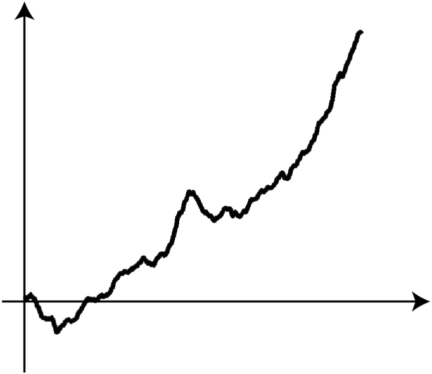
Introduction to Lévy-type processes I

Franziska Kühn
(Technische Universität Dresden)

Workshop “Nonlocal Operators and Markov Processes I”
Wroclaw-Dresden 2020
October 26, 2020

Getting started

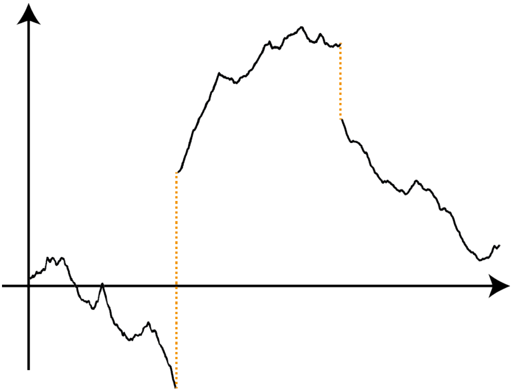


Brownian motion with drift:

$$X_t = bt + \sigma B_t,$$

with $b \in \mathbb{R}$ and $\sigma \geq 0$.

Getting started: Lévy processes



Lévy process:

$$X_t = bt + \sigma B_t + J_t,$$

where J_t is a jump process with stationary and independent increments.

Getting started: Lévy processes

The jumps $\Delta X_t := X_t - X_{t-}$ of a Lévy process can be uniquely characterized by a measure ν on $\mathbb{R}^d \backslash \{0\}$. Idea:

$$\mathbb{E}(\#\{s \in (0,t] : \Delta X_s \in A\}) = t\nu(A).$$

Note:

$$\nu(A) = 0 \implies \text{no jumps of height } A \text{ occur, i.e. } \Delta X_s \notin A \text{ for any } s$$

Consequence: any Lévy process can be characterized by a triplet (b, σ^2, ν), the so-called Lévy triplet.

Getting started: Lévy processes

Because of the independence and stationarity of the increments, Lévy processes are homogeneous in space. In particular, its jumping behaviour does not depend on its current position, i.e.

$$\text{likelihood to jump from 'old position' } X_{t-} \text{ to 'new position' } X_t = X_{t-} + y$$

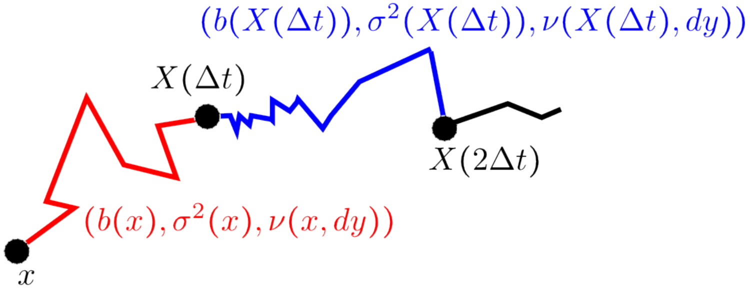
does not depend on t or X_{t-}.

$\rightsquigarrow$ generalization: Lévy-type processes

Getting started: From Lévy to Lévy-type processes

Idea: Lévy-type processes behave locally like Lévy processes but the triplet may depend on the current position of the process,

$$\text{Lévy process} \rightsquigarrow \text{Lévy-type process}$$
$$\text{triplet } (b, \sigma^2, \nu) \rightsquigarrow \text{family of triplets } (b(x), \sigma^2(x), \nu(x, dy)),\ x \in \mathbb{R}^d$$



Mind: this is no rigorous definition/construction.

Plan

Structure of the remaining talk:

1. some basics on Markov processes and semigroups,
2. Feller processes and integro-differential operators,
3. examples of Feller processes.

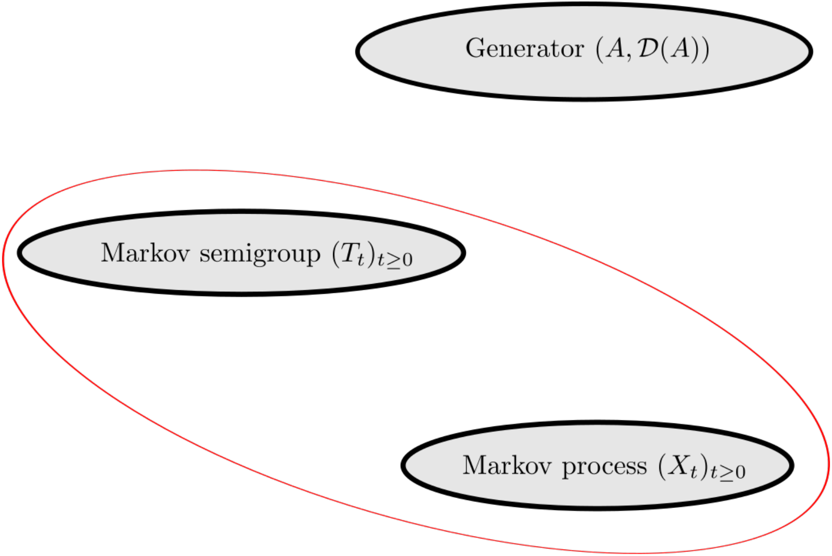
Central objects

Generator $(A, \mathcal{D}(A))$

Markov semigroup $(T_t)_{t\geq 0}$

Markov process $(X_t)_{t\geq 0}$

Central objects



Semigroup

Markov semigroup $(T_t)_{t\geq 0}$

Markov process $(X_t)_{t\geq 0}$

Definition (Markov semigroup)

A family of linear operators $T_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d)$, $t \geq 0$, is a *Markov semigroup* if

1. $T_0 = \mathrm{id}$, $T_{t+s} = T_t T_s$ ("semigroup property"),
2. $T_t f \geq 0$ for all $f \geq 0$, ("positive"),
3. $\|T_t f\|_\infty \leq \|f\|_\infty$ ("contractive").

