function*, i.e.

$$f(u) = \alpha u + \int_0^\infty (1-e^{-uy})\,\mu(dy), \quad u\geq 0,$$

for $\alpha\geq 0$ and a measure μ on $(0,\infty)$ such that $\int_0^\infty \min\{1,y\}\,\mu(dy) < \infty$. By Bernstein's theorem, cf. [24, Theorem 3.2], this is equivalent to (2).

If $(S_t)_{t\geq 0}$ is a subordinator with Laplace exponent f and $(B_t)_{t\geq 0}$ an independent Brownian motion, then the *subordinated Brownian motion* $(B_{S_t})_{t\geq 0}$ is again a Lévy process, and its characteristic exponent is given by $\psi(\xi) = f(|\xi|^2)$. A comprehensive treatment of completely monotone functions, Bernstein functions and subordination is given in [24].

3. Results

We will prove the following extended version of Schoenberg's theorem.

Theorem 3.1. *Let $f:[0,\infty)\to[0,\infty)$. The following statements are equivalent.*

(i) $\mathbb{R}^k\ni\xi\mapsto f(|\xi|^2)$ *is a continuous negative definite function for all* $k\geq 1$.

(ii) $\mathbb{R}^k\ni\xi\mapsto f(|\xi|^2)$ *is a continuous negative definite function for any* $k = 1+2n$, $n\in\mathbb{N}_0$.

(iii) f *is a Bernstein function.*

(iv) *For some (all)* $k\geq 1$ *there exists a* k*-dimensional Lévy process* $(X_t)_{t\geq 0}$ *with characteristic exponent* $\psi(\xi) := f(|\xi|^2)$ *and*

$$\mathbb{P}(X_t\in B) = e^{-ct}\delta_0(B) + \int_B p_t(x)\,dx, \quad B\in\mathcal{B}(\mathbb{R}^k),\ t>0, \tag{9}$$

for some constant $c\in[0,\infty]$ *and a rotationally invariant function* $p_t:\mathbb{R}^k\to[0,\infty)$ *such that* $p_t(\sqrt{\cdot})$ *is completely monotone.*

The proof of Theorem 3.1 actually shows that f is a Bernstein function if, and only if, there exist infinitely many $n \in \mathbb{N}$ such that $\mathbb{R}^k \ni \xi \mapsto f(|\xi|^2)$ is a continuous negative definite function for $k = n$ and $k = n + 2$. Moreover, if $(S_t)_{t\geq 0}$ is a subordinator with Laplace exponent f, then the Lévy process $(X_t)_{t\geq 0}$ is subordinate to a Brownian motion, i.e. it is, in distribution, a time-changed Brownian motion $(B_{S_t})_{t\geq 0}$. The next corollary reveals how the density function p_t^k and the Lévy measure ν_k in different dimensions are related. As before, we use sub- and superscripts to indicate the dimension. We continue using the notation of Theorem 3.1.

Corollary 3.2. *Let $f : [0, \infty) \to [0, \infty)$ and suppose that one (hence all) of the conditions of Theorem 3.1 holds.*

(i) *The rotationally invariant function $p_t = p_t^k$ satisfies*

$$p_t^k(r) = -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}p_t^{k-2}(r) \quad \textit{for all} \quad r > 0,\ t > 0,\ k \geq 3.$$

(ii) *The Lévy measure ν_k of the k-dimensional Lévy process $(X_t)_{t\geq 0}$ has a rotationally invariant density m_k with respect to k-dimensional Lebesgue measure; moreover, $m_k(\sqrt{\cdot})$ is completely monotone and satisfies*

$$m_k(r) = -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}m_{k-2}(r) \quad \textit{for all} \quad r > 0,\ k \geq 3.$$

If we formally define an operator T by $T := -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}$, then Corollary 3.2(i) reads $p_t^k = Tp_t^{k-2}$. Theorem 3.3 shows that a similar relation holds for the generator:

$$A_k u = TA_{k-2}(T^{-1}u)$$

for any smooth rotationally invariant function u with compact support. This means that the operators A_{k-2} and A_k are intertwined.

