Politecnico di Milano
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QUANTUM TRAJECTORIES: MEMORY AND FEEDBACK

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Ad Annalia Gallo, che mi ha convinto ad intraprendere questa strada.
Ai miei genitori, che la hanno resa possibile.
Ad Antonio e Vincenzina, che mi sono stati vicino.

Contents

Contenuto del lavoro. Scopo del lavoro è includere memoria e retroazione nella formulazione della teoria delle misurazioni quantistiche in tempo continuo basata su equazioni differenziali stocastiche. Più precisamente ci limitiamo al caso diffusivo con spazio di Hilbert finito dimensionale, come in [3] dove viene studiato il caso senza memoria. La teoria è basata su due tipi di equazioni stocastiche in spazi di Hilbert collegate da una trasformazione di Girsanov, una lineare e una non lineare, quest'ultima detta equazione di Schrödinger stocastica. La formulazione viene poi estesa al livello degli operatori statistici, dove viene introdotta una "master equation stocastica".

In primo luogo vogliamo costruire la teoria matematica necessaria alla modellazione del caso che studiamo: un modo naturale per inserire memomoria nel modello è quello di ammettere che i coefficienti dell'equazione di Schrödinger stocastica lineare siano processi stocastici e non solo funzioni deterministiche del tempo. La letteratura garantisce esistenza ed unicità per la soluzione di tale equazione in casi molto generali e tentare di riadattare tali risultati al nostro caso risulta essere una strada di difficile percorrenza. Pertanto, quello che scegliamo di fare è estendere i risultati per EDS a coefficienti deterministici. A tal fine proponiamo delle condizioni sui coefficienti molto stringenti, ma che risultano idonee a trattare alcuni modelli di interesse fisico.

Innanzi tutto, per estendere la teoria classica delle EDS in uno spazio di Hilbert finito dimensionale complesso abbiamo bisogno di nozioni di calcolo stocastico complesso. Inoltre, la formulazione assiomatica della meccanica quantistica coinvolge, oltre allo spazio di Hilbert, gli operatori che agiscono su di esso. Dunque abbiamo bisogno di lavorare con alcune norme sia nello spazio di Hilbert che nello spazio degli operatori che vi agiscono. Tutte queste nozioni sono raccolte nel Capitolo 1, per fissare le idee.

I risultati teorici per l'esistenza e l'unicità della soluzione dell'equazione di Schrödinger stocastica lineare, sotto le suddette condizioni, sono invece presentati nel Capitolo 2.

Nel Capitolo 3 introduciamo il propagatore dell'equazione di Schrödinger stocastica lineare di cui mostriamo l'invertibilità, per garantire che la soluzione di questa sia quasi certamente diversa da zero; tale proprietà ci permette di definire gli stati normalizzati del sistema quantistico nello spazio di Hilbert. Proviamo poi che la norma al quadrato degli stati non normalizzati è un processo di densità di probabilità. Pertanto, utilizzando una trasformazione di Girsanov, attraverso tale densità, intoduciamo le probabilità fisiche dell'esperimento di misurazione. Mostriamo poi che, sotto tali probabilità, gli stati normalizzati del sistema soddisfano una EDS non lineare.

Nel Capitolo 4 diamo la formulazione della meccanica quantistica in termini di operatori statistici e otteniamo un'equazione che può essere interpretata come master equation stocastica lineare per l'evoluzione del sistema quando lo stato iniziale di questo è un operatore statistico. La traccia della soluzione è ancora un processo di densità di probabilità che può essere utilizzato per introdurre le probabilità fisiche sempre attraverso una trasformazione di Girsanov. La formulazione in termini di operatori statistici premette sia di considerare la situazione in cui sussista un certo grado di incertezza sullo stato iniziale del sistema, dovuta ad esempio ad una qualche procedura di preparazione attuata sullo stesso sistema, sia di considerare fenomeni di tipo dissipativo.

Gli effetti dissipativi Markoviani sono introdotti nel Capitolo 5. In altre parole riduciamo l'osservazione assumendo che siano presenti alcuni fenomeni dissipativi (non osservati e che non introducono memoria) dovuti all'interazione del sistema col mondo esterno. Utilizzando questa ulteriore assunzione, che è di tipo fisico, otteniamo un'equazione lineare che, per l'osservazione ridotta, ha ancora l'interpretazione di master equation stocastica lineare. Anche in questo caso possiamo dare le probabilità fisiche, sotto le quali otteniamo una EDS non lineare per l'evoluzione degli stati normalizzati. Il processo di densità è ancora la traccia degli stati non normalizzati. Alla fine di questo capitolo introduciamo il propagatore della master equation stocastica lineare per l'osservazione ridotta.

Nella prima parte del Capitolo 6 mostriamo come la teoria sviluppata rientri nella formulazione assiomatica della meccanica quantistica e costruiamo quelli che si chiamano "gli strumenti" associati alla misurazione continua. Diamo anche una legge di composizione per "gli strumenti random". Nella seconda parte, definiamo cosa bisogna intendere per osservabile del sistema quantistico da analizzare. Infine, utilizzando l'operatore caratteristico e il funzionale caratteristico, ricaviamo delle formule per il calcolo della media e dei momenti secondi di un'osservabile sotto le probabilità fisiche.

Una applicazione della teoria costruita viene data nel Capitolo 7 dove presentiamo un modello fisico per la rilevazione su un atomo a due livelli, sia nel caso eterodino che in quello omodino. Lo spunto del modello è preso da [3], dove viene studiato un modello di rilevazione della stessa categoria, ma l'atomo a due livelli è stimolato da un laser perfettamente monocromatico e coerente, così come lo sono i laser presenti negli apparati di rilevazione. In questa tesi vogliamo inserire degli effetti di memoria nel sistema, ammettendo che il laser stimolante e quelli usati per la rilevazione abbiano una fase stocastica. Sviluppato il modello con questi effetti dissipativi non Markoviani, calcoliamo media e momenti secondi per l'output che verranno utilizzati per fornire l'espressione esplicita dello spettro dell'output stesso, sia nel caso omodino che in quello eterodino. Alla fine del capitolo, diamo poi delle proposte per introdurre memoria anche nei fenomeni dissipativi dovuti all'interazione del sistema col mondo esterno ed alcune proposte pre introdurre anche la retroazione. Questi ultimi argomenti non sono ulteriormente sviluppati in quanto vogliono essere solo delle proposte per un lavoro futuro.

Abstract. The aim of this work is to include memory and feedback in the theoretical formulation of quantum measurements in continuous time. More precisely, we focus our attention only in the diffusive case, in a finite dimensional Hilbert space, as they did in [3] where the memoryless case has been studied. The theory is based on two kind of equations in the Hilbert space, which are tied together by a Girsanov transformation: one is a linear SDE and the other one, known as stochastic Schrödinger equation, is non linear. The formulation can be extended in the case of statistical operators, where a "stochastic master equation" is given.

The first step that we have to achieve is the construction of the mathematical framework needed for the models that we shall study. A natural way to insert memory in the model is to allow that the coefficients in the linear stochastic Schrödinger equation are stochastic processes and not only deterministic functions of time. The existent literature guarantees the existence and the uniqueness of the solution of such an equation in very general cases, that are difficult to adapt. Therefore, we chose to extend the results obtained in the theory of classical differential equations with deterministic coefficients. To reach this goal we propose conditions on the coefficients that are very strong, but they turn out to be suitable to study some models of physical interesrt.

First of all, we need notions of stochastic calculus for complex processes. Furthermore, it is well known that quantum mechanics involves both complex Hilbert spaces and operators acting on them. For this reason, we have to work with norms either in the Hilbert space or in the space of its operator. We gather these notions in Chapter 1 to fix the ideas.

The theoretical results about the existence and the uniqueness of the solution of the linear stochastic Schrödinger equation are presented in Chapter 2.

In Chapter 3, we define the propagator of the linear stochastic Schrödinger equation and we show that it is, almost surely, an invertible operator. Thanks to this result, we can claim that the solution of the equation is almost surely non zero, thus we can define the normalised states of the quantum system in the Hilbert space. Then, we show that the square norm of the non normalised states is a probability density process and we use it to define a new probability measure by mean of a Girsanov transformation. This new probability has the interpretation of physical probability of the measuring experiment. We show that under the physical probability the normalised states satisfy a non linear SDE.

In Chapter 4 we give the formulation of quantum mechanic in terms of statistical operators and we obtain an equation which has the interpretation of linear stochastic master equation for the evolution of the non normalised states when the initial condition is a statistical operator. The trace process of the solution of this equation is still a density process and then, by mean of a Girsanov transformation, we can define the physical probability also in this case. The formulation in terms of statistical operators allows to consider situations in which we have an uncertainty on the initial state of the system because, for example, of some preparing procedure carried out on the system itself. Furthermore, in this formalism, we can consider dissipative phenomena.

The Markovian dissipation is introduced in Chapter 5. In other words, we reduce the observation by assuming that the system feels the effect of some dissipative phe-

nomenena of Markov type (they are not observed and do not introduce memory), due to the interaction of the system with the external world. With this further assumption of physical type, we end up with a linear SDE for the non normalised states of the reduced observation which still has the interpretation of linear stochastic master equation. Then, we define the physical probability also in this case and we obtain a non linear SDE for the normalised states. The density process is still the trace of the solution of the reduced linear stochastic master equation. We conclude this chapter by defining the propagator of the solution of the reduced linear stochastic master equation.

In the first part of Chapter 6, we shoe that the developed theory falls within the assiomatic formulation of quantum mechanics and built up the so called "instruments" of the continuous measurement. Then, a composition law for the "random instruments" is given. In the second part of this chapter, we define the observables of our system and, by using the characteristic operator and the characteristic functional, we obtain the formulas to compute their mean and their second moments under the physical probability.

An application of the theory that we have constructed, is given in Chapter 7, where we present a physical model for the homodyne and heterodyne detection on a two levels atom. The model draws on [3] where a model of the same kind is studied but the detection apparatus and the stimulating lasers are perfectly monochromatic and coherent. In this work we want to put in the system some memory effects by allowing that the involved lasers have a stochastic phase. Once the model with non Markovian effects has been developed, we compute mean and second moments of the detected current under the physical probability and we use them to give the explicit expression of the spectrum of the output in both the heterodyne and the homodyne cases. We conclude this treatment giving some proposals to consider memory effects also in the dissipative phenomena due to the interaction of the system with the external world and some ideas to introduce also the feedback. These latter topics are not further developed and explored here because they are just proposals for a future work.

Introduction

In the last years, stochastic wave function methods for the description of open quantum systems have received considerable attention. These approaches have been mainly motivated by quantum open system theory and by continuous measurements. We speak of continuous measurement when one or more observables of a quantum system are followed with continuity in time. These kinds of measurements are very common in the experimental practice: typical cases are the various forms of photon detection, namely homodyne and heterodyne detection, that we shall study in this work. In this kind of continuous measurement procedure, the atom is stimulated by a laser (the stimulating laser), and the output light is measured by means of detectors, each of them using another laser (local oscillator) and a half transparent mirror (the beam splitter), to produce interference. Then, the observable of the measurement is obtained subtracting the two currents coming from the photocounters. This set up with two photocounters reduces the noise in the final current. The scheme of the heterodyne/homodyne detection is depicted in the following figure

The statements of a quantum theory about an observable are of probabilistic nature and, so, it is natural that a quantum theory of continuous measurements gives rise to stochastic processes. Moreover, a continuously observed quantum system is certainly open.

In [3] the approach to quantum open system theory based on classical diffusive SDEs is presented, with particular emphasis on continuous measurements. In that book, the studied case is the diffusive one without memory: the linear stochastic Schrödinger equation of the wave function $\psi(t)$ has deterministic and bounded coefficients.

The aim of this work is to extend some of the results obtained in [3] in the case of dissipative phenomena in both the case of Markov and non Markov type, thus we want to insert some memory effects in the models stated in the book.