If $(T_t)_{t\geq 0}$ has the Feller property and is strongly continuous on $C_\infty(\mathbb{R}^d)$, then $(T_t)_{t\geq 0}$ is a *Feller semigroup*.

Markov process

$$\begin{tikzcd} \text{Markov semigroup } (T_t)_{t\geq 0} \arrow[r, shift right, "\text{Feller}"'] & \text{Markov process } (X_t)_{t\geq 0} \arrow[l, shift right] \end{tikzcd}$$

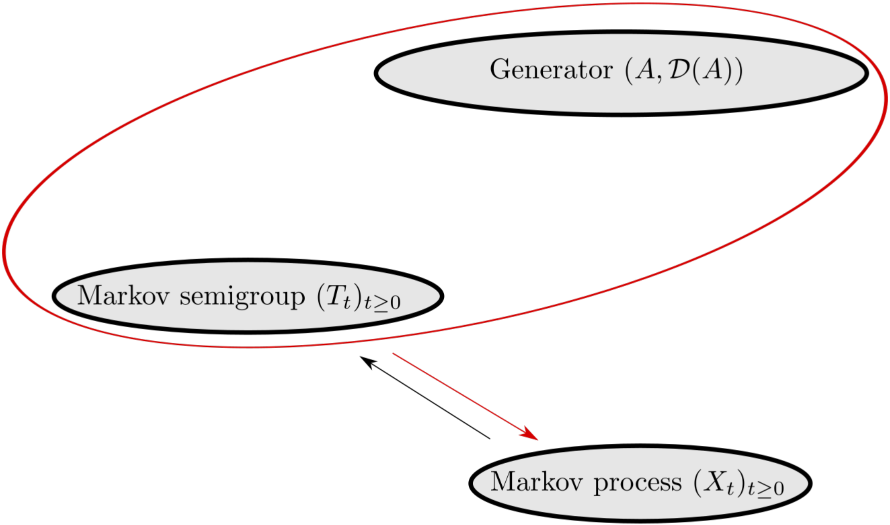
Lemma

If $(X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$ is a Markov process, then $T_t f(x) = \mathbb{E}^x f(X_t)$ defines a Markov semigroup.

Question

Is the converse true, i.e. is possible to construct for every Markov semigroup $(T_t)_{t\geq 0}$ a Markov process $(X_t)_{t\geq 0}$ with $T_t f(x) = \mathbb{E}^x f(X_t)$?

Answer: Yes, if $(T_t)_{t\geq 0}$ is a Feller semigroup.


Generator $(A, \mathcal{D}(A))$
Markov semigroup $(T_t)_{t\geq 0}$
Markov process $(X_t)_{t\geq 0}$

Infinitesimal generator

Idea: Generator A is the derivative of T_t with respect to t

Background:

$$T_t T_s = T_{t+s} \qquad T_0 = \mathrm{id}$$

is an "operator version" of the Cauchy–Abel equation

$$u(t)u(s) = u(t+s), \qquad u(0) = 1.$$

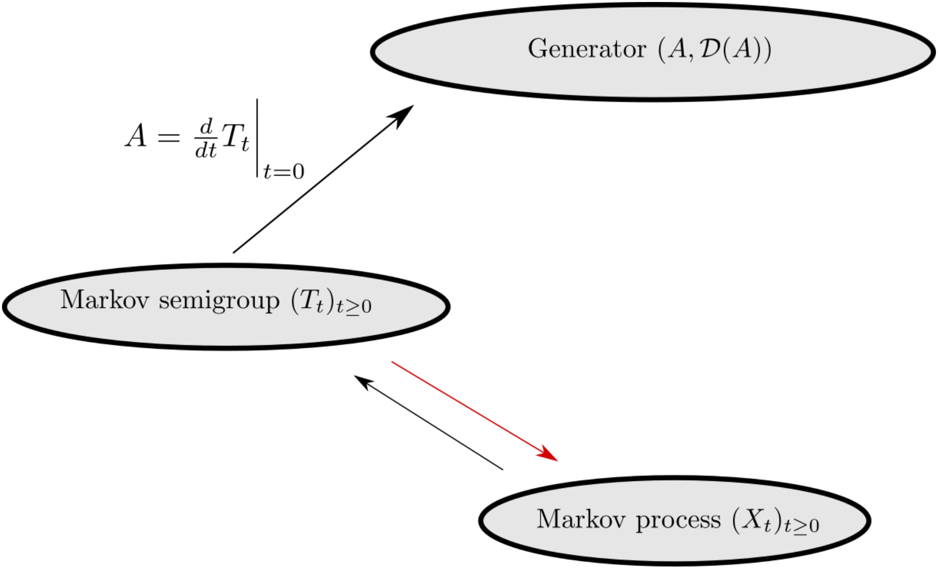
Solution (for u 'nice'):

$$u(t) = e^{ct} \qquad \text{with } c = \frac{d}{dt}u(t)\Big|_{t=0} = \lim_{t\to 0}\frac{u(t)-1}{t}.$$