Theorem 3.3. *Let $(X_t^k)_{t\geq 0}$ be a k-dimensional Lévy process with rotationally invariant characteristic exponent $\psi_k(\xi) = \psi(|\xi|)$, $\xi \in \mathbb{R}^k$, $k \geq 3$. Then there exists a $(k-2)$-dimensional Lévy process $(X_t^{k-2})_{t\geq 0}$ with characteristic exponent $\psi_{k-2}(\xi) := \psi(|\xi|)$, $\xi \in \mathbb{R}^{k-2}$. If we denote by A_{k-2} and A_k the generator of $(X_t^{k-2})_{t\geq 0}$ and $(X_t^k)_{t\geq 0}$, respectively, then both $A_k u$ and $A_{k-2}u$ are rotationally invariant and*

$$A_k u(r) = \frac{1}{r}\frac{d}{dr}A_{k-2}\left(\int_0^\bullet su(s)\,ds\right)(r), \quad r > 0,$$

for any smooth rotationally invariant function u with compact support, cf. (7).

Finally, we derive the following result on subordinated Brownian motion.

Corollary 3.4. *Let $(B_t)_{t\geq 0}$ be a k-dimensional Brownian motion and $(S_t)_{t\geq 0}$ a subordinator with Laplace exponent f. The subordinated Brownian motion $X_t := B_{S_t}$ satisfies $\mathbb{P}(X_t \in dx) = e^{-ct}\delta_0(dx) + p_t(x)\,dx$ with c and p_t as in 3.1(iv).*

(i) *The following statements are equivalent.*
   (a) *$c = \infty$ (i.e. $\mathbb{P}(X_t = 0) = 0$ for some $t > 0$) and $\lim_{r\to 0} p_t(r) < \infty$ for small $t > 0$.*

(b) $\mathbb{E}(S_t^{-k/2}) < \infty$ *for small* $t > 0$.
(c) $\psi(\xi) := f(|\xi|^2)$ *satisfies the Hartman–Wintner condition* (HW).

(ii) $c < \infty$ (*i.e.* $\mathbb{P}(X_t = 0) > 0$ *for some* $t > 0$) *if, and only if,* $(X_t)_{t\geq 0}$ *is a compound Poisson process.*

Corollary 3.4(ii) implies that a subordinate Brownian motion is either a compound Poisson process or absolutely continuous for all $t > 0$. Zabczyk [32] proved, more generally, that this holds for any Lévy process with a rotationally invariant characteristic exponent. Moreover, (the proof of) Corollary 3.4(i) shows that for a subordinator $(S_t)_{t\geq 0}$ with Laplace exponent f we have

$$\mathbb{E}(S_t^{-\kappa}) < \infty \quad \text{for some (all) } \kappa > 0 \text{ and small (all) } t > 0$$

if, and only if, f satisfies the Hartman–Wintner condition $\lim_{r\to\infty} f(r)/\log r = \infty$. For a discussion of the Hartman–Wintner condition (HW) see Knopova & Schilling [12].

4. Proof of Theorem 3.1

In order to prove Schoenberg's original statement (the equivalence of 3.1(i) and 3.1(iii)) we will first focus on functions f satisfying the Hartman–Wintner condition

$$\lim_{r\to\infty} \frac{f(r)}{\log r} = \infty,$$

and then extend the result using an approximation argument. The key tool is the following proposition which is of independent interest. It is inspired by a publication by Kulczycki & Ryznar [16] where the implication "(ii)⇒(i)" is used to obtain gradient estimates of transition densities for Lévy processes.

Proposition 4.1. *Let* $(X_t^k)_{t\geq 0}$ *be a* k*-dimensional Lévy process with rotationally invariant characteristic exponent* $\psi_k(\xi) = \psi(|\xi|)$, $\xi \in \mathbb{R}^k$. *If* ψ *satisfies the Hartman–Wintner condition, then the following statements are equivalent.*

(i) *There exists a* $(k+2)$*-dimensional Lévy process* $(X_t^{k+2})_{t\geq 0}$ *with characteristic exponent* $\psi_{k+2}(\xi) := \psi(|\xi|)$, $\xi \in \mathbb{R}^{k+2}$.

(ii) *The rotationally invariant density* p_t^k *of* X_t^k *satisfies* $\frac{d}{dr} p_t^k(r) \leq 0$ *for all* $t > 0$.

(iii) X_t^k *is unimodal isotropic for all* $t > 0$.