All the results that we shall obtain need notions of stochastic calculus in complex Hilbert spaces. Furthermore, we have to work with norms in the finite dimensional complex Hilbert space and in the space of the operators acting on it. We gather these topics in Chapter 1 to fix the ideas.

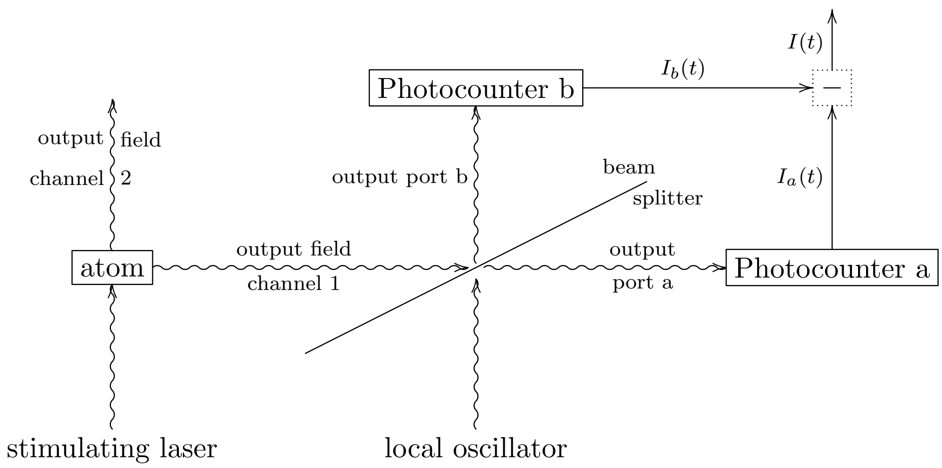

Figure 1: Balanced heterodyne/homodyne detection

In Chapter 2, we fix the space of the quantum system, that is the finite dimensional Hilbert space $\mathcal{H} := \mathbb{C}^n$. Then, we chose a reference stochastic basis $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{Q}\big)$ with a d dimensional Wiener process $W = (W_1, \dots, W_d)$ and we postulate the following linear stochastic Schrödinger equation for the wave function $\psi(t)$:

$$\begin{cases} \mathrm{d}\psi(t) = -\mathrm{i}\left[H(t) + \dfrac{1}{2}\displaystyle\sum_{j=1}^{d} R_j^*(t)R_j(t)\right]\psi(t)\mathrm{d}t + \displaystyle\sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t), \quad t \geq u \\ \\ \psi(u) = \psi_u, \qquad\qquad \psi_u \in L^2\big(\Omega, \mathcal{F}_u, \mathbb{Q}; \mathcal{H}\big), \end{cases}$$

which is the evolution of the non normalised states of the quantum system.

This equation is structurally the same one stated in [3] but, while in that work $H(t)$ and $\{R_j(t)\}_{j=1}^d$ are deterministic operators, in this treatment they are $M_n(\mathbb{C})$ valued stochastic processes, where $M_n(\mathbb{C})$ is the space of the n dimensional square matrices acting on $\mathcal{H}$. $H(t,\omega)$ is a self-adjoint operator, the effective Hamiltonian of the system. We assume that these stochastic processes are progressive with respect to the reference filtration $\{\mathcal{F}_t\}_t$.

To ensure the existence and the uniqueness of the solution of such an equation, there exist very general conditions on the coefficients: in [4] an example is given for processes in a general infinite dimensional separable Hilbert space. In this paper the driving process is not a Wiener process but a general semimartingale, and jumps in the equation are allowed: in [13] there are theorems that guarantee the existence and the uniqueness of the linear stochastic Schrödinger equation in this general case. Other conditions on the coefficients are given in [16]: in this case general stochastic calculus is used, but now the state space of the process is a finite dimensional space.

To avoid the general case sketched above, which is very difficult to develop and apply, we to choose extend the simpler results for existence and uniqueness of the solution of classical SDEs with deterministic coefficients. So, we assume

$$\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\left\|\sum_{j=1}^{d}R_j^*(t,\omega)R_j(t,\omega)\right\|\leq L(T)<\infty\,,\qquad\forall T>0\,,$$

$$\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\|H(t,\omega)\|\leq M(T)<\infty\,,\qquad\forall T>0\,,$$

where $L(T)$ and $M(T)$ are positive real numbers. Although these are very strong, they are suitable to study some quantum physical models. Furthermore, they allow to obtain L^p estimates for the solution of the linear stochastic Schrödinger equation $\psi(t)$.

In Chapter 3 we introduce the propagator of the linear stochastic Schrödinger equation, the two times-$M_n(\mathbb{C})$ valued process $A=\{A(t,s)\}_{t\in[s,T]}$. The existence of this process is guaranteed by the linear structure of the equation. Moreover we obtain a closed equation for $A(t,s)$, whose solution exists and is unique. The propagator A fulfills the typical composition law of an evolution. Then, we show that the matrix $A(t,s,\omega)$ is almost surely invertible, i.e. his kernel almost surely contains the null vector of $\mathcal{H}$ only. This result shows that the process $\psi(t)$ is almost surely non zero and then we can normalise it by using its euclidean norm to obtain the process $\hat{\psi}(t):=\psi(t)/\|\psi(t)\|$. This process is interpreted as the state of the quantum system in the Hilbert space. By assuming that the initial condition ψ_u is normalised ω by ω, that is $\|\psi_u(\omega)\|=1$, $\forall\omega\in\Omega$, we can prove that the square norm of $\psi(t)$ is a positive, mean one martingale and, so, we can use this process to introduce a consistent family of probability laws, say $\{\mathbb{P}^t_{\psi_u}\}_{t\in[u,T]}$, which is absolutely continuous with respect to the reference probability $\mathbb{Q}$. Under the physical probabilites the process $\hat{\psi}(t):=\psi(t)/\|\psi(t)\|$ satisfies a non linear SDE. Furthermore, these probabilities can be interpreted as the physical probabilities of the the measurement experiment. The output of our measurement is a functional of the Wiener process W: using a Girsanov transformation we can divide the output W in two components; a noise $\widehat{W}$, which is a Wiener process under the physical probabilities, and a signal v, which is a bounded square integrable process. Let us stress that the signal and the noise turn out to be correlated.

In Chapter 4, we generalise the Hilbert space formulation to the statistical operator case. Indeed, the theory in the Hilbert space can be generalised to the context in which the initial state is a mixture of random vectors in $\mathcal{H}$. In this way, we can introduce a further uncertainty on the initial state of the system, which can be generated, for example, by a preparation procedure on the system itself. To reach this goal, it is useful to formulate the description of quantum mechanics in the language of the statistical operators. Moreover, this generalisation is very suited to treat open systems and dissipative dynamics. Then, we define the process $\sigma=\{\sigma(t)\}_{t\in[u,T]}$ as

$$\sigma(t):=\sum_{\beta}|\psi^\beta(t)\rangle\langle\psi^\beta(t)|\,.$$

This process is a square integrable and its value space is $M_n(\mathbb{C})$. Moreover, with a further assumption of normalisation of the demixture of the initial state, we can prove that the trace of σ is a mean one, square integrable martingale. In other words this is a density process and, also in the statistical formulation, we can introduce the physical probabilities of the measurement. Then, we obtain a closed stochastic differential equation for σ. This equation turns out to be dependent on the statistical operator which is the initial state, but independent of the particular decomposition of the statistical operator in therms of pure states. So, we can say that the equation fulfilled by σ can be interpreted as a linear stochastic master equation, i.e. as the evolution equation of the quantum system when the initial condition is a statistical operator. At the end of this chapter we observe that the mean of σ, that is the a priori state of the quantum optical system, does not satisfy a closed equation, and this is because of the randomness of the coefficients R_j and H; in [3] a closed equation for the mean of σ is obtained, and this equation is the master equation.

By using the formalism of statistical operators, we insert dissipative effects in the model. The considerations about this phenomenon are carried out in Chapter 5, where we chose the form of the observed process by assuming that the output of the measurement is a function of the first $m \leq d$ components of the Wiener process W, and not of the other $m - d$ ones. Then, the $m - d$ ignored components are those representing dissipative phenomena, due to the interaction of the system with the external world, which are not observed.

To mathematically cope with this situation we introduce the two times filtration $\{\mathcal{E}_t^s\}_{t \geq s}$, that is the natural augmented filtration of the increments of the first m components of the Wiener process W. Then, by considering the conditional expectation of the process σ, with respect to the filtration $\{\mathcal{E}_t^0\}_{t \geq 0}$, we introduced the process ϱ:

$$\varrho(t) := \mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{E}_t^0].$$

Moreover, we introduce the statistical operators valued process $\hat{\varrho}$ as the normalisation of the process ϱ with respect to its trace: we can think to $\hat{\varrho}$ as the process of the a posteriori states of the system, that is $\hat{\varrho}(t)$ is the state of the system once the measuring experiment has been carried out on it, supposing that the trajectory of the output up to the time t has been observed.

If we want to obtain a closed stochastic differential equation for the process ϱ, as we did for the process σ, we must state some further assumptions. Indeed, the introduced quantities feel the effect of the randomness of the coefficients H and $\{R_j\}_{j=1}^d$. These are physical assumptions and they consist of the hypothesis that the dissipative phenomena do not directly influence the output of the measurement and that do not introduce memory effects: they are of Markov type. This physical hypothesis is inserted in the mathematical model by requiring that the coefficients of the SDEs we are considering are adapted with respect to the filtration $\{\mathcal{E}_t^0\}_{t \geq 0}$. This further assumption enables us to state the closed equation for ϱ, which can be again interpreted as a linear stochastic master equation. Furthermore, we show once again that the process $\mathrm{Tr}\{\varrho\}$ is a density process which can be used to introduce the physical probabilities and that the a posteriori states $\hat{\varrho}$ satisfy a non linear SDE.

A very important tool for the development of this work is the propagator Λ of the linear stochastic master equation for ϱ. As the propagator A, this is a two-time process, but now its value space is the space of the maps acting on $M_n(\mathbb{C})$. Furthermore, this process fulfills the composition law typical of an evolution. Finally, we state a non-linear closed stochastic differential equation for the process $\hat{\varrho}$, which can be interpreted as the evolution of the a posteriori state of the system under the physical probability.

In the first part of Chapter 6 we introduce the instruments of the continuous measurement. These are fundamental theoretical tools: they actually allow to interpret all the previous framework as a measuring experiment. First, we introduce the "non random instruments". They are the mathematical objects which allow us to represent the measurement. If the initial state of the system is known, the non random instruments permit to give the physical probabilities for the possible outcome of the observation and the state of the system conditioned by the result of the measurement itself. Moreover, they can represent instantaneous observations or observations which have some temporal duration, as in our case.

It is well known that every measuring procedure carried out on a quantum system causes a change of its state. For this reason, if we want to put into effect another measuring procedure after the first one, we must know the state of the system after the first measurement, conditioned by its outcome. In other words, it is necessary to give the transformation of the pre-measurement state into the post-measurement one, conditioned by an arbitrary event which can occur in the experiment, in both the cases in which the occurrence is observed or not. By the way, the outcome of the first measurement is not known before the end of the experiment itself. For this reason, the instruments representing a successive measuring procedure are stochastic and we introduce the "random instruments". A very important theoretical result for our work, is the composition law of the random instruments because it guarantees that in a sequence of measurements on a quantum system, each one represented by an instrument, the temporal order is respected and so the temporal causality principle.

In the second part of Chapter 6 we define the observables of the quantum system. Indeed, the "moral" output is the singular process $\dot{W}$. To give a rigorous sense to the first temporal derivative of the Wiener process, we have to intend it as a generalised process. Then, the observables of our quantum system are suitable smooth functional of W. By using the characteristic operator and the characteristic functional, we give the formula for the mean of an observable and for its two time second moments.