Operator valued:

$$T_t = e^{tA}$$

$$\implies Af = \frac{d}{dt}T_t f\Big|_{t=0} = \lim_{t\to 0}\frac{T_t f - f}{t}$$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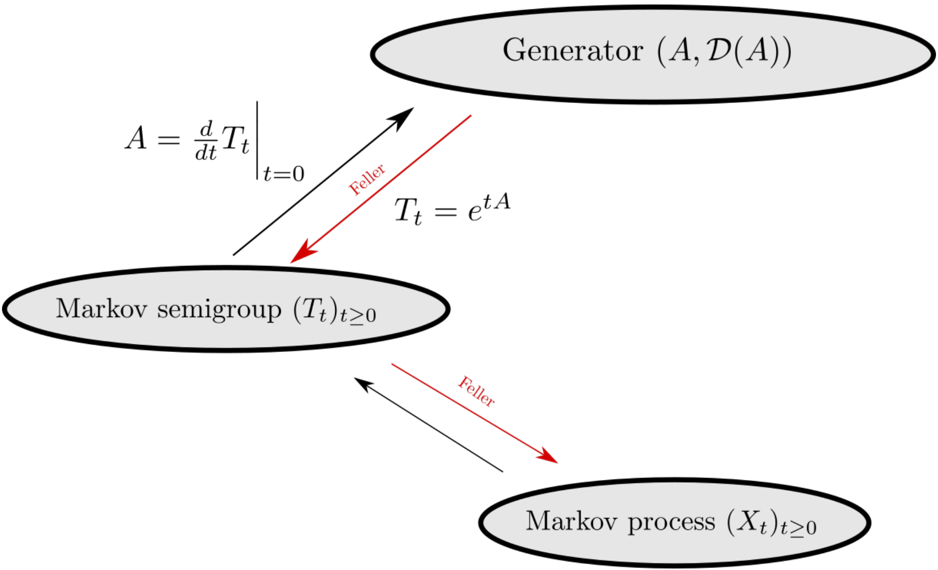
Generator $(A, \mathcal{D}(A))$
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Feller
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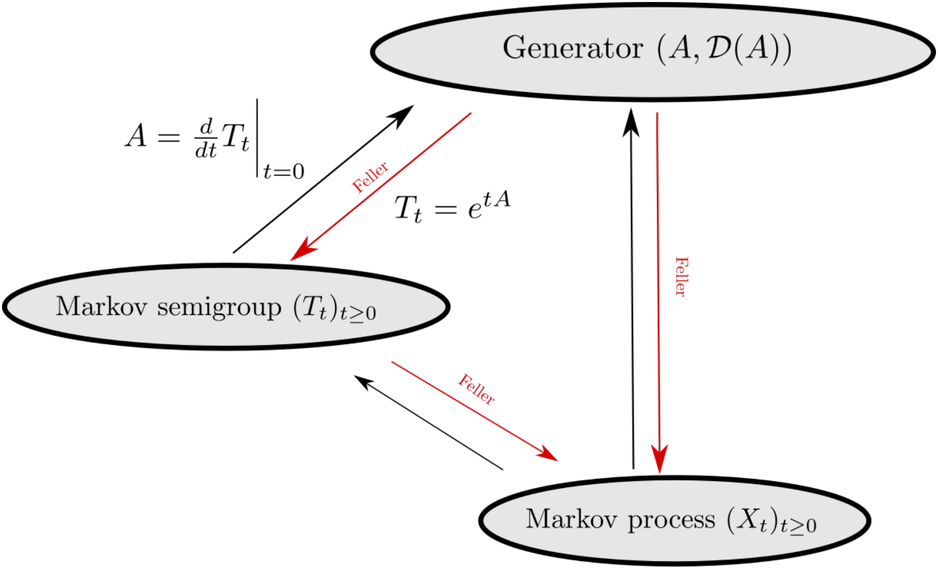
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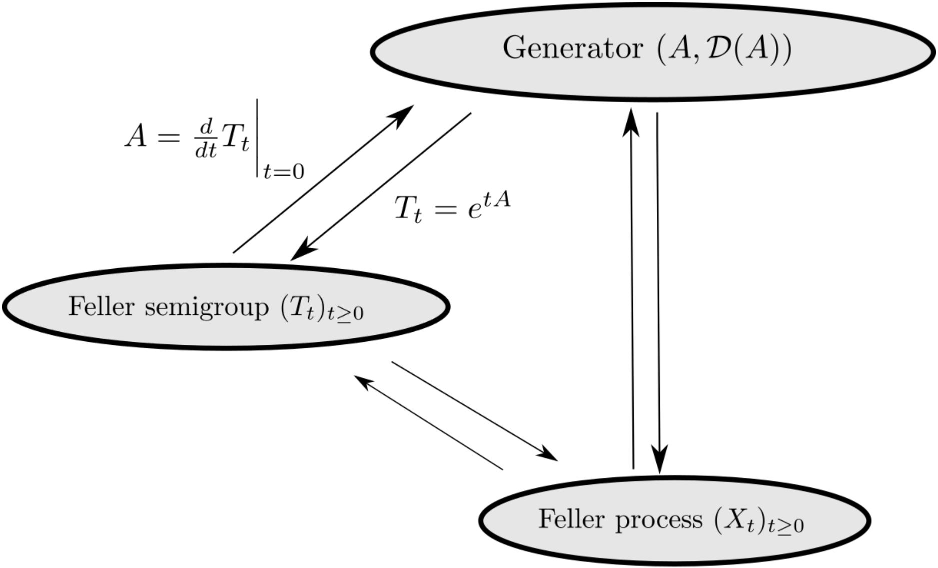
Generator $(A, \mathcal{D}(A))$
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Generator of a Feller semigroup

Theorem (Courrège-von Waldenfels)

If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ then

$$\begin{aligned} Af(x) &= b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\cdot\nabla^2 f(x)) \\ &\quad + \int_{y\neq 0} \big(f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{(0,1)}(|y|)\big)\, \nu(x,dy) \end{aligned}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$; here $(b(x), Q(x), \nu(x,dy))$ is a Lévy triplet for each $x \in \mathbb{R}^d$.