If one (hence all) of the conditions is satisfied, then

$$p_t^{k+2}(r) = -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr} p_t^k(r) \quad \textit{for all} \quad r > 0. \tag{10}$$

Wolfe [31] and Medgyessy [19] have shown that X_t^k is unimodal isotropic if, and only if, the Lévy measure ν_k is unimodal isotropic, see also Sato [22, Theorem 54.1]. Let us briefly give an intuitive explanation for pure-jump Lévy processes $(X_t^k)_{t\geq 0}$. It is known that the Lévy measure ν_k is the vague limit of $t^{-1}\mathbb{P}(X_t^k \in \cdot)$ as $t \to 0$, i.e.

$$\nu_k(B) = \lim_{t\to 0} \frac{1}{t}\mathbb{P}(X_t^k \in B) = \lim_{t\to 0} \frac{1}{t}\int_B p_t^k(y)\, dy \tag{11}$$

for any Borel set $B \in \mathcal{B}(\mathbb{R}^k \setminus \{0\})$ with no mass at the boundary $\nu_k(\partial B) = 0$, see e.g. [11, Remark 6.12] or [15, Corollary 3.3], and therefore ν_k is unimodal isotropic as the vague limit of unimodal isotropic distributions. On the other hand, if ν_k is unimodal isotropic, then the truncated measure $\mu_\epsilon := \nu_k(\cdot \cap B(0,\epsilon)^c)$ is unimodal isotropic for each $\epsilon > 0$, and the associated compound Poisson process $X_t^{k,\epsilon}$ has a distribution of the form

$$e^{-\lambda_\epsilon t} \sum_{m=0}^{\infty} \frac{t^m}{m!} \mu_\epsilon^{*m} \quad \text{with} \quad \lambda_\epsilon = \nu_k(B(0,\epsilon)^c)$$

which implies that $X_t^{k,\epsilon}$ is unimodal isotropic; hence, $X_t^k = \lim_{\epsilon \to 0} X_t^{k,\epsilon}$ is unimodal isotropic.

Proof of Proposition 4.1. Because of the growth condition (HW),

$$p_t^{k+2}(x) := \frac{1}{(2\pi)^{k+2}} \int_{\mathbb{R}^{k+2}} e^{-ix\cdot\xi} e^{-t\psi(|\xi|)}\, d\xi$$

is well-defined and, in fact, infinitely often differentiable. Moreover, p_t^{k+2} is rotationally invariant and, as $\psi(0) = 0$, we have

$$\int_{\mathbb{R}^{k+2}} p_t^{k+2}(x)\, dx = (\mathcal{F}_{k+2}^{-1} p_t^{k+2})(0) = e^{-t\psi(0)} = 1.$$

Because of (5), the relation (10) holds.

(i)⇒(ii): By definition, p_t^{k+2} is the density of X_t^{k+2}; in particular, $p_t^{k+2} \geq 0$. Because of (10), we get $\frac{d}{dr} p_t^k(r) \leq 0$.

(ii)⇒(i): (10) shows $p_1^{k+2} \geq 0$, and therefore p_1^{k+2} is a density function of a probability measure, say μ, on $\mathbb{R}^{k+2}$. By construction, we have

$$\left((\mathcal{F}_{k+2}^{-1} p_{1/n}^{k+2})(\xi)\right)^n = e^{-\psi_{k+2}(\xi)} = e^{-\psi(|\xi|)} = (\mathcal{F}_{k+2}^{-1} p_1^{k+2})(\xi)$$

for all $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^{k+2}$. This shows that $\mu(dx) = p_1^{k+2}(x)\, dx$ is infinitely divisible. Consequently, there exists a $(k+2)$-dimensional Lévy process $(X_t^{k+2})_{t\geq 0}$ such that $X_1^{k+2} \sim \mu$,

$$\mathbb{E} e^{i\xi\cdot X_1^{k+2}} = e^{-\psi_{k+2}(\xi)}.$$

(ii)⇔(iii): Since X_t^k is absolutely continuous – due to the growth condition (HW) –, this follows directly from the definition of a unimodal isotropic distribution. □