We present in Chapter 7 a quantum optical system as an example of application of the theory we have so far constructed. We take essentially the model given in [3], but we introduce random phases in the local oscillator and in the stimulating laser; this makes the model more realistic, with imperfections, and introduces non Markov effects . In this way we have to restrict further our filtration: we have to introduce the filtration generated by the increments of observed components of the Wiener process W, that are the first $\bar{m} \leq m$, and we use the components from $\bar{m}+1$ to m to introduce memory, that is dissipation phenomena which are not of Markov type. The components from $m+1$ to d are used for dissipative phenomena of Markov type. The quantum mechanical system is a two levels atom stimulated by a not perfectly coherent and not perfectly monochromatic laser. The choice of a two level atom fixes the Hilbert space, thus $\mathcal{H} \equiv \mathbb{C}^2$. The cases of homodyne and heterodyne detection are considered and in both of them we have a local oscillator whose phase is stochastic too. The Markovian dissipative phenomena are the dephasing and a thermal bath. To fix the model, we chose the form of the coefficient H and $\{R_j\}_{j=1}^d$ and we define the observable of the measuring experiment. Then, we carry out the computations, that now are different from those made in [3], to obtain the mean and the second moments. Furthermore, we explicitly calculate the spectrum of the output current and we give some graphical examples. In this context with memory significant differences appear, due to the new randomness introduced, with respect to the memoryless case in [3]. This is a simple model to show the effects of non Markovian terms on the system.

We use the proposal of [5] to introduce some memory in the thermal bath. In this treatment we do not tray to carry out the computation but we only rise some suggestions for a future work. We conclude, giving two proposals to insert the feedback in our model, either in the homodyne detection or in the heterodyne case.

1

Complex processes and stochastic calculus

In this work we shall use general notions of stochastic calculus and of quantum mechanics. For this reason we collect in this short chapter some properties of the processes that will be considered. Furthermore, we shall expose some properties of norms defined on the finite dimensional Hilbert space and on the space of linear operators acting on it (matrices). We refer to [2] for stochastic processes and to [3] for norms.

1.1 Useful norms

Let us consider a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$. In this section we report the most important properties of some norms on $\mathcal{H}$ and on the space of the $n \times n$-square matrices: the axiomatic formulation of quantum mechanics actually involves separable complex Hilbert spaces (of finite dimension in this our case) and operators acting on them.

The p-norm of a vector of $\mathbb{C}^n$ is

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \quad \forall x \in \mathbb{C}^n,$$

where $|(\cdot)|$ is the modulus of a complex number. Furthermore, we denote by $\|(\cdot)\|$ the Euclidian norm on $\mathbb{C}^n$, that is the previous one with $p = 2$.

The following proposition gathers some inequalities needed in our presentation.

Proposition 1.1. *Let x be a vector in $\mathbb{C}^n$. Then, we have*

$$\|x\|_p^p = \sum_{i=1}^n |x_i|^p \leq \|x\|_2^p, \quad p \geq 2; \tag{1.1}$$

$$\|x\|_2^p \leq n^{(p-2)/2} \sum_{i=1}^n |x_i|^p = n^{(p-2)/2} \|x\|_p^p; \quad p \geq 2, \tag{1.2}$$

$$\left| \sum_{i=1}^n x_i \right|^p \leq \left(\sum_{i=1}^n |x_i| \right)^p = \|x\|_1^p \leq n^{p-1} \sum_{i=1}^n |x_i|^p = n^{p-1} \|x\|_p^p, \quad p \geq 1. \tag{1.3}$$

Proof. Equation (1.2) is obtained from Jensen inequality:

$$\sum_{i=1}^{p} |x_i|^p = n \sum_{i=1}^{p} \frac{|x_i|^p}{n} = n \sum_{i=1}^{p} \frac{\left(|x_i|^2\right)^{p/2}}{n} \geq n^{1-p/2} \left(\sum_{i=1}^{n} |x_i|^2 \right)^{p/2} .$$

Equation (1.3) is a direct consequence of Hölder inequality:

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} n^{\frac{1}{q}} ,$$

noticing that $p\,, q \geq 1 : \frac{1}{p} + \frac{1}{q} = 1$ we obtain $q = \frac{p}{p-1}$; the thesis follows by raising the last inequality to the p-th power.

To prove the estimate (1.1), we first show the following statement:

$$\|x\|_p \leq \|x\|_q \,, \quad \forall x \in \mathbb{C}^n \,,\; \forall q,\, p \in \mathbb{R} : p \geq q > 0.$$

In other words, we want to obtain that the function $f(\beta) := \|(\cdot)\|_\beta$, $\beta > 0$, is strictly decreasing and, to reach this aim we shall use the concavity and the increasing properties of the logarithm.

Then we define the function

$$f(\beta) := \left(\sum_{i=1}^{n} |x_i|^\beta \right)^{\frac{1}{\beta}} = \exp\left\{ \frac{1}{\beta} \log \sum_{i=1}^{n} |x_i|^\beta \right\} , \quad \beta > 0.$$

By calculating the derivative with respect to β we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\, f(\beta) = \exp\left\{ \frac{1}{\beta} \log \sum_{i=1}^{n} |x_i|^\beta \right\} \left[-\frac{1}{\beta^2} \log \sum_{i=1}^{n} |x_i|^\beta + \frac{\sum_{i=1}^{n} |x_i|^\beta \log |x_i|}{\beta \sum_{j=1}^{n} |x_j|^\beta} \right] .$$

Let us study the sign of the derivative:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\beta}\, f(\beta) \leq 0 \Longleftrightarrow & \left[-\frac{1}{\beta^2} \log \sum_{i=1}^{n} |x_i|^\beta + \frac{\sum_{i=1}^{n} |x_i|^\beta \log |x_i|}{\beta \sum_{j=1}^{n} |x_j|^\beta} \right] \leq 0 \\ & \Longleftrightarrow \frac{\sum_{i=1}^{n} |x_i|^\beta \log |x_i|^\beta}{\sum_{j=1}^{n} |x_j|^\beta} \leq \log \sum_{i=1}^{n} |x_i|^\beta . \end{aligned}$$

Defining $y_i := |x_i|^\beta$, the last of the previous inequalities becomes

$$\frac{\sum_{i=1}^{n} y_i \log y_i}{\sum_{j=1}^{n} y_j} \leq \log \sum_{i=1}^{n} y_i .$$

If we define

$$\lambda_i := \frac{y_i}{\sum_{j=1}^{n} y_j} ,$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\, f(\beta) \leq 0 \Longleftrightarrow \sum_{i=1}^{n} \lambda_i \log y_i \leq \log \sum_{i=1}^{n} y_i .$$

Furthermore $\lambda_i \in [0,1]$ and the sum on i of λ_i is equal to one. From the logarithm concavity it results

$$\sum_{i=1}^{n} \lambda_i \log y_i \leq \log \sum_{i=1}^{n} \lambda_i y_i .$$

Then, observing that

$$\sum_{i=1}^{n} \lambda_i y_i \leq \sum_{i=1}^{n} y_i$$

and using the increasing property of the logarithm, it follows that the derivative of $f(\beta)$ is a negative function. By this statement we obtain that $f(\beta)$ is decreasing in β. □

We denote the space of the $n \times n$-square matrices with components in $\mathbb{C}$ with $M_n(\mathbb{C})$. The natural norm in this space is the infinity norm, defined by

$$\|A\| = \|A\|_\infty := \sup_{\psi \in \mathcal{H} : \|\psi\|=1} \|A\psi\| , \quad \forall A \in M_n(\mathbb{C}).$$

It is usual to endow the space $M_n(\mathbb{C})$ with the Hilbert-Schmidt norm and the trace norm, defined respectively by

$$\|A\|_2 := \sqrt{\operatorname{Tr}\{A^*A\}} = \sqrt{\sum_{i,j=1}^{n} |A_{ij}|^2} , \qquad \|A\|_1 := \operatorname{Tr}\{\sqrt{A^*A}\} , \quad \forall A \in M_n(\mathbb{C}).$$

We point out that all these norms are the same for an operator and for its adjoint and that if A is a positive definite matrix, then its trace-norm equals its trace.

Furthermore the following relations hold:

$$\|A\| \leq \|A\|_2 \leq \|A\|_1 \leq n\|A\| , \quad \forall A \in M_n(\mathbb{C}),$$

$$|\langle \varphi | A \psi \rangle| \leq \|A\| \|\varphi\| \|\psi\| , \quad \forall A \in M_n(\mathbb{C}) , \quad \forall \varphi, \psi \in \mathcal{H} ,$$

$$|\operatorname{Tr}\{AB\}| \leq \begin{cases} \|A\| \|B\|_1 , \\ \|A\|_2 \|B\|_2 , \end{cases} \qquad \forall A, B \in M_n(\mathbb{C}) .$$

1.2 Some properties of stochastic processes

The fundamental notion is that one of *filtration*. Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. If we chose as time space $\mathbb{R}_+$, we say that a filtration in $\mathcal{F}$ is an increasing family of sub-σ-algebras: $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} , \ t > s \geq 0$ [2, Definition 1.1 p. 21]. A filtration is said to satisfy usual conditions [3, p. 264] if

1. it is right continuous , that is $\mathcal{F}_t = \mathcal{F}_{t^+}$, where

$$\mathcal{F}_{t^+} := \bigcap_{s : s > t} \mathcal{F}_s ,$$

2. if we define $\mathcal{N} := \{A \in \mathcal{F} : \mathbb{Q}(A) = 0\}$, then $\mathcal{N} \subset \mathcal{F}_0$.

If a general stochastic basis $(\Omega, \mathcal{F}, \{\tilde{\mathcal{F}}_t\}_t, \mathbb{Q})$ is given, it is possible to construct a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathbb{Q})$ in usual conditions by setting $\mathcal{F}_t := \overline{\mathcal{F}}_{t^+}$, $\overline{\mathcal{F}}_0 := \tilde{\mathcal{F}}_0 \vee \mathcal{N}$.

Another important notion is that one of *natural filtration of a stochastic process*. If we consider a random variable X from a measurable space $(\Omega, \mathcal{F})$ in the space $(\Xi, \mathcal{E})$ we call σ-algebra generated by X the σ-algebra

$$\sigma(X) := \{X^{-1}(A) : A \in \mathcal{E}\}.$$

Then, if the process $X = \{X(t)\}_{t \geq 0}$ is given, its natural filtration is $\{\mathcal{F}_t^X\}_t$ where

$$\mathcal{F}_t^X := \sigma\{X(r),\, 0 \leq r \leq t\}.$$

Every process is adapted to its natural filtration.

In the following treatment we pay attention to the following particular classes of processes. For progressive processes, see [2].

Definition 1.1 ([2, Definition 7.10, p. 139])**.** Let $X = \{X(t)\}_{t \in [\alpha, \beta]}$ be a process with states in a complex matrix space $\mathcal{K}$.

1. We say that X is in $\Lambda^2([\alpha, \beta]; \mathcal{K})$ if, for all i, j, the one-dimensional process $\{X_{ij}(t)\}_t$ is progressive and
$$\mathbb{Q}\left[\int_\alpha^\beta |X_{ij}(t)|^2 \, \mathrm{d}t < \infty\right] = 1\,.$$

2. We say that X is in $\mathcal{M}^2([\alpha, \beta]; \mathcal{K})$ if, for all i, j, the one-dimensional process $\{X_{ij}(t)\}_t$ is progressive and
$$\mathbb{E}_{\mathbb{Q}}\left[\int_\alpha^\beta |X_{ij}(t)|^2 \, \mathrm{d}t\right] < \infty\,.$$

The analogous definitions are given when $\mathcal{K}$ is a vector space and the components of X have a single index.

A particular example of stochastic process is the Wiener process defined as follows.