Alternative representation as a pseudo-differential operator:

$$Af(x) = -q(x,D)f(x) := -\int_{\mathbb{R}^d} q(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$$

with negative definite *symbol*

$$q(x,\xi) = -ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{y\neq 0} (1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{(0,1)}(|y|))\, \nu(x,dy)$$

Generator of a Feller semigroup

Key for the proof: The Feller generator $(A, \mathcal{D}(A))$ satisfies the *positive maximum principle*, i.e.

$$f \in \mathcal{D}(A) \subseteq C_\infty(\mathbb{R}^d),\ f(x_0) = \sup_{x \in \mathbb{R}^d} f(x) \implies Af(x_0) \leq 0.$$

This follows from

$$Af(x_0) = \lim_{t \to 0} \frac{T_t f(x_0) - f(x_0)}{t}.$$

Generator of a Feller semigroup

Corollary

Let $(X_t)_{t\geq 0}$ be a Feller process with generator $(A,\mathcal{D}(A))$. If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ is a core for $(A,\mathcal{D}(A))$, then $(X_t)_{t\geq 0}$ is uniquely determined by

- *its symbol $q(x,\xi)$, $x,\xi \in \mathbb{R}^d$,*
- *by its characteristics $(b(x),Q(x),\nu(x,dy))$, $x \in \mathbb{R}^d$.*

The idea to study and characterize Feller processes using the symbol $q(x,\xi)$ goes back to W. Hoh, N. Jacob and R. Schilling.

How to calculate the symbol in practice?

$$
\begin{aligned}
-q(x,\xi) &= \lim_{t\to 0} \frac{1}{t}\left(\mathbb{E}^x e^{i\xi(X_t - x)} - 1\right) \\
&= \frac{d}{dt}\mathbb{E}^x e^{i\xi(X_t-x)}\bigg|_{t=0} \\
&= e^{-i\xi x}\left(Ae^{i\xi\bullet}\right)(x)
\end{aligned}
$$

Mind: In general,

$$
\mathbb{E}^x\left(e^{i\xi(X_t-x)}\right) \neq e^{-tq(x,\xi)}.
$$

Generator of a Feller semigroup

- The symbol can be used to characterize many distributional properties and sample path properties of a Feller process ($\rightsquigarrow$ tomorrows talk).
- Thanks to the result by Courrège & von Waldenfels, we can use probabilistic methods to study properties of a large class of integro-differential operators.

Kolmogorov equation

Theorem

Let $(X_t)_{t\geq 0}$ be a Feller process with generator $(A, \mathcal{D}(A))$. If $f \in \mathcal{D}(A)$, then

$$u(t,x) := T_t f(x) := \mathbb{E}^x f(X_t), \qquad t \geq 0,\ x \in \mathbb{R}^d,$$

solves the Kolmogorov equation

$$\frac{\partial}{\partial t} u(t,x) - A_x u(t,x) = 0 \qquad u(0,x) = f(x). \tag{$\star$}$$

Possible approach for constructing a solution to $(\star)$ for a given Lévy-type operator A:

1. Construct a Feller process $(X_t)_{t\geq 0}$ such that $Af(x) = \frac{d}{dt}\mathbb{E}^x f(X_t)|_{t=0}$
2. Study the domain $\mathcal{D}(A)$ of the generator. Hope: $\mathcal{D}(A)$ is 'large'.

More about this in tomorrow's talk.

Examples

Next: examples of Feller processes.

- Lévy processes,
- processes of variable order,
- solutions to Lévy-driven SDEs

Examples: Lévy processes as Feller processes

Every Lévy process is a Feller process. Semigroup:

$$T_t f(x) = \mathbb{E} f(x + X_t).$$

Note: $(T_t)_{t \geq 0}$ is translation invariant, i.e.

$$(T_t f)(x) = (T_t f(x + \cdot))(0).$$

In fact:

Theorem

A càdlàg process $(X_t)_{t \geq 0}$ is a Lévy process if, and only if, it is a Feller process with translation invariant, conservative semigroup $(T_t)_{t \geq 0}$.

Because of the translation invariance, the symbol and the characteristics of a Lévy process do not depend on x.

Generator of a Lévy process

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2}\operatorname{tr}(Q \cdot \nabla^2 f(x))$$
$$+ \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|))\, \nu(dy)$$

for $f \in C_c^\infty(\mathbb{R}^d)$, where (b, Q, ν) is the *Lévy triplet*. Equivalently:

$$Af(x) = -\int_{\mathbb{R}^d} \psi(\xi) e^{ix \cdot \xi} \hat{f}(\xi)\, d\xi,$$

where

$$\psi(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot Q\xi + \int_{y \neq 0} (1 - e^{iy \cdot \xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|))\, \nu(dy)$$

is the *characteristic exponent*.

Examples: Lévy processes

Brownian motion with drift

- triplet: $b \in \mathbb{R}^d$, $Q = \mathrm{id}$, $\nu = 0$
- characteristic exponent: $\psi(\xi) = -ib \cdot \xi + \frac{1}{2}|\xi|^2$
- generator: $Af(x) = b \cdot \nabla f(x) + \frac{1}{2}\Delta f(x)$

Poisson process with intensity $\lambda \in (0, \infty)$

- triplet: $b = 0$, $Q = 0$, $\nu = \lambda\delta_1$,
- characteristic exponent: $\psi(\xi) = \lambda(1 - e^{i\xi})$,
- generator: $Af(x) = \lambda(f(x+1) - f(x))$

Examples: Lévy processes

isotropic α-stable Lévy process with index of stability $\alpha \in (0,2)$:

- triplet: $b = 0$, $Q = 0, \nu(dy) = c_{d,\alpha} \frac{1}{|y|^{d+\alpha}}\, dy$
- characteristic exponent: $\psi(\xi) = |\xi|^\alpha$
- generator: fractional Laplacian

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &:= c_{d,\alpha} \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \frac{1}{|y|^{d+\alpha}}\, dy, \end{aligned}$$

cf. Kwaśnicki [9] for detailed discussion of further definitions

Examples: process of variable order

Intuition: the process behaves locally like an isotropic stable Lévy process but the index α depends on the current position of the process, i.e.

$$q(x,\xi) = |\xi|^{\alpha(x)}, \qquad x,\xi \in \mathbb{R}^d.$$

Question

Under which assumptions exists a Feller process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$?