Remark 4.2. Proposition 4.1 shows that we need an additional assumption on the growth behaviour of the density p_t^k of the (k-dimensional) Lévy process to ensure the existence of a Lévy process in dimension $k+2$. This assumption is not needed to construct Lévy processes in lower dimensions. *Indeed:* Let $(X_t^k)_{t\geq 0}$ be a k-dimensional Lévy process with rotationally invariant characteristic exponent $\psi_k(\xi) = \psi(|\xi|)$, and fix $d \leq k-1$. Denote by

$$\pi_d : \mathbb{R}^k \to \mathbb{R}^d, x = (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_d)$$

the projection onto the first d coordinates. Since

$$\mathbb{E}e^{i\xi\cdot\pi_d(X_t^k)} = \mathbb{E}e^{i\tilde{\xi}\cdot X_t^k} = e^{-t\psi(|\tilde{\xi}|)} = e^{-t\psi(|\xi|)}, \quad \xi \in \mathbb{R}^d,$$

for $\tilde{\xi} := (\xi, 0, \dots, 0) \in \mathbb{R}^k$, it is not difficult to see that $X_t^d := \pi_d(X_t^k)$ defines a d-dimensional Lévy process with characteristic exponent $\psi_d(\xi) = \psi(|\xi|)$, $\xi \in \mathbb{R}^d$.

Proposition 4.1 can be used to derive gradient estimates for the semigroup; they are not needed for the proof of Schoenberg's theorem but are of independent interest.

Corollary 4.3. *Let $(X_t)_{t\geq 0}$ be a one-dimensional Lévy process with characteristic exponent $\psi(\xi) = f(|\xi|^2)$ for some Bernstein function f. If f satisfies the Hartman–Wintner condition, then the semigroup $P_t u(x) := \mathbb{E}u(x + X_t) = \int u(x+y)p_t(y)\,dy$ satisfies the gradient estimate*

$$\left|\frac{d}{dx}P_t u(x)\right| \leq 4\|u\|_\infty \|p_t\|_\infty \tag{12}$$

for all bounded Borel measurable functions $u : \mathbb{R} \to \mathbb{R}$.

Remark 4.4. (i) Using a convolution argument, it is not difficult to extend Corollary 4.3 to Lévy processes whose characteristic exponent ψ satisfies $\psi(\xi) \geq cf(|\xi|^2)$ for some Bernstein function f and $c > 0$; see [2, Lemma 3.1].

(ii) Clearly, $\|p_t\|_\infty \leq \int_{\mathbb{R}} e^{-t\psi(\xi)}\,d\xi$; we can estimate the integral if we have additional information on the growth of f, see e.g. [25].

(iii) In [14, Lemma 4.5], (10) is used to obtain gradient estimates in terms of moments.

(iv) It is possible to iterate (12) to derive estimates for derivatives of higher order, cf. [14, Lemma 4.1] for details.

Proof of Corollary 4.3. It follows from Proposition 4.1 that $(X_t)_{t\geq 0}$ is unimodal isotropic, and therefore the density p_t (which exists because of (HW)) is unimodal. Using exactly the same reasoning as in [2, proof of Theorem 3.4] we conclude that

$$\int_{\mathbb{R}} |p_t(x+y) - p_t(x)|\,dx \leq 4|y|\,\|p_t\|_\infty, \quad t > 0.$$

Applying Fatou's lemma we get

$$\int_{\mathbb{R}} |p_t'(x)|\,dx \leq \liminf_{|y|\to 0} \int_{\mathbb{R}} \left|\frac{p_t(x+y) - p_t(x)}{y}\right|\,dx \leq 4\|p_t\|_\infty.$$

Since

$$\mathbb{E}u(x + X_t) = \int_{\mathbb{R}} u(x+y)p_t(y)\,dy = \int_{\mathbb{R}} u(y)p_t(y-x)\,dy,$$

a variant of the differentiation lemma for parameter-dependent integrals, cf. [14, Proposition A.1] or [23, Problem 14.20], yields

$$\frac{d}{dx}\mathbb{E}u(x+X_t) = -\int_{\mathbb{R}} u(y)p_t'(y-x)\,dy.$$

Note that the differentiation lemma is indeed applicable since the map

$$x \mapsto \int_{\mathbb{R}} u(y) p_t'(y-x)\, dy$$

is continuous: it is the convolution of a Lebesgue-integrable function with a bounded function, see e.g. [23, Theorem 15.8(ii)]. Hence,

$$\left| \frac{d}{dx} \mathbb{E} u(x + X_t) \right| \leq \|u\|_\infty \int_{\mathbb{R}} |p_t'(y-x)|\, dy \leq 4\|u\|_\infty \|p_t\|_\infty. \qquad \square$$

We are now ready to prove the first part of Schoenberg's theorem.