Definition 1.2 ([2, Definition 2.3, p. 32])**.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathbb{Q})$ be a stochastic basis. A d-dimensional *Wiener process* $W \equiv \{W_j(t), t \geq 0, j = 1, \ldots, d\}$ is a continuous, $\mathbb{R}^d$-valued, adapted process with the following properties:

(i) $W(0) = 0$ a.s.;

(ii) for $0 \leq s < t < +\infty$ the increment $W(t) - W(s)$ is normal with vector of means 0 and covariance matrix $(t-s)\mathbb{1}$;

(iii) for $0 \leq s < t < +\infty$ the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

It would be equivalent to define a one-dimensional Wiener process and to say that a d-dimensional Wiener process is a collection of d independent one-dimensional Wiener processes.

Next result is needed in the proof of the existence and uniqueness of the solution of the equations that we shall consider.

Theorem 1.2 ([2, Theorem 6.12, p. 117]). *Let us consider the stochastic basis* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{Q})$ *endowed with the one-dimensional standard continuous Wiener process* $W = \{W(t)\}_{t\geq 0}$. *Then, if the processes* X, $X_n \in \Lambda^2([a,b];\mathbb{C})$, $n \geq 1$, *are such that*

$$\lim_{n\to\infty} \int_a^b |X_n(t) - X(t)| \,\mathrm{d}t = 0\,,$$

it follows that

$$\lim_{n\to\infty} \int_a^b X_n(t)\,\mathrm{d}W(t) = \int_a^b X(t)\mathrm{d}W(t).$$

All these limits are in the probability $\mathbb{Q}$.

1.3 Doob's inequalities and consequences

Theorem 1.3 ([2, Theorem 4.20, p. 75]). *Let* $M(t)$ *be a real right continuous martingale and* $p > 1$*; then*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} |M(t)|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}_{\mathbb{Q}}\left[|M(T)|^p\right].$$

Corollary 1.4. *If* $M(t)$ *is a right continuous complex martingale and* $p \geq 2$*, then*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} |M(t)|^p\right] \leq \frac{1}{2}\left(\frac{\sqrt{2}\,p}{p-1}\right)^p \mathbb{E}_{\mathbb{Q}}\left[|M(T)|^p\right]. \tag{1.4}$$

Proof. By using the relations (1.3) and (1.1), we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} |M(t)|^p\right] &\leq \mathbb{E}_{\mathbb{Q}}\left[\left(\sup_{t\in[0,T]} (\operatorname{Re} M(t))^2 + \sup_{t\in[0,T]} (\operatorname{Im} M(t))^2\right)^{p/2}\right] \\
&\leq 2^{(p-2)/2}\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} |\operatorname{Re} M(t)|^p + \sup_{t\in[0,T]} |\operatorname{Im} M(t)|^p\right]
\end{aligned}$$

$$\leq 2^{\frac{p}{2}-1}\left(\frac{p}{p-1}\right)^p \left(\mathbb{E}_{\mathbb{Q}}\left[|\operatorname{Re} M(T)|^p\right] + \mathbb{E}_{\mathbb{Q}}\left[|\operatorname{Im} M(T)|^p\right]\right) \leq \frac{1}{2}\left(\frac{\sqrt{2}\,p}{p-1}\right)^p \mathbb{E}_{\mathbb{Q}}\left[|M(T)|^p\right].$$

□

Theorem 1.5 ([2, Theorem 7.13, p. 141]). *Let us suppose* $\{\varphi_j\}_{j=1}^d \subset \mathcal{M}^2([0,T];\mathcal{H})$. *It follows*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} \left\|\sum_{j=1}^d \int_0^t \varphi_j(s)\mathrm{d}W_j(s)\right\|^2\right] \leq 4\sum_{j=1}^d \int_0^T \mathbb{E}_{\mathbb{Q}}\left[\|\varphi_j(s)\|^2\right] \mathrm{d}s.$$

Proof. The difference between this proposition and the cited one is that now the integrands processes are complex. On the other hand, when $p = 2$ the Doob's inequality holds in the complex case as well as in the real one, then the proof [2, Theorem 7.13, p. 141] has not to be changed. □

Proposition 1.6 ([2, Proposition 7.18, p. 144]). *Let $\{\varphi_j\}_{j=1}^d$ be a set of stochastic processes in $\Lambda^2([0,T],\mathcal{H})$ and $p \geq 2$. Then, we have*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left\|\sum_{j=1}^d\int_0^t \varphi_j(s)\mathrm{d}W_j(s)\right\|_2^p\right] \leq c(p,d,n,T)\sum_{j=1}^d\int_0^T \mathbb{E}_{\mathbb{Q}}\left[\|\varphi_j(s)\|_2^p\right]\mathrm{d}s,$$

where $c(p,d,n,T) := \frac{1}{2}\left(\frac{p^3}{p-1}\right)^{p/2} d^{p-1}(nT)^{(p-2)/2}$.

Proof. Because of the fact that our processes are complex and not real, the mentioned propositions holds with $m = 2n$. From the second last passage in the proof of [2] we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left\|\sum_{j=1}^d\int_0^t \varphi_j(s)\mathrm{d}W_j(s)\right\|_2^p\right] \leq c(p,d,n,T)\sum_{j=1}^d\sum_{i=1}^n\int_0^T \mathbb{E}_{\mathbb{Q}}\left[|\left(\varphi_j(s)\right)_i|^p\right]\mathrm{d}s.$$

We achieve the thesis using the estimate (1.1). □

2

The linear stochastic Schrödinger equation

2.1 The linear evolution equation in $\mathcal{H}$

Assumption 2.1. We consider the stochastic basis in usual conditions $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{Q}\big)$, where the σ-algebra $\mathcal{F}$ is

$$\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t \,.$$

We assume that a standard d-dimensional continuous Wiener process $W = (W_1, W_2, \ldots, W_d)$ is given in this basis.

Let $\mathcal{H} = \mathbb{C}^n$ be the Hilbert space of a quantum system we are interested in.

2.1.1 The equation

We assume the following stochastic differential equation for the evolution of the $\mathcal{H}$-valued process $\psi = \{\psi(t)\}_{t\geq u}$, with initial condition at time $u \geq 0$:

$$\begin{cases} \mathrm{d}\psi(t) = K(t)\psi(t)\mathrm{d}t + \displaystyle\sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t), & t \geq u \\ \psi(u) = \psi_u, & \psi_u \in L^2\big(\Omega, \mathcal{F}_u, \mathbb{Q}; \mathcal{H}\big). \end{cases} \tag{2.1}$$

We call the previous equation *linear stochastic Schrödinger equation.*

Assumption 2.2. Let us suppose that $K = \{K(t)\}_{t\geq 0}$ and $R_j = \{R_j(t)\}_{t\geq 0}$, $j = 1, \ldots, d$, are $M_n(\mathbb{C})$-valued progressive processes and that Assumption 2.1 holds. We postulate the following structure for K:

$$K(t,\omega) = -\mathrm{i}H(t,\omega) - \frac{1}{2}\sum_{j=1}^{d} R_j^*(t,\omega)R_j(t,\omega),$$

where $H = \{H(t)\}_{t\geq 0}$ is a process such that $H(t,\omega)$ is a self-adjoint operator on $\mathcal{H}$, *the effective Hamiltonian* of the system.

We shall speak of solution and uniqueness of the equation (2.1) in the sense specified in the next definition.

Definition 2.1. If we consider Eq. (2.1), in the given stochastic basis, we call *solution* in $[u,T]$ an adapted continuous process $\psi = \{\psi(t)\}_{t\in[u,T]}$ such that

$$\psi(t) = \psi_u + \int_u^t K(s)\psi(s)\mathrm{d}s + \sum_{j=1}^d \int_u^t R_j(s)\psi(s)\mathrm{d}W_j(s)\,, \quad \mathbb{Q}\text{-a.s., } \forall t \geq u. \tag{2.2}$$

Besides, we say that the solution of (2.1) is *pathwise unique* if any two continuous and adapted processes ψ and φ satisfying Eq. (2.2) are indistinguishable, that is

$$\mathbb{Q}\Big[\psi(t) = \varphi(t)\,,\ \forall t \in [u,T]\Big] = 1\,.$$

2.1.2 Conditions for existence and uniqueness of the solution

We want to give now some sufficient conditions on the operator-valued processes $\{R_j(t)\}_{j=1}^d$ and $H(t)$ to ensure the existence and the uniqueness of the solution of Eq. (2.1).

Assumption 2.3. Let Assumption 2.2 holds and let us assume that

$$\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\left\|\sum_{j=1}^d R_j^*(t,\omega)R_j(t,\omega)\right\| \leq L(T) < \infty\,, \qquad \forall T>0\,;$$

$$\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\|H(t,\omega)\| \leq M(T) < \infty\,, \qquad \forall T>0\,;$$

Remark 2.1. From Assumption 2.3 it immediately follows that

$$\|K(t,\omega)\| \leq M(T) + \frac{1}{2}\,L(T) =: L_1(T). \tag{2.3}$$

On the other hand, the R_js satisfy

$$\|R_j(t,\omega)\| \leq \sqrt{L(T)}\,, \quad \forall j-1,\dots,d\,. \tag{2.4}$$

Indeed, we have

$$\begin{aligned}
\|R_j(t,\omega)x\|^2 &\leq \sum_j \|R_j(t,\omega)x\|^2 \\
&= \sum_j \langle R_j(t,\omega)x|R_j(t,\omega)x\rangle = \sum_j \Big\langle x|R_j^*(t,\omega)R_j(t,\omega)x\Big\rangle \\
&= \Big\langle x\Big|\sum_j R_j^*(t,\omega)R_j(t,\omega)x\Big\rangle \leq \|x\|^2\left\|\sum_j R_j^*(t,\omega)R_j(t,\omega)\right\| \leq \|x\|^2\,L(T)\,,
\end{aligned}$$

which gives (2.4) because $\|R_j(t,\omega)\| = \sup_{x:\|x\|=1}\|R_j(t,\omega)x\|$.

2.1.3 Existence and uniqueness of the solution of the linear equation

By using the conditions that we introduced in the previous section, we readapt the theorems of [2] to our context: we have actually to modify standard results in such a way that they hold in the case of stochastic differential equations, for $\mathcal{H}$-valued processes, with non deterministic drift and diffusion. We point out that these results are part of more general theorems which are more difficult to apply. Furthermore, we shall obtain some simplifications in readjusting these theorems, due to the linear structure of Eq. (2.1).

We define now the following linear operator, acting on progressive processes,

$$S_u[\varphi](t) := \varphi(u) + \int_u^t K(s)\varphi(s)\,\mathrm{d}s + \sum_{j=1}^{d} \int_u^t R_j(s)\varphi(s)\,\mathrm{d}W_j(s). \tag{2.5}$$

Proposition 2.1 ([2, Lemma 8.6, p. 163]). *Let us consider the process $\varphi \in \mathcal{M}^2([0,T];\mathcal{H})$, $t \in [u,T]$. Then, from Assumption 2.3, it comes out that*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \left\| \int_u^s K(r)\varphi(r)\,\mathrm{d}r \right\|^2\right] \le (t-u)L_1(t)^2 \int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|\varphi(s)\|^2\right]\mathrm{d}s\,, \tag{2.6}$$

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \left\| \sum_{j=1}^{d}\int_u^s R_j(r)\varphi(r)\mathrm{d}W_j(r) \right\|^2\right] \le 4L(t) \int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|\varphi(s)\|^2\right]\mathrm{d}s\,, \tag{2.7}$$

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \|S_u[\varphi](s)\|^2\right] \le 3\mathbb{E}_{\mathbb{Q}}\left[\|\varphi(u)\|^2\right] + C(t)\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|\varphi(s)\|^2\right]\mathrm{d}s\,, \tag{2.8}$$

where $C(t) := 3\left[tL_1(t)^2 + 4L(t)\right] \le C(T)$.