Answer: $\alpha : \mathbb{R}^d \to (0,2]$ is bounded away from zero and Hölder continuous, cf. [1, 6, 7] (in dimension $d = 1$, Dini continuity is enough [1]).

Generator: 'fractional Laplacian of variable order'

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha(x)/2} f(x) \\ &= c_{d,\alpha(x)} \int_{y\neq 0} \left(f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{(0,1)}(|y|)\right) \frac{1}{|y|^{d+\alpha(x)}}\, dy. \end{aligned}$$

Examples

More generally, one can ask:

Question

Given the characteristic exponent ψ of a Lévy process and a mapping $\alpha : \mathbb{R}^d \to (0,1)$, then under which assumptions defines

$$q(x,\xi) := \psi(\xi)^{\alpha(x)}, \qquad x, \xi \in \mathbb{R}^d,$$

the symbol of a Feller process?

Particular case: $\alpha(x) = \alpha$ constant $\rightsquigarrow$ 'classical' subordination [11]

General case: $\rightsquigarrow$ symbols of variable order / variable order subordination [4, 7]

Examples: Lévy-driven SDE

For a Lévy process $(L_t)_{t\geq 0}$ consider the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_{t-})\,dL_t.$$

Theorem (Schilling, Schnurr [10])

If σ, b are bounded and Lipschitz continuous, then the (unique) solution $(X_t)_{t\geq 0}$ is a Feller process with symbol

$$q(x,\xi) = -ib(x)\cdot\xi + \psi(\sigma(x)^T\xi), \qquad x,\xi \in \mathbb{R}^d,$$

where ψ is the characteristic exponent of $(L_t)_{t\geq 0}$.

The assumptions on b, σ can be relaxed [8].

Lévy-driven SDE which is not Feller

Example (Schilling, Schnurr [10])

Let $(N_t)_{t\geq 0}$ be a Poisson process. Then the solution to the SDE

$$dX_t = -X_{t-}\, dN_t, \qquad X_0 = x,$$

is **not** a Feller process.

Reason: The solution equals x until the first jump of $(N_t)_{t\geq 0}$ occurs; after the first jump it is 0. In particular,

$$\mathbb{P}^x(|X_t| \leq R) \geq \mathbb{P}^x(X_t = 0) \geq \mathbb{P}(N_t \geq 1) > 0, \qquad t > 0,$$

and so

$$\lim_{|x|\to\infty} \mathbb{P}^x(|X_t| \leq R) > 0.$$

This shows that the semigroup of $(X_t)_{t\geq 0}$ does not satisfy the Feller property.

Examples: (Generalized) Ornstein-Uhlenbeck process

Let $L_t = (U_t, V_t)$ be a two-dimensional Lévy process with Lévy triplet (b, Q, ν). The (unique) solution to the SDE

$$dX_t = X_{t-}\, dU_t + dV_t, \qquad X_0 = x,$$

is called *generalized Ornstein–Uhlenbeck process*. Classical OU process: $U_t = bt$ and $V_t = \sigma B_t$.

Theorem (Behme & Lindner; K.)

$(X_t)_{t\geq 0}$ is a Feller process if, and only, if $\nu(\{-1\} \times \mathbb{R}) = 0$. In particular, the classical Ornstein-Uhlenbeck process is a Feller process.

References I

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References II

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[8] Kühn, F.: Solutions of Lévy-driven SDEs with unbounded coefficients as Feller processes. *Proc. Amer. Math. Soc.* **146** (2018), 3591–3604.

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Introduction to Lévy-type processes II

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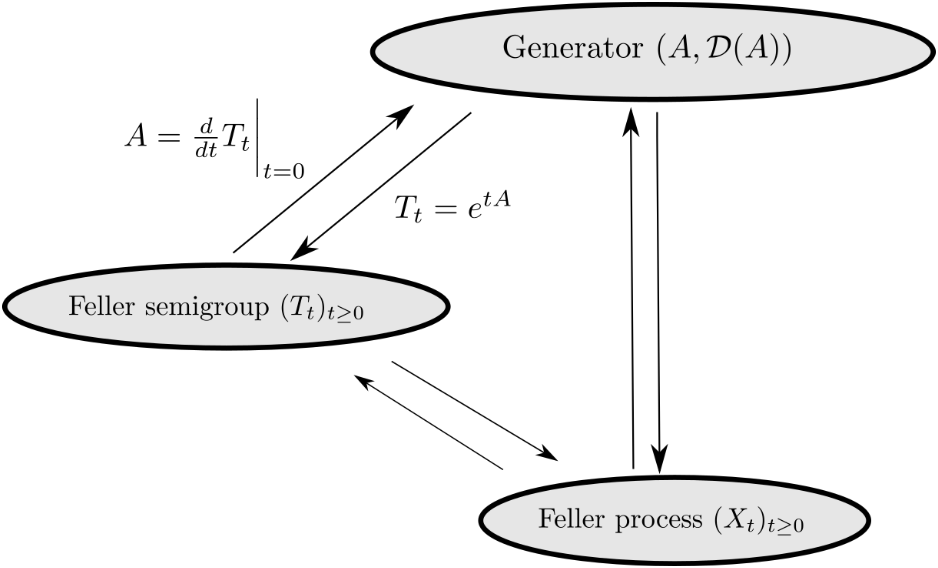
(Technische Universität Dresden)