Proof of Theorem 3.1. We will prove the equivalence of (i), (ii) and (iii).

The direction (i)⇒(ii) is clear. For (ii)⇒(iii) we assume first that f satisfies the Hartman–Wintner condition, i.e. $\lim_{r\to\infty} f(r)/\log r = \infty$. By assumption, $\psi_k(\xi) := \psi(|\xi|) := f(|\xi|^2)$, $\xi \in \mathbb{R}^k$, is a continuous negative definite function for $k = 1+2n$, $n \in \mathbb{N}_0$, satisfying (HW). In particular, there exists a k-dimensional Lévy process $(X_t^k)_{t\geq 0}$ with characteristic exponent ψ_k. Because of (HW), X_t^k has a density p_t^k with respect to Lebesgue measure. In particular, (i) in Proposition 4.1 holds for any $k = 1+2n$. If we set $g_t^k(r) := p_t^k(2\sqrt{r})$, then by (10),

$$\frac{d}{dr} g_t^k(r) = \frac{1}{\sqrt{r}} \frac{d}{ds} p_t^k(s) \bigg|_{s=2\sqrt{r}} \overset{(10)}{=} \frac{1}{\sqrt{r}} (-2\pi s p_t^{k+2}(s)) \bigg|_{s=2\sqrt{r}} = -4\pi g_t^{k+2}(r).$$

Iterating this procedure, we obtain

$$\frac{d^n}{dr^n} g_t^1(r) = (-4\pi)^n g_t^{1+2n}(r) \quad \text{for all} \quad n \in \mathbb{N}, r > 0.$$

As $g_t^{1+2n} \geq 0$ for $n \in \mathbb{N}_0$, this proves that g_t^1 is completely monotone, i.e. there exists a finite measure μ_t on $(0,\infty)$ such that

$$g_t^1(r) = \int_{(0,\infty)} e^{-rs}\, \mu_t(ds), \quad r \geq 0.$$

Applying Fubini's theorem we find

$$\begin{aligned} e^{-tf(r^2)} = \int_{\mathbb{R}} e^{ixr} p_t^1(|x|)\, dx &= \int_{\mathbb{R}} e^{ixr} \left(\int_{(0,\infty)} e^{-s|x|^2/4} \mu_t(ds) \right) dx \\ &= \int_{(0,\infty)} \int_{\mathbb{R}} e^{ixr} e^{-s|x|^2/4}\, dx\, \mu_t(ds) \\ &= \sqrt{4\pi} \int_{(0,\infty)} \frac{1}{\sqrt{s}} e^{-r^2/s}\, \mu_t(ds). \end{aligned} \tag{13}$$

This identity shows that $r \mapsto e^{-tf(r)}$ is the Laplace transform of a finite measure, hence, completely monotone. In view of (1) this implies that

$$-\frac{1}{t} \frac{d}{dr} e^{-tf(r)} = f'(r) e^{-tf(r)}$$

is completely monotone. Letting $t \to 0$ we conclude that f' is completely monotone, and so f is a Bernstein function; see [24, Theorem 3.7] for an alternative proof that $e^{-tf(\cdot)} \in \mathcal{CM}$ implies that f is a Bernstein function.

If f does not satisfy the Hartman–Wintner condition, we set

$$f_\epsilon(r) := f(r) + \epsilon r.$$

Note that $\mathbb{R}^k \ni \xi \mapsto f_\epsilon(|\xi|^2)$ is a continuous negative definite function for any $k = 1+2n$, $n \in \mathbb{N}_0$ and $\epsilon > 0$. As $f \geq 0$, f_ϵ obviously satisfies the Hartman–Wintner condition. The first part of this proof shows that f_ϵ is a Bernstein function. Consequently, f is a Bernstein function as the pointwise limit of Bernstein functions, cf. [24, Corollary 3.8(ii)].