Proof. Equation (2.6) is proved by applying the Hölder inequality, the integrand positivity and Assumption 2.3:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \left\| \int_u^s K(r)\varphi(r)\,\mathrm{d}r \right\|^2\right] &\le \mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} (s-u)\int_u^s \|K(r)\varphi(r)\|^2\,\mathrm{d}r\right] \\
&= (t-u)\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \|K(r)\varphi(r)\|^2\,\mathrm{d}r\right] \le (t-u)L_1(t)^2\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|\varphi(r)\|^2\right]\mathrm{d}r.
\end{aligned}$$

We prove now the statement in Eqs. (2.6). From Assumption 2.3, because R_j maps $\mathcal{H}$ into itself, it is bounded in t and ω and $\varphi \in \mathcal{M}^2([0,T];\mathcal{H})$, we get that

$R_j(t)\varphi(t) \in \mathcal{M}^2([0,T];\mathcal{H})$. Then, from Theorem 1.5 we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]}\left\|\sum_{j=1}^{d}\int_u^s R_j(r)\varphi(r)\,\mathrm{d}W_j(r)\right\|^2\right] \leq 4\sum_{j=1}^{d}\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|R_j(s)\varphi(s)\|^2\right]\mathrm{d}s$$

$$= 4\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \sum_{j=1}^{d}\|R_j(s)\varphi(s)\|^2\,\mathrm{d}s\right] = 4\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \left\langle\varphi(s)\middle|\sum_{j=1}^{d}R_j^*(s,\omega)R_j(s,\omega)\varphi(s)\right\rangle\right]$$

$$\leq 4\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \|\varphi(s)\|^2\left\|\sum_{j=1}^{d}R_j^*(s,\omega)R_j(s,\omega)\right\|\mathrm{d}s\right] \leq 4L(t)\mathbb{E}_{\mathbb{Q}}\left[\int_u^t \|\varphi(s)\|^2\,\mathrm{d}s\right].$$

In conclusion we can say that Eq. (2.6) and (2.7) hold. Equation (2.8) comes out by the facts that $t-u\leq t$ and

$$\|S_u[\varphi](t)\|^2 \leq 3\left\{\|\varphi(u)\|^2 + \left\|\int_u^t K(s)\varphi(s)\,\mathrm{d}s\right\|^2 + \left\|\sum_{j=1}^{d}\int_u^t R_j(s)\varphi(s)\,\mathrm{d}W_j(s)\right\|^2\right\}.$$

□

Lemma 2.2 ([2, Theorem 8.7, p. 164])**.** *Let $\psi = \{\psi(t)\}_{t\geq u}$ be a solution of Eq. (2.1) with coefficients under Assumption 2.3 and square integrable initial condition ψ_u. Then*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]}\|\psi(t)\|^2\right] \leq 3\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right]\exp\{TC(T)\}\,, \tag{2.9}$$

and, for this reason, $\psi\in\mathcal{M}^2([u,T];\mathcal{H})$.

Proof. Let τ_N be the stopping time defined by

$$\tau_N := \inf\{t\in[u,T] : \|\psi(t)\| \;\geq\; N\}\,.$$

In other words, we can say that τ_N is the exit time of the process ψ from the open ball with radius equal to N. Then, we set $\psi_N(t) := \psi(t\wedge\tau_N)$. Obviously, the process ψ is continuous because it is a solution of our equation: it comes out from Definition 2.1 that ψ_N is a solution in $[u,\tau_N]$. Moreover, on $\{\tau_N > u\}$ it results $\|\psi_N\|\leq N$, where the equality, for continuity, is reached in τ_N. On $\{\tau_N = u\}$ we have $\psi_N = \psi_u$. Then we can deduce that $\psi_N \in \mathcal{M}^2([0,T];\mathcal{H})$ and $\mathbb{Q}$-a.s. we can write

$$\psi_N(t) = \psi_u + \int_u^t K(s)\psi_N(s)\,\mathbb{1}_{[u,\tau_N]}(s)\,\mathrm{d}s + \sum_{j=1}^{d}\int_u^t R_j(s)\psi_N(s)\,\mathbb{1}_{[u,\tau_N]}(s)\,\mathrm{d}W_j(s).$$

We now introduce the coefficients

$$K^N(t) := K(s)\mathbb{1}_{[u,\tau_N]}(t)\,, \qquad R_j^N(t) := R_j(t)\mathbb{1}_{[u,\tau_N]}(t)$$

and, from them, the operator

$$S_u^N[\varphi](t) := \varphi(u) + \int_u^t K^N(s)\varphi(s)\,\mathrm{d}s + \sum_{j=1}^d \int_u^t R_j^N(s)\varphi(s)\,\mathrm{d}W_j(s)\,.$$

In this way we can write

$$\psi_N(t) = S_u^N[\psi_N](t)\,.$$

By the fact that the new processes K^N e R_j^N are bounded by the same constants of the old ones and because $\psi_N \in \mathcal{M}^2$, we can apply Eq. (2.8) to get

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \|\psi_N(s)\|^2\right] \le 3\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right] + C(T)\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\sup_{r\in[u,s]} \|\psi_N(r)\|^2\right]\mathrm{d}s\,.$$

It comes out from the Gronwall Lemma that

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]} \|\psi_N(t)\|^2\right] \le 3\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right]\exp\{TC(T)\}.$$

By the definition of ψ_N and of τ_N we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,\tau_N\wedge T]} \|\psi(t)\|^2\right] \le 3\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right]\exp\{TC(T)\}.$$

By the continuity of ψ it follows that $\sup_{t\in[u,\tau_N\wedge T]}\|\psi(t)\|^2 = N^2$ on $\{\tau_N \le T\}$. By the definition of τ_N and Markov inequality ([2, formula (0.6), p. 3], with $\beta = 2$ and $\delta = N$), we have

$$\begin{aligned}\mathbb{Q}\left[u < \tau_N \le T\right] &= \mathbb{Q}\left[\sup_{t\in(u,\tau_N\wedge T]} \|\psi(t)\|^2 \ge N\right] \\ &\le \frac{\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in(u,T]}\|\psi_N(t)\|^2\right]}{N^2} \le 3\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right]\frac{\exp\{TC(T)\}}{N^2}.\end{aligned}$$

Moreover, because ψ_u is square-integrable, we get $\mathbb{Q}[\tau_N = u] = \mathbb{Q}[\|\psi_u\| \ge N] \xrightarrow[N\to\infty]{} 0$. Therefore

$$\mathbb{Q}\left[\tau_N \le T\right] \xrightarrow[N\to\infty]{} 0\,, \qquad \tau_N\wedge T \xrightarrow[N\to\infty]{} T\,, \quad \mathbb{Q}\text{-a.s.}$$

Then

$$\liminf_{N\to\infty} \sup_{t\in[u,\tau_N\wedge T]} \|\psi(t)\|^2 = \lim_{N\to\infty} \sup_{t\in[u,\tau_N\wedge T]} \|\psi(t)\|^2 = \sup_{t\in[u,T]} \|\psi(t)\|^2\,, \qquad \mathbb{Q}\text{-a.s.,}$$

and, because $\sup_{t\in[u,\tau_N\wedge T]}\|\psi(t)\|^2 \ge 0$, $\forall N$, we can invoke Fatou Lemma to conclude that

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]} \|\psi(t)\|^2\right] \le \liminf_{N\to\infty}\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,\tau_N\wedge T]} \|\psi(t)\|^2\right] \le 3\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right]\exp\{TC(T)\},$$

which is the statement. □

We are now ready to prove the following theorem.

Theorem 2.3. *Under Assumption 2.3, Eq.* (2.1) *with square integrable initial condition admits a solution in* $\mathcal{M}^2([u,T];\mathcal{H})$, $\forall T > u$. *Furthermore, pathwise uniqueness holds.*

Proof. In what follows we shall take $0 \leq u \leq t \leq T$.

Uniqueness

Let ψ and ϕ be two solutions of Eq. (2.1) and $\eta := \psi - \phi$. Because of the linearity of (2.1), η is solution with null initial condition; therefore, by relation (2.9), η belongs to $\mathcal{M}^2$ and $\eta(t) = 0$, $\forall t \in [u,T]$, $\mathbb{Q}$-a.s.

Existence

The proof of existence is based on the Picard iteration method.

We set $\psi_0(t) \equiv \psi_u$ and we define $\psi_m(t) = S_u[\psi_{m-1}](t)$ for any integer $m \geq 1$. Then, we show by induction that, for some positive constant R (eventually T-dependent), it results

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \|\psi_{m+1}(s) - \psi_m(s)\|^2\right] \leq \frac{\big(R(t-u)\big)^{m+1}}{(m+1)!}, \qquad \forall t \in [u,T]. \tag{2.10}$$

Let us start with $m = 0$. From the triangular inequality and the Hölder's one, we have

$$\|\psi_1(s) - \psi_u\|^2 \leq 2\left\|\int_u^s K(r)\psi_u \mathrm{d}r\right\|^2 + 2\left\|\sum_{j=1}^d \int_u^s R_j(r)\psi_u \,\mathrm{d}W_j(r)\right\|^2.$$

If we take the expectation and we use relations (2.6) and (2.7) we get the inequality

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \|\psi_1(s) - \psi_u\|^2\right] \leq 2(t-u)\left(tL_1(t)^2 + 4L(t)\right)\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right] \leq R(t-u)\,,$$

with $R \geq 2\left(TL_1(T)^2 + 4L(T)\right)\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^2\right]$.

Let us assume that the thesis holds for m and we prove it for $m+1$. First we have

$$\sup_{s\in[u,t]} \|\psi_{m+1}(s) - \psi_m(s)\|^2 \leq 2\sup_{s\in[u,t]}\left\|\int_u^s K(r)\left(\psi_m(r) - \psi_{m-1}(r)\right)\mathrm{d}r\right\|^2$$
$$+ 2\sup_{s\in[u,t]}\left\|\sum_{j=1}^d \int_u^s R_j(r)\left(\psi_m(r) - \psi_{m-1}(r)\right)\mathrm{d}W_j(r)\right\|^2.$$

If we take the expectation, using the same reasoning as before and the inductive hypothesis, we obtain $\forall t \in [u,T]$,

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]} \|\psi_{m+1}(s)-\psi_m(s)\|^2\right]$$
$$\leq 2(TL_1(T)^2+4L(T))\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|(\psi_m(s)-\psi_{m-1}(s))\|^2\right]\mathrm{d}s$$
$$\leq 2(TL_1(T)^2+4L(T))\int_u^t \frac{R^m s^m}{m!}\,\mathrm{d}s = 2(TL_1(T)^2+4L(T))R^m\frac{(t-u)^{m+1}}{(m+1)!}$$

and the statement in (2.10) is proved if we choose $R = 2(TL_1(T)^2+4L(T))(1 \vee \mathbb{E}_{\mathbb{Q}}[\|\psi_u\|^2])$. Now, from Markov inequality we obtain:

$$\mathbb{Q}\left[\sup_{t\in[u,T]}\|\psi_{m+1}(t)-\psi_m(t)\|^2 > \frac{1}{2^m}\right] \leq 2^{2m}\frac{(RT)^{m+1}}{(m+1)!}\,.$$

By the summability of the right hand side we are allowed to use Borel-Cantelli Lemma

$$\mathbb{Q}\left[\sup_{t\in[u,T]}\|\psi_{m+1}(t)-\psi_m(t)\|^2 > \frac{1}{2^m} \qquad \text{for infinite elements}\right] = 0\,,$$

which allows us to say that for almost all $\omega \in \Omega$ there exists a positive integer $m_0 = m_0(\omega)$ such that

$$\sup_{t\in[u,T]}\|\psi_{m+1}(t)-\psi_m(t)\|^2 \leq \frac{1}{2^m}\,, \qquad \forall m \in \mathbb{N} : m \geq m_0\,.$$

Therefore, we have obtained that, for a fixed ω, the sequence of the partial sums

$$\psi_u + \sum_{i=0}^{m-1}[\psi_{i+1}(t)-\psi_i(t)] = \psi_m(t)$$

is almost surely, uniformly convergent, on $[u,T]$; we denote the limit by $\psi(t)$. This limit turns out to be continuous for all t and the process $\psi := \{\psi(t)\}_{t\geq u}$ belongs to $\Lambda^2([u,T];\mathcal{H})$. Moreover, because the operators $K(t,\omega)$ and $R_j(t,\omega)$ are linear and bounded in t and ω, we can write, uniformly on $[u,T]$ and almost surely on Ω,

$$\lim_{m\to\infty} K(t)\psi_m(t) = K(t)\psi(t)\,, \qquad \lim_{m\to\infty} R_j(t)\psi_m(t) = R_j(t)\psi(t)\,.$$

In particular, in probability, we have

$$\lim_{m\to\infty}\int_u^T [R_j(t)\psi_m(t) - R_j(t)\psi(t)]\,\mathrm{d}t = 0\,, \qquad j = 1,\dots,d\,.$$

From Theorem 1.2, because the limits are considered by components, we can take the limit in $\psi_m(t) = S_u[\psi_{m-1}](t)$ and we reach the equality

$$\psi(t) = \psi_u + \int_u^t K(s)\psi(s)\,\mathrm{d}s + \sum_{j=1}^d \int_u^t R_j(s)\psi(s)\,\mathrm{d}W_j(s) = S_u[\psi](t)\,,$$

that proves that the process $\psi := \{\psi(t)\}_{t\geq u}$ is a solution of (2.1). By reminding Lemma 2.2 we can conclude that $\psi \in \mathcal{M}^2([u,T];\mathcal{H})$. □

2.2 L^p estimates

The aim of this section is to generalise the estimates obtained in the previous one for the solution of Eq. (2.1); we need these results in next chapters. For this reason we widen Lemma 2.2 to the case L^p obtaining a variant of [2, Proposition 8.15, p. 172].