Workshop “Nonlocal Operators and Markov Processes I”
Wroclaw-Dresden 2020
October 27, 2020

Outline

1. Existence: from the Lévy-type operator to the process
2. Distributional and sample path properties of Feller processes
3. Domains of Lévy and Feller generators

Reminder


Generator $(A, \mathcal{D}(A))$
$A = \frac{d}{dt} T_t \Big|_{t=0}$
$T_t = e^{tA}$
Feller semigroup $(T_t)_{t \geq 0}$
Feller process $(X_t)_{t \geq 0}$

Reminder

Lévy-type operators

$$Af(x) = b(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q(x) \cdot \nabla^2 f(x)) \\ + \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \, \nu(x, dy)$$

...appear as infinitesimal generators of Feller processes. We can write, equivalently, A as a pseudo-differential operator

$$Af(x) = -q(x, D) f(x) := -\int_{\mathbb{R}^d} q(x, \xi) e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi$$

with continuous negative definite symbol

$$q(x, \xi) = -ib(x) \cdot \xi + \frac{1}{2} \xi \cdot Q(x) \xi + \int_{y \neq 0} (1 - e^{iy \cdot \xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|)) \, \nu(x, dy).$$

We know: There is a one-to-one correspondence between Feller processes and Feller semigroups. How to construct Feller processes (or semigroups)?

Aim: Construct Feller process starting from the Lévy-type operator (. . . or the symbol $q(x, \xi)$ . . . or the characteristics $(b(x), Q(x), \nu(x, dy))$).

Existence: from the symbol to the process

Question

Given the symbol q of a Lévy-type operator (or its characteristics), then does there exist a Feller process $(X_t)_{t\geq 0}$ such that

$$Af(x) = \lim_{t\to 0} \frac{\mathbb{E}^x f(X_t) - f(x)}{t} \overset{!}{=} -q(x,D)f(x), \quad x \in \mathbb{R}^d,\ f \in C_c^\infty(\mathbb{R}^d)$$

Answer:

- no, in general.
- yes **if** $q(x,\xi) = q(\xi)$. Reason: There is a 1-1 correspondence between Lévy processes and continuous negative definite functions.

Existence: from the symbol to the process

Find necessary/sufficient conditions for the existence. Aim: The existence result should

- . . . apply to a large class of symbols,
- . . . require only mild regularity of the symbol with respect to the x-variable.

Why is the regularity with respect to x important? Recall: The symbol of the solution to a Lévy-driven SDE

$$dX_t = \sigma(X_{t-})\, dL_t$$

is $q(x, \xi) = \psi(\sigma(x)^T \xi)$. Consequently:

$$\text{regularity of } q(\bullet, \xi) \leftrightsquigarrow \text{regularity of } \sigma(\bullet)$$

Existence: from the symbol to the process

Some possible approaches:

1. via Hille–Yoshida theorem,
2. via Dirichlet forms,
3. via parametrix construction,
4. via martingale problems,
5. via SDEs,
6. via random time changes.

Survey: [2].

Existence results via Hille-Yoshida

Theorem

The closure $(A, \mathcal{D}(A))$ of a linear operator $A : \mathcal{D} \to C_\infty(\mathbb{R}^d)$ is the generator of a Feller semigroup if, and only if,

1. $\mathcal{D} \subseteq C_\infty(\mathbb{R}^d)$ *is dense,*
2. $(A, \mathcal{D})$ *satisfies the positive maximum principle,*
3. $(\lambda - A)(\mathcal{D}) \subseteq C_\infty(\mathbb{R}^d)$ *is dense for some* $\lambda > 0$.

We want to take $A := -q(\cdot, D)$ on $\mathcal{D} := C_c^\infty(\mathbb{R}^d)$. The third condition is hard to check. Need solution, say, $u \in C_c^\infty(\mathbb{R}^d)$ to

$$\lambda u(x) + q(x, D)u(x) = f(x)$$

for $f \in C_\infty(\mathbb{R}^d)$ from a dense subset.

Okay if $Q(\bullet)$ is uniformly elliptic and bounded. Further results using (Hoh's) symbolic calculus typically require a high degree of smoothness of $x \mapsto q(x, \xi)$, see e.g. [8, 9].

Existence results via martingale problems

Observation: If $(X_t)_{t\geq 0}$ is a Feller process with generator A, then

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Af(X_s)\,ds, \qquad t \geq 0,$$

is a martingale for every $f \in \mathcal{D}(A)$.

Plan:

1. Construct $(X_t)_{t\geq 0}$ such that $(M_t^f)_{t\geq 0}$ is a martingale for $Af(x) = -q(x,D)f(x)$, $f \in C_c^\infty(\mathbb{R}^d)$.
2. Show that the solution is a Feller process, e.g. by showing uniqueness of the solution to the martingale problem [13].

Remark: Existence of solutions to the martingale problem is, in general, easy but uniqueness is difficult.

Existence results via parametrix construction

Observation: If $(X_t)_{t\geq 0}$ has a 'nice' transition density $p(t,x,y)$, then p solves the Cauchy problem

$$\frac{\partial}{\partial t}p(t,x,y) - A_x p(t,x,y) = 0.$$

Plan:

1. Construct the fundamental solution to the Cauchy problem for $A = -q(x,D)$ (by freezing coefficients).
2. Show that the solution is the transition density of a Feller process.

Idea traces back to Levi [18], Hadamard [7] and Feller [4], see also Friedman [5]. More: talk by A. Kulik this friday.

Next: use the symbol to study distributional properties and sample path properties of Feller processes.

Important tool: a maximal inequality.