(iii)⇒(i): Since f is a Bernstein function, there exists a subordinator $(S_t)_{t\geq 0}$ with Laplace transform e^{-tf}. If $(B_t)_{t\geq 0}$ is a k-dimensional Brownian motion, then $X_t := B_{S_t}$ is a k-dimensional Lévy process with characteristic exponent $f(|\xi|^2)$. This implies that $\mathbb{R}^k \ni \xi \mapsto f(|\xi|^2)$ is a continuous and negative definite function. □

It remains to prove the equivalence of (iii) and (iv) in Theorem 3.1. To this end, we recall a result on the distribution of subordinated Brownian motion.

Lemma 4.5. *Let $(B_t)_{t\geq 0}$ be a k-dimensional Brownian motion and $(S_t)_{t\geq 0}$ a subordinator with Laplace exponent f. Then the distribution of $X_t := B_{S_t}$ equals*

$$\mathbb{P}(X_t \in B) = \mathbb{P}(S_t = 0)\delta_0(B) + \int_B p_t(x)\,dx, \quad B \in \mathcal{B}(\mathbb{R}^k),\ t > 0;$$

here $p_t : \mathbb{R}^k \to [0,\infty)$ is a rotationally invariant function and $p_t(\sqrt{\cdot}) \in \mathcal{CM}$.

Proof. Since $(S_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are independent, we have

$$\begin{aligned}
\mathbb{P}(X_t \in B) &= \int_{[0,\infty)} \mathbb{P}(B_s \in B)\,\mathbb{P}(S_t \in ds) \\
&= \delta_0(B)\mathbb{P}(S_t = 0) + \int_{(0,\infty)} \left(\int_B \frac{1}{(2\pi s)^{k/2}} \exp\left(-\frac{|y|^2}{2s}\right) dy \right) \mathbb{P}(S_t \in ds) \\
&= \delta_0(B)\mathbb{P}(S_t = 0) + \int_B p_t(y)\,dy
\end{aligned}$$

for

$$p_t(y) := \int_{(0,\infty)} \frac{1}{(2\pi s)^{k/2}} \exp\left(-\frac{|y|^2}{2s}\right) \mathbb{P}(S_t \in ds). \tag{14}$$

The fact that $p_t(\sqrt{\cdot})$ is completely monotone follows directly from this representation and the differentiation lemma for parameter-dependent integrals. □

Proof of Theorem 3.1, equivalence of (iii) *&* (iv). (iii)⇒(iv): As f is a Bernstein function, there exists a subordinator $(S_t)_{t\geq 0}$ with Laplace exponent f. The subordinated Brownian motion $X_t := B_{S_t}$ is a Lévy process with characteristic exponent $\psi(\xi) = f(|\xi|^2)$ and, by Lemma 4.5,

$$\mathbb{P}(X_t \in B) = \mathbb{P}(S_t = 0)\delta_0(B) + \int_B p_t(x)\,dx, \quad B \in \mathcal{B}(\mathbb{R}^k),$$

for the rotationally invariant non-negative function p_t defined in (14) satisfying $p_t(\sqrt{\cdot}) \in \mathcal{CM}$. By the Markov property of $(S_t)_{t\geq 0}$, we have for any $s \leq t$

$$\mathbb{P}(S_t = 0) = \mathbb{E}\left[\mathbb{P}(z + S_{t-s} = 0)\big|_{z=S_s}\right] = \mathbb{E}\left[\mathbf{1}_{\{S_s=0\}}\mathbb{P}(S_{t-s} = 0)\right]$$

as $(S_t)_{t\geq 0}$ has non-decreasing sample paths. Consequently, $f(t) := \mathbb{P}(S_t = 0)$ satisfies $f(t+s) = f(t)f(s)$. Since f is right-continuous, this implies, by the Cauchy–Abel functional equation, $\mathbb{P}(S_t = 0) = f(t) = e^{-ct}$ for some $c \in [0, \infty]$, see e.g. [11, Theorem A.1] for a proof.