Theorem 2.4. *Under Assumption 2.3 for Eq.* (2.1) *the following estimate L^p, $p \geq 2$, holds:*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]}\|\psi(t)\|^p\right] \leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^p\right]\exp\{TP(T)\},$$

where

$$P(T) := 3^{p-1}(T^{p-1}L_1(T)^p + dL(T)^{p/2}c(p,d,n,T)),$$

and $c(p,d,n,T)$ is the constant defined in Proposition 1.6.

Proof. We need the same inequalities of Lemma 2.2 and, so, we use the same notation. We have

$$\psi_N(t) = \psi_u + \int_u^t K^N(s)\psi_N(s)\,\mathrm{d}s + \sum_{j=1}^d \int_u^t R_j^N(s)\psi_N(s)\,\mathrm{d}W_j(s)$$

and

$$\|\psi_N(t)\|^p \leq 3^{p-1}\|\psi_u\|^p + 3^{p-1}\left\|\int_u^t K^N(s)\psi_N(s)\,\mathrm{d}s\right\|^p + 3^{p-1}\left\|\sum_{j=1}^d\int_u^t R_j^N(s)\psi_N(s)\,\mathrm{d}W_j(s)\right\|^p.$$

Applying the Hölder inequality and Proposition 1.6 we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]}\|\psi_N(s)\|^p\right] \\
&\leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^p\right] + 3^{p-1}T^{p-1}L_1(T)^p\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|\psi_N(s)\|^p\right]\mathrm{d}s \\
&\quad + 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]}\left\|\sum_{j=1}^d\int_u^t R_j^N(s)\psi_N(s)\,\mathrm{d}W_j(s)\right\|^p\right] \\
&\leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^p\right] + 3^{p-1}T^{p-1}L_1(T)^p\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\|\psi_N(s)\|^p\right]\mathrm{d}s \\
&\quad + 3^{p-1}c(p,d,n,T)\mathbb{E}_{\mathbb{Q}}\left[\sum_{j=1}^d\int_u^t\left\|R_j^N(s)\psi_N(s)\right\|^p\mathrm{d}s\right]
\end{aligned}$$

$$\leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^p\right]+\Big(3^{p-1}T^{p-1}L_1(T)^p+dL(T)^{p/2}c(p,d,n,T)\Big)\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\big\|\psi_N(s)\big\|^p\right]\mathrm{d}s$$
$$\leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\|\psi_u\|^p\right]+P(T)\int_u^t \mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,r]}\|\psi_N(r)\|^p\right]\mathrm{d}r\,.$$

From here, by Gronwall Lemma, we have

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{s\in[u,t]}\|\psi_N(s)\|^p\right]\leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\Big[\|\psi_u\|^p\Big]\exp\{TP(T)\}\,.$$

The statement follows from Fatou Lemma, in the same way of Lemma 2.2. □

3

The propagator and the square norm of the state

3.1 The propagator of the linear equation

We denote the set of the times with $[0, T]$ and we allow that every element of this set can be choose as starting time, which we name u. The running time is $t \in [u, T]$. The final time T is arbitrary; it is introduced simply to have a finite time interval in which existence and uniqueness are given. For simplicity sake we write here again the linear evolution equation of the system state

$$\begin{cases} \mathrm{d}\psi(t) = K(t)\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t), & t \geq u \\ \psi(u) = \psi_u, & \psi_u \in L^2\big(\Omega, \mathcal{F}_u, \mathbb{Q}; \mathcal{H}\big) . \end{cases}$$

The goal of this section is to define the fundamental solution, or the propagator of (2.1). In Chapter 2 we showed that, for all initial condition ψ_u in $L^2\big(\Omega, \mathcal{F}_u, \mathbb{Q}; \mathcal{H}\big)$, existence and pathwise uniqueness hold for the solution of Eq. (2.1). We want to define now the application that associates an initial conditions with the correspondent solution of Eq. (2.1), that is the application $\psi_u \mapsto \psi(t)$. Because of the linear structure of the equation, this application is represented by a matrix, for all $t \in [u, T]$ and almost surely in ω. In other words we are interested in defining the process $A = \{A(t, u)\}_{t \in [u,T]}$, whose states space is $M_n(\mathbb{C})$, such that $A(t, u; \omega)$ is the matrix of the almost surely defined application $\psi_u(\omega) \mapsto \psi(t, \omega)$.
It is clear from Remark 3.1 below that the stochastic differential equation for the $M_n(\mathbb{C})$-valued process is

$$\begin{cases} \mathrm{d}A(t, u) = K(t)A(t, u)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)A(t, u)\mathrm{d}W_j(t), & t \geq u \geq 0 \\ A(u, u) = \mathbb{1}. \end{cases} \tag{3.1}$$

Let us stress that A is a quantity depending on two times: $u \in [0, T)$, $t \in [u, T]$. We can say that we have an adapted process with running time $t \in [u, T]$ for every choice of the initial time u.

Remark 3.1. Let us assume that Eq. (3.1) admits a unique solution, the process $A = \{A(t,u)\}_{t\in[u,T]}$. This process represents the mentioned application. Indeed the equation for $A\psi_u = \{A(t,u)\psi_u\}_{t\in[u,T]}$ is

$$\begin{cases} \mathrm{d}A(t,u)\psi_u = K(t)A(t,u)\psi_u \mathrm{d}t + \sum_{j=1}^{d} R_j(t)A(t,u)\psi_u \mathrm{d}W_j(t), \quad t \geq u \geq 0 \\ A(u,u)\psi_u = \psi_u \,. \end{cases} \tag{3.2}$$

By Theorem 2.3 this equation has a unique solution. Then we can claim that the processes ψ and $A\psi_u$ are indistinguishable because they are solutions of the same equation, whose solutions are trajectory unique, with the same initial condition, that is

$$A(t,u)\psi_u = \psi(t)\,, \quad \forall t \in [u,T]\,, \quad \mathbb{Q}\text{-a.s.}$$

We have also $A(t,s)\psi(s) = \psi(t)$ for $0 \leq u \leq s \leq t \leq T$.

We give now the following theorem

Theorem 3.1. *If Assumption 2.3 holds, there exists a unique solution of the SDE* (3.1)*. Furthermore, for a fixed* $p \geq 2$ *, the following* L^p *estimate holds:*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]} \|A(t,u)\|_2^p\right] \leq 3^{p-1} n^{p/2} \exp\{TP(T)\} < \infty\,,$$

where $P(T)$ *has been defined in Theorem 2.4; in particular the solution belongs to* $\mathcal{M}^2([u,T]; M_n(\mathbb{C}))$.

Proof. First we observe that, if Eq. (3.1) has a solution, then pathwise uniqueness holds. Indeed, if $B = \{B(t,u)\}_{t\in[u,T]}$ were another solution, pathwise uniqueness would not hold for Eq. (3.2). Then, for some initial condition, the solution of Eq. (2.1) would not be unique. This contrasts with the results that we obtained in Chapter 2.

To show the existence of the propagator we proceed in a constructive way. So, let $\{e_i\}_{i=1}^n \subseteq \mathcal{H}$ be such that $e_{ik} = \delta_{ik}\,, i = 1, \ldots, n\,; k = 1, \ldots, n$, being δ_{ik} the Kronecker symbol. From Theorem 2.3 the equation

$$\psi_i(t) = e_i + \int_u^t K(s)\psi_i(s)\mathrm{d}s + \sum_{j=1}^{d} \int_u^t R_j(s)\psi_i(s)\mathrm{d}W_j(s)\,, \quad \forall t \in [u,T]\,, \ \mathbb{Q}\text{-a.s.}$$

has a unique solution. Then we can define, $\mathbb{Q}$-almost surely and componentwise, the process $\{A(t,u)\}_{t\in[u,T]}$, by setting

$$[A(t,u)]_{ik} = \Big(\psi_i(t)\Big)_k\,, \quad i,k = 1, \ldots, n, \ \forall t \in [u,T], \ \mathbb{Q}\text{-a.s.}$$

and this proves the existence of all the elements of the propagator A.

Then, from Theorem 2.4 we have the inequality

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]} \|\psi_i(t)\|^p\right] \leq 3^{p-1}\mathbb{E}_{\mathbb{Q}}\left[\|e_i\|^p\right] \exp\{TP(T)\} = 3^{p-1} \exp\{TP(T)\}\,.$$

By definition of the 2-norm, we have

$$\|A(t,u)\|_2^p := \left(\sum_{i=1}^n \sum_{k=1}^n |A_{ki}(t,u)|^2\right)^{p/2} = \left(\sum_{i=1}^n \sum_{k=1}^n |(\psi_i(t))_k|^2\right)^{p/2}$$
$$= \left(\sum_{i=1}^n \|\psi_i(t)\|^2\right)^{p/2} = \left|\sum_{i=1}^n \|\psi_i(t)\|^2\right|^{p/2} \leq n^{(p-2)/2} \sum_{i=1}^n \|\psi_i(t)\|^p \,.$$

For this reason

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]} \|A(t,u)\|_2^p\right] \leq n^{(p-2)/2} \sum_{i=1}^n \mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]} \|\psi_i(t)\|^p\right]$$
$$\leq 3^{p-1} n^{p/2} \exp\{TP(T)\} < \infty \,.$$

□

Definition 3.1. We name the solution of Eq. (3.1) *fundamental solution* or *propagator* of the linear equation (2.1)

We collect some properties of the propagator in the following proposition.

Proposition 3.2. *Let us take* $0 \leq u \leq s \leq t \leq T$. *Then we have*

1. $\psi(t) = A(t,u)\psi_u \,,\ \psi(t) = A(t,s)\psi(s)$;

2. $A(t,s)A(s,u) = A(t,u) \,,\ t \geq u \geq r \geq 0 \,.$

More precisely both the processes $t \mapsto A(t,s)A(s,u)$ *and* $t \mapsto A(t,u)$ *and the processes* $t \mapsto A(t,u)\psi_u$ *and* $t \mapsto \psi(t)$ *are mutually indistinguishable.*

Proof. The first claim has been proved in Remark 3.1.