Maximal inequality

Theorem (Schilling '98 [23])

If $(X_t)_{t\geq 0}$ is a Feller process with symbol $q(x,\xi)$, then

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| \geq r\right) \leq ct \sup_{|y-x|\leq r}\sup_{|\xi|\leq r^{-1}}|q(y,\xi)|$$

for all $r>0$ and $t\geq 0$; here $c>0$ is an absolute constant.

Alternative interpretation: estimate on first exit time

$$\mathbb{P}^x(\tau_r^x \leq t) \leq ct \sup_{|y-x|\leq r}\sup_{|\xi|\leq r^{-1}}|q(y,\xi)|.$$

Idea of proof: clever application of Dynkin's formula

$$\mathbb{E}^x f(X_t) - f(x) = \mathbb{E}^x \int_0^t Af(X_s)\,ds.$$

Examples

1. $(L_t)_{t\geq 0}$ Lévy process with char. exponent ψ:

$$\mathbb{P}^x\left(\sup_{s\leq t}|L_s - x| \geq r\right) \leq ct \sup_{|\xi|\leq r^{-1}} |\psi(\xi)|, \qquad t > 0.$$

2. $dX_t = \sigma(X_{t-})\, dL_t$ weak solution of Lévy-driven SDE:

$$\mathbb{P}^x\left(\sup_{s\leq t}|X_s - x| \geq r\right) \leq ct \sup_{|y-x|\leq r} \sup_{|\xi|\leq r^{-1}} |\psi(\sigma(x)^T \xi)|.$$

Non-explosion in finite time

$$\mathbb{P}^x \left(\sup_{s \leq t} |X_s - x| \geq r \right) \leq ct \sup_{|y-x| \leq r} \sup_{|\xi| \leq r^{-1}} |q(y,\xi)|$$

Observation:

$$\mathbb{P}^x \left(\sup_{s \leq t} |X_s - x| \geq r \right) \xrightarrow{r \to \infty} \mathbb{P}^x \left(\text{life time of } X \text{ is} \leq t \right).$$

Consequence: If

$$\lim_{r \to \infty} \sup_{|y-x| \leq r} \sup_{|\xi| \leq r} |q(y,\xi)| = 0,$$

then $(X_t)_{t \geq 0}$ does not explode in finite time. A bit more is true ...

(Non-)Explosion in finite time

Theorem (Wang '11 [25])

Let $(X_t)_{t\geq 0}$ be a Feller process with symbol q which is locally bounded

$$\sup_{|x|\leq r} \sup_{|\xi|\leq 1} |q(x,\xi)| < \infty, \qquad r > 0.$$

If

$$\liminf_{r\to\infty} \sup_{|y-x|\leq r} \sup_{|\xi|\leq r^{-1}} |q(y,\xi)| < \infty,$$

for all $x \in \mathbb{R}^d$, then $(X_t)_{t\geq 0}$ is conservative, i.e. it does not explode in finite time.

There is a generalization for solutions to martingale problems, cf. [14].

Example

Corollary

Consider the Lévy-driven SDE

$$dX_t = \sigma(X_{t-})\, dL_t, \qquad X_0 = x.$$

If σ *satisfies the linear growth condition* $|\sigma(x)| \leq C(1+|x|)$, *then any weak solution to the SDE does not explode in finite time.*

Recall: $q(x,\xi) = \psi(\sigma(x)^T \xi)$ is the symbol of $(X_t)_{t\geq 0}$, where ψ is the characteristic exponent of ψ.

Existence of moments

Question

Given a Feller process $(X_t)_{t\geq 0}$ and a function $f \geq 0$, then under which assumptions is $\mathbb{E}^x f(X_t) < \infty$?

For f submultiplicative,

$$f(x+y) \leq c f(x) f(y), \qquad x, y \in \mathbb{R}^d,$$

there is a sufficient condition in terms of the characteristics $(b(x), Q(x), \nu(x, dy))$.

Ex.: $f(x) = \max\{|x|^\alpha, 1\}$, $f(x) = \exp(|x|^\beta)$, $\beta \in (0,1)$.

Theorem (K.' [12])

Let $(X_t)_{t\geq 0}$ be a Feller process with characteristics $(b(x), Q(x), \nu(x,dy))$ satisfying

$$\sup_{x\in\mathbb{R}^d}\left(|b(x)|+|Q(x)|+\int \min\{1,|y|^2\}\,\nu(x,dy)\right)<\infty.$$

If $f\in C^2$ is a non-negative submultiplicative function, then

$$\sup_{x\in\mathbb{R}^d}\int_{|y|\geq 1} f(y)\,\nu(x,dy)<\infty$$

implies

$$\sup_{s\leq t}\sup_{x\in\mathbb{R}^d}\mathbb{E}^x f(X_t-x)<\infty.$$

Example

For σ bounded and Lipschitz continuous consider

$$dX_t = \sigma(X_{t-})\, dL_t.$$

If the Lévy measure ν of $(L_t)_{t\geq 0}$ satisfies $\int_{|y|\geq 1} |y|^\alpha \, \nu(dy) < \infty$ ($\iff \mathbb{E}(|L_t|^\alpha) < \infty$) for some $\alpha > 0$, then

$$\sup_{s\leq t} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x(|X_t - x|^\alpha) < \infty.$$

Existence of moments: alternative approach

Observation:

$$\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y \geq r)\, dr$$

for any random variable $Y \geq 0$. Thus,

$$\mathbb{E}^x \left(\sup_{s \leq t} |X_s - x|^p \right) = \int_0^\infty \mathbb{P}^x \left(\sup_{s \leq t} |X_s - x| \geq r^{1/p} \right) dr.$$

$\rightsquigarrow$ combine with maximal inequality to get sufficient conditions for the existence of fractional moments in terms of the symbol, cf. [3, 12]

Asymptotic behaviour of sample paths

Theorem (Schilling '98 [23])