For the converse direction (iv)⇒(iii), we can argue exactly as in the proof of (13) in the step "(ii)⇒(iii)." □

5. Proof of Corollary 3.2, Theorem 3.3 & Corollary 3.4

Proof of Corollary 3.2. (i) The proof of Theorem 3.1 shows that p_t^k is given by (14), i.e.

$$p_t^k(r) = \int_{(0,\infty)} \frac{1}{(2\pi s)^{k/2}} \exp\left(-\frac{r^2}{2s}\right) \mathbb{P}(S_t \in ds).$$

If we differentiate p_t^k with respect to r, then we obtain

$$\frac{d}{dr}p_t^k(r) = -r\int_{(0,\infty)} \frac{1}{s}\frac{1}{(2\pi s)^{k/2}} \exp\left(-\frac{r^2}{2s}\right) \mathbb{P}(S_t \in ds) = -2\pi r p_t^{k+2}(r).$$

(ii) Since the Lévy measure $\nu_k(dy)$ is the vague limit of $t^{-1}p_t^k(y)\,dy$, cf. (11), the assertion follows formally from (i) by dividing both sides by t^{-1} and letting $t \to 0$. For a rigorous argument we note that the density m_k of the Lévy measure ν_k is given by

$$m_k(r) = \int_{(0,\infty)} \frac{1}{(2\pi s)^{k/2}} \exp\left(-\frac{r^2}{2s}\right) \mu(ds)$$

where μ denotes the Lévy measure of the subordinator $(S_t)_{t\geq 0}$, cf. [22, Theorem 30.1]. Now the claim follows using exactly the same calculation as in (i). □

Proof of Theorem 3.3. For the existence of the process $(X_t^{k-2})_{t\geq 0}$ see Remark 4.2. Since both A_k and A_{k-2} are pseudo-differential operators with rotationally invariant symbols, cf. (6), it is obvious that $A_k u$ and $A_{k-2}u$ are rotationally invariant for any smooth rotationally invariant u with compact support. By (5) and (6), we have

$$\begin{aligned} A_k u(r) = -\mathcal{F}_k^{-1}(\psi \cdot \mathcal{F}_k u)(r) &= 2\pi \frac{1}{r}\frac{d}{dr}\mathcal{F}_{k-2}^{-1}(\psi \cdot \mathcal{F}_k u)(r) \\ &= -2\pi \frac{1}{r}\frac{d}{dr}A_{k-2}(\mathcal{F}_{k-2}^{-1}\mathcal{F}_k u)(r). \end{aligned}$$

If we can show that

$$-2\pi\mathcal{F}_{k-2}^{-1}(\mathcal{F}_k u)(r) = \int_0^r s \cdot u(s)\,ds - C =: v(r), \quad r > 0,$$

where $C := \int_0^\infty su(s)\,ds$, then the claim follows; note that $A_{k-2}(C\mathbf{1}_{\mathbb{R}^{k-2}}) = 0$ by the very definition of the generator. Applying (4), we find

$$\begin{aligned}\mathcal{F}_k u(r) &= \frac{1}{(2\pi)^{k/2} r^{k/2-1}} \int_0^\infty u(s) s^{k/2} J_{k/2-1}(sr)\,ds \\ &= \frac{1}{(2\pi)^{k/2} r^{k/2-1}} \int_0^\infty \left(\frac{d}{ds} v(s)\right) s^{k/2-1} J_{k/2-1}(sr)\,ds.\end{aligned}$$

Since v has compact support, the integration by parts formula yields

$$\mathcal{F}_k u(r) = -\frac{1}{(2\pi)^{k/2} r^{k/2-1}} \int_0^\infty v(s) s^{k/2-2} \left(\left[\frac{d}{2} - 1\right] J_{k/2-1}(sr) + s\frac{d}{ds} J_{k/2-1}(sr)\right) ds.$$

As

$$\frac{d}{dz} J_{k/2-1}(z) = -\left(\frac{d}{2} - 1\right) \frac{1}{z} J_{k/2-1}(z) + J_{k/2-2}(z),$$

cf. [20, (10.6.2)], we get

$$\mathcal{F}_k u(r) = -\frac{1}{(2\pi)^{k/2} r^{(k-2)/2-1}} \int_0^\infty v(s) s^{(k-2)/2} J_{(k-2)/2-1}(sr)\,du = -\frac{1}{2\pi} \mathcal{F}_{k-2} v(r).$$