Let us prove the second statement. Because of the fact that Eq. (3.1) admits a (unique) solution we can define the process $B = \{B(t,u)\}_{t\in[u,T]}$ as

$$B(t) := \begin{cases} A(t,u)\,, & u \leq t < r \\ A(t,s)A(s,u)\,, & t \geq r > u \,. \end{cases}$$

For $t < r$ we have

$$B(t) = A(t,u) = \mathbb{1} + \int_u^t K(q)A(q,u)\mathrm{d}q + \sum_{j=1}^d \int_u^t R_j(q)A(q,u)\mathrm{d}W_j(q)$$
$$= \mathbb{1} + \int_r^t K(q)B(q)\mathrm{d}q + \sum_{j=1}^d \int_r^t R_j(q)B(q)\mathrm{d}W_j(q) \,.$$

On the other hand, when $t \geq r$,

$$
\begin{aligned}
B(t) = A(t,s)A(s,r) \\
&= \left(\mathbb{1} + \int_s^t K(q)A(q,s)\mathrm{d}q + \sum_{j=1}^d \int_s^t R_j(q)A(q,s)\mathrm{d}W_j(q) \right) A(s,r) \\
&= A(s,r) + \int_s^t K(q)A(q,s)A(s,r)\mathrm{d}q + \sum_{j=1}^d \int_s^t R_j(q)A(q,s)A(s,r)\mathrm{d}W_j(q) \\
&= \mathbb{1} + \int_r^s K(q)A(q,r)\mathrm{d}q + \sum_{j=1}^d \int_r^s R_j(q)A(q,r)\mathrm{d}W_j(q) \\
&\quad + \int_s^t K(q)A(q,s)A(s,r)\mathrm{d}q + \sum_{j=1}^d \int_s^t R_j(q)A(q,s)A(s,r)\mathrm{d}W_j(q) \\
&= \mathbb{1} + \int_r^t K(q)B(q)\mathrm{d}q + \sum_{j=1}^d \int_r^t R_j(q)B(q)\mathrm{d}W_j(q)\,,
\end{aligned}
$$

Then the continuous processes A and B fulfill the same equation with the same initial condition and, so, they are indistinguishable and in particular

$$
A(t,u) = A(t,s)A(s,u), \qquad \mathbb{Q}\text{-a.s.} \tag{3.3}
$$

□

We show now that the solution of Eq. (2.1), the process ψ, is almost surely non zero. To reach this goal we choose to prove that the process A is such that the matrix $A(t,u;\omega)$ is almost surely invertible or equivalently that its null space almost surely contains the null vector of $\mathcal{H}$. In this way we are guaranteed that, for all initial non zero conditions only $\psi_u \in L^2$, the solution of Eq. (2.1) is almost surely non zero. Once this property has been proved, we can almost surely normalise the solution of Eq. (2.1).
We proceed in a heuristic way before we formalise the previous concepts: our aim is actually to determine a stochastic differential equation for the $M_n(\mathbb{C})$-valued process $B := \{B(t,u)\}_{t\in[u,T]}$, such that $B(t,u;\omega)$ is the inverse of the adjoint of $A(t,u;\omega)$.

Heuristic considerations

Let us consider the process A; the heuristic idea of differential is that of increment from t to $t + \mathrm{d}t$, $\mathrm{d}t > 0$:

$$
\mathrm{d}A(t,u) = A(t+\mathrm{d}t,u) - A(t,u)\,.
$$

It turns out from Eq. (3.3) that

$$
A(t+\mathrm{d}t,u) = A(t+\mathrm{d}t,t)A(t,u)
$$

and the previous relation becomes

$$
\mathrm{d}A(t,u) = [A(t+\mathrm{d}t,t) - \mathbb{1}]A(t,u)\,.
$$

By the equation for A with $t = u$, we obtain

$$A(t+\mathrm{d}t,t) - \mathbb{1} = K(t)A(t,t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)A(t,t)\mathrm{d}W_j(t)$$
$$= K(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\mathrm{d}W_j(t)\,.$$

We want to write $A(t+\mathrm{d}t,t)$ as the exponential of an appropriate process C, that is in the form $A(t+\mathrm{d}t,t) = e^{\mathrm{d}C}$. If we expand the process $e^{\mathrm{d}C}$ in its Taylor series up to the second order, we have

$$e^{\mathrm{d}C} = \mathbb{1} + \mathrm{d}C + \frac{1}{2}(\mathrm{d}C)^2\,.$$

Let us suppose that the process C is of the form

$$\mathrm{d}C(t) = C_1(t)\mathrm{d}t + \sum_{j=1}^{d} C_2^{(j)}(t)\mathrm{d}W_j(t);$$

then, the latter expression becomes

$$e^{\mathrm{d}C} = \mathbb{1} + \mathrm{d}C + \frac{1}{2}(\mathrm{d}C)^2 = \mathbb{1} + C_1(t)\mathrm{d}t + \sum_{j=1}^{d} C_2^{(j)}(t)\mathrm{d}W_j(t) + \frac{1}{2}\sum_{j=1}^{d}(C_2^{(j)}(t))^2\mathrm{d}t\,.$$

So, the appropriate choice of C is

$$\begin{cases} C_1(t) + \dfrac{1}{2}\displaystyle\sum_{j=1}^{d}(C_2^{(j)}(t))^2 = K(t) \\ \\ C_2^{(j)}(t) = R_j(t)\,,\ j = 1,\dots,d\,, \end{cases}$$

and we can write

$$A(t+\mathrm{d}t,t) = \exp\left\{\left[K(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^2\right]\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\mathrm{d}W_j(t)\right\}.$$

Being the exponential of an infinitesimal quantity, $A(t+\mathrm{d}t,t)$ is invertible and the same holds for its adjoint. Then we can define $B(t+\mathrm{d}t,t)$ as

$$B(t+\mathrm{d}t,t) := \Big[A^*(t+\mathrm{d}t,t)\Big]^{-1}\,.$$

We can deduce the relation

$$B(t+\mathrm{d}t,t) = \exp\left\{\left[\frac{1}{2}\sum_{j=1} R_j^*(t)^2 - K^*(t)\right]\mathrm{d}t - \sum_{j=1}^{d} R_j^*(t)\mathrm{d}W_j(t)\right\}$$
$$\approx \mathbb{1} + \left[\sum_{j=1}^{d} R_j^*(t)^2 - K^*(t)\right]\mathrm{d}t - \sum_{j=1}^{d} R_j^*(t)\mathrm{d}W_j(t)\,.$$

If $A^*(t,u)$ is an invertible matrix we have

$$B(t+\mathrm{d}t,u) = \Big[A^*(t+\mathrm{d}t,u)\Big]^{-1} = \Big[\big(A(t+\mathrm{d}t,t)A(t,u)\big)^*\Big]^{-1} = B(t+\mathrm{d}t,t)B(t,u);$$

so,

$$\mathrm{d}B(t,u) = B(t+\mathrm{d}t,u) - B(t,u) = [B(t+\mathrm{d}t,u) - \mathbb{1}]B(t,u)\,.$$

Therefore

$$\mathrm{d}B(t,u) = \left[\sum_{j=1}^{d} R_j^*(t)^2 - K^*(t)\right] B(t,u)\mathrm{d}t - \sum_{j=1}^{d} R_j^*(t)B(t,u)\mathrm{d}W_j(t)\,.$$

This is the candidate equation for the process $B := \{B(t,u)\}_{t\in[u,T]}$ such that $B(t,u)$ is the inverse of the adjoint of the process $A(t,u)$ for all u and t.

Back to the rigorous developments

Let us consider the stochastic differential equation

$$\begin{cases} \mathrm{d}B(t,u) = \left[\sum_{j=1}^{d} R_j^*(t)^2 - K^*(t)\right] B(t,u)\mathrm{d}t - \sum_{j=1}^{d} R_j^*(t)B(t,u)\mathrm{d}W_j(t) \\ B(u,u) = \mathbb{1} \end{cases} \tag{3.4}$$

This equation admits a unique solution as stated in the following proposition.

Proposition 3.3. *Under Assumption 2.3 the following inequality holds:*

$$\sup_{\omega\in\Omega}\sup_{t\in[s,T]} \left\| \sum_{j=1}^{d} R_j^*(t,\omega)^2 - K^*(t,\omega) \right\| \le \sqrt{2dL(T)^2 + 2L_1(T)^2} < \infty\,. \tag{3.5}$$

Moreover, a pathwise unique solution of the SDE (3.4) *exists.*

Proof. First of all, the following inequalities hold:

$$\begin{aligned} \left\| \sum_{j=1}^{d} R_j^*(t)^2 - K^*(t) \right\|^2 &\le 2d\sum_{j=1}^{d} \left\|R_j^*(t)^2\right\|^2 + 2\left\|K^*(t)\right\|^2 \\ &\le 2d\sum_{j=1}^{d} \left\|R_j^*(t)^2\right\|^2 + 2L_1(T)^2 \quad \forall t\in[u,T]\,,\quad \omega\in\Omega. \end{aligned}$$

Let us estimate the sum on the right hand side. Let x be in $\mathcal{H}$ and $y_j := R_j^*(t)x$.

Therefore

$$\sum_{j=1}^{d} \left\| R_j^*(t)^2 x \right\|^2 = \sum_{j=1}^{d} \left\| R_j^*(t) y_j \right\|^2 = \sum_{j=1}^{d} \big\langle R_j^*(t) y_j \big| R_j^*(t) y_j \big\rangle$$
$$= \sum_{j=1}^{d} \big\langle y_j \big| R_j(t) R_j^*(t) y_j \big\rangle \leq \sum_{j=1}^{d} \|y_j\| \left\| R_j(t) R_j^*(t) y_j \right\|$$
$$\leq \sum_{j=1}^{d} \|y_j\|^2 \left\| R_j(t) R_j^*(t) \right\| \leq \sum_{j=1}^{d} \|y_j\|^2 \|R_j(t)\|^2 \leq L(T) \sum_{j=1}^{d} \|y_j\|^2$$
$$= L(T) \sum_{j=1}^{d} \left\| R_j^*(t) x \right\|^2 = L(T) \sum_{j=1}^{d} \big\langle x \big| R_j(t) R_j^*(t) x \big\rangle = L(T) \Big\langle x \Big| \sum_{j=1}^{d} R_j(t) R_j^*(t) x \Big\rangle$$
$$\leq L(T) \|x\|^2 \left\| \sum_{j=1}^{d} R_j(t) R_j^*(t) \right\| = L(T) \|x\|^2 \left\| \sum_{j=1}^{d} R_j^*(t) R_j(t) \right\|$$
$$\leq L(T)^2 \|x\|^2 , \quad \forall t \in [u,T] , \ \forall \omega \in \Omega , \ \forall x \in \mathcal{H} .$$

If we consider the operator norm we obtain

$$\sum_{j=1}^{d} \left\| R_j^*(t)^2 \right\|^2 \leq L(T)^2 , \quad \forall t \in [u,T] , \ \forall \omega \in \Omega.$$

Then, we have

$$\left\| \sum_{j=1}^{d} R_j^*(t)^2 - K^*(t) \right\| \leq \sqrt{2dL^2(T) + 2L_1^2(T)} , \quad \forall t \in [u,T] , \ \forall \omega \in \Omega ,$$

and Eq. (3.5) follows.