Let $(X_t)_{t\geq 0}$ be a Feller process with symbol q such that

$$\sup_{|y-x|\leq r^{-1}} \sup_{|\xi|\leq r} |q(y,\xi)| \leq Cr^{\beta}, \qquad r > 1,$$

for some $\beta \in (0,2]$. Then

$$\limsup_{t\to 0} \frac{1}{t^{1/\lambda}} \sup_{s\leq t} |X_s - x| = 0 \quad \mathbb{P}^x\text{-a.s. for } \lambda > \beta.$$

Idea: Apply the Borel–Cantelli lemma and show that

$$\sum_{k\geq 1} \mathbb{P}^x \left(\sup_{s\leq t_k} |X_s - x| > t_k^{1/\lambda} \right) < \infty \qquad \text{with } t_k = 2^{-k}$$

Further results on sample paths of Feller processes:

- study the p-variation of sample paths of Feller processes [19, 20]
- study the Besov regularity of sample paths of Feller processes [24, 2]
- obtain upper and lower functions for sample paths of Feller processes, see e.g [10, 11]
- recurrence & transience, see e.g. [1, 21, 22],
- Hausdorff dimension of images $X(E)$, $E \subseteq [0, \infty)$ [2]

Domain of a Feller generator

Question

How 'large' is the domain $\mathcal{D}(A)$ of the generator of a Feller process?

Motivation:

- Solve equations of the form

$$\partial_t u(t,x) - A_x u(t,x) = 0 \qquad u(0,x) = f(x)$$

  for a large class of functions f.
- extend Itô's formula by relaxing the regularity assumptions on f.

Example: Generator of Brownian motion

$$Af(x) = \frac{1}{2}\Delta f(x), \qquad f \in C_c^\infty(\mathbb{R}^d).$$

Well-known: $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(A)$.

Question

Is it true that $\mathcal{D}(A) \subseteq C^2(\mathbb{R}^d)$, i.e. is any $f \in \mathcal{D}(A)$ twice differentiable?

Answer: Yes, for $d = 1$; no, in general (counterexample by Günther). However,

$$\mathcal{D}(A) \subseteq \mathcal{C}_b^2(\mathbb{R}^d) \quad \text{Hölder–Zygmund space}$$

Summary:

$$\mathcal{C}_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_b^2(\mathbb{R}^d).$$

$C_b^k(\mathbb{R}^d) \subset \mathcal{C}_b^k(\mathbb{R}^d)$ for $k \in \mathbb{N}$ and $C_b^\alpha(\mathbb{R}^d) = \mathcal{C}_b^\alpha(\mathbb{R}^d)$ for $\alpha \notin \mathbb{N}$

Example: Generator of isotropic stable Lévy process

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x), \qquad f \in C_c^\infty(\mathbb{R}^d) \\ &= c_{d,\alpha} \int_{y\neq 0} (f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{(0,1)}(|y|))|y|^{-d-\alpha}\, dy. \end{aligned}$$

Question

What regularity is needed to ensure $f \in \mathcal{D}(A)$?

Answer: $f \in \mathcal{C}_\infty^\beta(\mathbb{R}^d)$ or $f \in C_\infty^\beta(\mathbb{R}^d)$ for some $\beta > \alpha$.

Question

How regular are the functions $f \in \mathcal{D}(A)$?

Answer: $f \in \mathcal{D}(A) \implies f \in \mathcal{C}_b^\alpha(\mathbb{R}^d)$. Idea: use smoothing property of the semigroup/resolvent

Summary: $$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcup_{\beta>\alpha} \mathcal{C}_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_b^\alpha(\mathbb{R}^d).$$

Domain of the generator of a Lévy process

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2}\operatorname{tr}(Q \cdot \nabla^2 f(x)) + \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|))\, \nu(dy) \tag{1}$$

Inclusions of the form

$$C_\infty^\beta(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_b^\gamma(\mathbb{R}^d)$$

can be obtained, more generally, for Lévy generators. How to find β, γ?

- choice of β: roughly speaking, all expressions in (1) need to be well-defined for $f \in C^\beta$, see e.g. [17]
- choice of γ: (try to) establish a gradient estimate $\int |\nabla p_t(x)|\, dx \leq ct^{-1/\gamma}$, $t \in (0,1)$, for the transition density p_t of the Lévy process [15]

Domain of a Feller generator

$$Af(x) = b(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q(x) \cdot \nabla^2 f(x)) + \int_{y \neq 0} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \, \nu(x, dy)$$

The regularity of $f \in \mathcal{D}(A)$ typically depends on the space variable x, e.g.

$$|f(x+h) - f(x)| \leq C|h|^{\lambda}, \quad |h| \leq 1,$$

for some $\lambda = \lambda(x)$
$\rightsquigarrow$ Use Hölder(–Zygmund) spaces of variable order to describe regularity accurately.

sufficient conditions for $f \in \mathcal{D}(A)$ are discussed in [17]

Processes of variable order

Consider

$$Af(x) = c(x)\int_{y\neq 0}\left(f(x+y)-f(x)-y\nabla f(x)\mathbb{1}_{(0,1)}(|y|)\right)\frac{1}{|y|^{d+\alpha(x)}}\,dy$$
$$= -(-\Delta)^{\alpha(x)/2}f(x)$$

for $\alpha:\mathbb{R}^d\to(0,2)$ Hölder continuous with $\inf_x \alpha(x)>0$.

Theorem (K.& Schilling [17, 16])

1. $\mathcal{C}_\infty^{\alpha(\cdot)+\varepsilon}(\mathbb{R}^d)\subseteq\mathcal{D}(A)$ *for* $\varepsilon>0$; *here* $\mathcal{C}^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)$ *is a Hölder–Zygmund space of variable order.*
2. $\mathcal{D}(A)\subseteq\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)$ *for* $\varepsilon>0$.

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