Consequently,

$$\mathcal{F}_{k-2}^{-1}(\mathcal{F}_k u)(r) = -\frac{1}{2\pi} v(r).$$

□

Proof of Corollary 3.4. (i) By (14) and the monotone convergence theorem, we have

$$\lim_{r\to 0} p_t(r) = \int_{(0,\infty)} \frac{1}{(2\pi s)^{k/2}}\,\mathbb{P}(S_t \in ds) = \frac{1}{(2\pi)^{k/2}} \mathbb{E}(S_t^{-k/2} \mathbf{1}_{\{S_t \neq 0\}}).$$

Moreover,

$$\mathbb{P}(S_t = 0) = \mathbb{P}(X_t = 0) = e^{-ct}.$$

From this the equivalence "(a)⇔(b)" follows easily. In order to prove "(b)⇔(c)" we use the following elementary identity

$$\frac{1}{y^\kappa} = \frac{1}{\Gamma(\kappa)} \int_0^\infty e^{-ry} r^{\kappa-1}\,dr, \quad y \geq 0,\ \kappa > 0$$

which entails

$$\mathbb{E}(S_t^{-k/2}) = \frac{1}{\Gamma(k/2)} \int_0^\infty e^{-tf(r)} r^{k/2-1}\,dr.$$

If $\psi(\xi) := f(|\xi|^2)$ satisfies the Hartman–Wintner condition, then obviously $\mathbb{E}(S_t^{-k/2}) < \infty$ for all $t > 0$. Conversely, suppose that $\mathbb{E}(S_t^{-k/2}) < \infty$ for sufficiently small $t > 0$. Introducing polar coordinates, we get

$$\int_1^\infty e^{-tf(r)} r^{-1/2}\,dr \leq \int_0^\infty e^{-tf(r)} r^{k/2-1}\,dr < \infty,$$

and so

$$\infty > \int_1^\infty e^{-tf(r)} r^{-1/2}\,dr = 4\int_1^\infty e^{-tf(s^4)} s\,ds.$$

Since $s \mapsto e^{-tf(s^4)}$ is a continuous function this implies $\lim_{s\to\infty} e^{-tf(s^4)}s = 0$. This, in turn, gives

$$\infty = -\lim_{s\to\infty} \log\left(e^{-tf(s^4)}s\right) = \lim_{s\to\infty}(tf(s^4) - \log s).$$

Consequently,

$$\frac{f(s^4)}{\log(s^2)} \geq \frac{1}{t}\frac{\log s}{2\log s} = \frac{1}{2t}$$

for $s > 0$ sufficiently large and $t > 0$ small. Letting $t \to 0$ proves $\lim_{s\to\infty}\frac{f(s^4)}{\log(s^2)} = \infty$, and this implies readily the assertion.

(ii) The direction "$\Leftarrow$" follows directly from the definition of a compound Poisson process. To prove "$\Rightarrow$", we define a stopping time $\tau := \inf\{t > 0; X_t \neq 0\}$. By the strong Markov property,

$$\mathbb{P}(X_t = 0, \tau < t) = \mathbb{E}\left[\mathbf{1}_{\{\tau<t\}}\mathbb{P}(z + X_{t-\tau} = 0)\big|_{z=X_\tau}\right] = 0$$

as $\mathbb{P}(x + X_s = 0) = 0$ for all $s > 0$ and $x \neq 0$, cf. (9). This shows

$$\mathbb{P}(X_t = 0) = \mathbb{P}(X_t = 0, \tau \geq t) = \mathbb{P}\left(\sup_{s\leq t}|X_s| = 0\right).$$

On the other hand, we have

$$\mathbb{P}\left(\sup_{s\leq t}|X_s| = 0\right) \leq \mathbb{P}(N_t(\mathbb{R}^k\setminus\{0\}) = 0) = e^{-t\nu(\mathbb{R}^k\setminus\{0\})};$$

here N denotes the jump measure (8) and ν the Lévy measure of $(X_t)_{t\geq 0}$. Combining both considerations and using that, by assumption and (9), $\mathbb{P}(X_t = 0) = e^{-ct}$ for some $c \in [0,\infty)$, we get $\nu(\mathbb{R}^k\setminus\{0\}) = c < \infty$. This proves that $(X_t)_{t\geq 0}$ is a compound Poisson process. □

For an alternative proof of Corollary 3.4(ii) see [22, Theorem 27.4].

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