By the bound given in (3.5), the statement regarding the existence and the uniqueness is a trivial consequence of Theorem 3.1. □

Let us rewrite Eq. (3.4) in integral form

$$B(t,u) = \mathbb{1} + \int_u^t \left[\sum_{j=1}^{d} R_j^*(q)^2 - K^*(q) \right] B(q,u) \mathrm{d}q - \sum_{j=1}^{d} \int_u^t R_j^*(q) B(q,u) \mathrm{d}W_j(q) . \quad (3.6)$$

Taking the adjoint of B (we recall that $\mathcal{H}$ is of finite dimension), we get

$$B^*(t,u) = \mathbb{1} + \int_u^t B^*(q,u) \left[\sum_{j=1}^{d} R_j(q)^2 - K(q) \right] \mathrm{d}q - \sum_{j=1}^{d} \int_u^t B^*(q,u) R_j(q) \mathrm{d}W_j(q) , \quad (3.7)$$

or, in differential form,

$$\begin{cases} \mathrm{d}B^*(t,u) = B^*(t,u) \left[\sum_{j=1}^{d} R_j(t)^2 - K(t) \right] \mathrm{d}t - \sum_{j=1}^{d} B^*(t,u) R_j(t) \mathrm{d}W_j(t) \\ B^*(u,u) = \mathbb{1} . \end{cases} \quad (3.8)$$

Proposition 3.4. *Under Assumption 2.3, the processes* $A := \{A(t,u)\}_{t\in[u,T]}$ *and* $B^* := \{B^*(t,u)\}_{t\in[u,T]}$, *solutions of Eqs.* (3.1) *and* (3.8) *respectively, are such that the matrices* $A(t,u;\omega)$ *and* $B^*(t,u;\omega)$ *are almost surely one the inverse of the other.*

Proof. It is possible to define, almost surely, the process $C = \{C(t,u)\}_{t\in[u,T]}$ by setting $C(t,u) = B^*(t,u)A(t,u)$. Obviously $C(u,u) = \mathbb{1}$. By applying Itô formula for products, we calculate the stochastic differential of C:

$$\begin{aligned} \mathrm{d}C(t,u) &= \big(\mathrm{d}B^*(t,u)\big)A(t,u) + B^*(t,u)(\mathrm{d}A(t,u)) + \big(\mathrm{d}B^*(t,u)\big)(\mathrm{d}A(t,u)) \\ &= B^*(t,u)\left\{\left[\sum_{j=1}^{d} R_j(t)^2 - K(t)\right]\mathrm{d}t - \sum_{j=1}^{d} R_j(t)\mathrm{d}W_j(t)\right\}A(t,u) \\ +B^*(t,u)&\left\{K(t)\mathrm{d}t+\sum_{j=1}^{d} R_j(t)\mathrm{d}W_j(t)\right\}A(t,u)-B^*(t,u)\left\{\sum_{j=1}^{d} R_j(t)^2\mathrm{d}t\right\}A(t,u) = 0\,. \end{aligned}$$

Therefore C is a constant process; by the initial condition $C(u,u) = \mathbb{1}$ we get $C(t,u) = \mathbb{1}$ or

$$B^*(t,u)A(t,u) \equiv \mathbb{1}\,, \quad 0 \le u \le t\,, \quad \mathbb{Q}\text{-a.s.} \tag{3.9}$$

So, for all t and u in $[0,T]$, $t \ge u$, we have

$$\det[B^*(t,u)]\det[A(t,u)] = 1, \quad \mathbb{Q}\text{-a.s.},$$

that is, $\det[B^*(t,u)] \neq 0 \neq \det[A(t,u)]$, $\mathbb{Q}$-a.s. Then, these matrices are almost surely invertible. Let $A^{-1}(t,u)$ be the inverse of $A(t,u)$: post-multiplying both the sides of Eq. (3.9) by this quantity we obtain, for all t and u,

$$B^*(t,u) = A^{-1}(t,u), \quad \mathbb{Q}\text{-a.s.}$$

□

Equation (3.8) is the evolution equation of the almost surely definite process $A^{-1} = \{A^{-1}(t,u)\}_{t\in[u,T]}$ and we rewrite it as

$$\begin{cases} \mathrm{d}A^{-1}(t,u) = A^{-1}(t,u)\left[\sum_{j=1}^{d} R_j(t)^2 - K(t)\right]\mathrm{d}t - \sum_{j=1}^{d} A^{-1}(t,u)R_j(t)\mathrm{d}W_j(t) \\ A^{-1}(u,u) = \mathbb{1}\,. \end{cases} \tag{3.10}$$

Therefore, we have proved that the process $A = \{A(t,u)\}_{t\in[u,T]}$ is such that $A(t,u)$ is almost surely invertible on $[u,T]$ and that its inverse satisfies Eq. (3.10).

Here, we rewrite Eq. (3.4) by using $A^{-1}(t,u)$:

$$\begin{cases} \mathrm{d}A^{-1^*}(t,u) = \left[\sum_{j=1}^{d} R_j^*(t)^2 - K^*(t)\right]A^{-1^*}(t,u)\mathrm{d}t - \sum_{j=1}^{d} R_j^*(t)A^{-1^*}(t,u)\mathrm{d}W_j(t) \\ A^{-1^*}(u,u) = \mathbb{1}\,. \end{cases} \tag{3.11}$$

We rewrite here the stochastic differential equation of the adjoint of the propagator:

$$\begin{cases} \mathrm{d}A^*(t,u) = A^*(t,u)K^*(t)\mathrm{d}t + \sum_{j=1}^{d} A^*(t,u)R_j^*(t)\mathrm{d}W_j(t), \quad t \geq u \geq 0 \\ A^*(u,u) = \mathbb{1}. \end{cases} \tag{3.12}$$

The solutions of Eqs. (3.10) and (3.12) exist by construction and the uniqueness can be proved by using the same techniques and reasoning as for the Eqs. (3.1) and (3.11).

3.2 The square norm of the solution

In Chapter 1 we showed that, under Assumption 2.3, the solution of Eq. (2.1) exists and that pathwise uniqueness holds. In the previous section we introduced the propagator of this equation, the $M_n(\mathbb{C})$-valued process A. Because of the almost sure invertibility of the propagator at every time, we can claim that, if the initial condition is a nonzero random variable, then the solution of Eq. (2.1) remains almost surely non zero and, so, we can introduce the quantities

$$m_j(t) := \big\langle \hat{\psi}(t) \big| (R_j(t) + R_j^*(t))\hat{\psi}(t) \big\rangle = 2\mathrm{Re}\big\langle \hat{\psi}(t) \big| R_j(t)\hat{\psi}(t) \big\rangle\,, \quad j = 1, \ldots, d\,, \tag{3.13}$$

where $\hat{\psi}$ is defined by

$$\hat{\psi}(t) := \frac{\psi(t)}{\|\psi(t)\|}\,, \quad \forall t \in [u,T]\,,\ \mathbb{Q}\text{-a.s.} \tag{3.14}$$

Remark 3.2. The quantities (3.13) are all bounded: it actually comes out that

$$\sum_{j=1}^{d} m_j(t)^2 \leq 4L(T)\,. \tag{3.15}$$

Let us prove this statement. We recall that the unidimensional orthogonal projection P_x of a vector of $\mathcal{H}$ on the ray containing x, $\|x\| = 1$, is:

$$\mathrm{P}_x := |x\rangle\langle x|\,. \tag{3.16}$$

Then,

$$\sum_{j=1}^{d} \big\langle x \big| \big(R_j(t) + R_j^*(t)\big)x \big\rangle^2 = \sum_{j=1}^{d} \Big(2\mathrm{Re}\big\langle x \big| R_j(t)x \big\rangle\Big)^2 \leq 4 \sum_{j=1}^{d} \Big| \big\langle x \big| R_j(t)x \big\rangle \Big|^2$$

$$= 4 \sum_{j=1}^{d} \big\langle R_j(t)x \big| x \big\rangle \big\langle x \big| R_j(t)x \big\rangle \leq 4\|x\|^2 \sum_{j=1}^{d} \big\langle R_j(t)x \big| \mathrm{P}_x R_j(t)x \big\rangle \leq 4\|x\|^2 \sum_{j=1}^{d} \|R_j(t)x\|^2$$

$$= 4\|x\|^2 \big\langle x \big| \sum_{j=1}^{d} R_j^*(t)R_j(t)x \big\rangle \leq 4\|x\|^4 \left\| \sum_{j=1}^{d} R_j^*(t)R_j(t) \right\| \leq 4L(T)\|x\|^4.$$

By replacing x with $\hat{\psi}(t)$ in the latter relation we get the thesis.

We show now that the process $\|\psi\|^2 := \{\|\psi(t)\|^2\}_{t\in[u,T]}$ is a square integrable martingale with respect to the stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{Q}\big)$. To reach this goal we apply the following general result.

Theorem 3.5 ([2, Proposition 7.19, p. 145])**.** *Let us consider the process Z defined by*

$$Z(t) := \exp\left\{\sum_{j=1}^{d}\int_u^t G_j(q)\mathrm{d}W_j(q) - \frac{1}{2}\int_u^t G_j(q)^2\mathrm{d}q\right\},$$

where the processes $G_j \in \Lambda^2([u,T];\mathbb{C})$, $j = 1,\ldots,d$, are such that there exists a constant K such that $\sum_{j=1}^d \int_u^T \|G_j(q)\|^2\,\mathrm{d}q \le K$. Then $\{Z(t)\}_{t\in[u,T]}$ is a complex square integrable martingale and, moreover,

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[u,T]}\|Z(t)\|^p\right] < \infty\,, \quad \forall p \ge 1.$$

By using this result we can prove the following theorem.

Theorem 3.6. *Under Assumption 2.3 the process $\|\psi\|^2 := \{\|\psi(t)\|^2\}_{t\in[u,T]}$ is a positive square integrable martingale that fulfils the Doléans equation*

$$\begin{cases} \mathrm{d}\,\|\psi(t)\|^2 = \displaystyle\sum_{j=1}^{d} m_j(t)\,\|\psi(t)\|^2\,\mathrm{d}W_j(t) \\ \|\psi(u)\|^2 = \|\psi_u\|^2\,, \end{cases} \tag{3.17}$$

whose solution is

$$\|\psi(t)\|^2 = \|\psi_u\|^2\exp\left\{\sum_{j=1}^{d}\int_u^t m_j(q)\mathrm{d}W_j(q) - \frac{1}{2}\int_u^t m_j(q)^2\mathrm{d}q\right\}. \tag{3.18}$$

Proof. By the fact that ψ is an Itô process we can calculate the stochastic differential of its square norm.

$$\begin{aligned}
\mathrm{d}\,\|\psi(t)\|^2 &= \mathrm{d}\langle\psi(t)|\psi(t)\rangle = \langle\mathrm{d}\psi(t)|\psi(t)\rangle + \langle\psi(t)|\mathrm{d}\psi(t)\rangle + \langle\mathrm{d}\psi(t)|\mathrm{d}\psi(t)\rangle \\
&= \Big\langle K(t)\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t)\Big|\psi(t)\Big\rangle \\
&\quad + \Big\langle\psi(t)\Big|K(t)\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t)\Big\rangle + \sum_{j=1}^{d}\langle R_j(t)\psi(t)\big|R_j(t)\psi(t)\rangle\mathrm{d}t \\
&= \Big\langle\psi(t)\Big|\Big(K(t) + K^*(t) + \sum_{j=1}^{d} R_j^*(t)R_j(t)\Big)\psi(t)\Big\rangle\mathrm{d}t \\
&\quad + \sum_{j=1}^{d}\langle\psi(t)\big|\big(R_j^*(t) + R_j(t)\big)\psi(t)\rangle\mathrm{d}W_j(t) = \sum_{j=1}^{d} m_j(t)\,\|\psi(t)\|^2\,\mathrm{d}W_j(t)\,,
\end{aligned}$$

where, in the last passage, we used Assumption 2.2 for the process K and the definition of $m_j(t)$. If we consider the process m_j as given, we obtain Eq. (3.17) whose solution is known and of the form (3.18). Furthermore, by Remark 3.2, we have

$$\sum_{j=1}^{d} \int_u^t m_j(q)^2 \mathrm{d}q \leq \int_u^t 4L(T)\mathrm{d}q = 4L(T)(T-u) \leq 4L(T)T.$$

The thesis follows by applying Theorem 3.5. □

3.3 A probability change

The aim of this section is to show that $\|\psi(t)\|^2$ can be used to introduce a new probability density, under which the standardized state $\hat{\psi}(t)$ satisfies a SDE, that we want to give. To proceed in this sense we introduce a further assumption.

Assumption 3.1. The initial condition of Eq. (2.1) is normalized ω by ω, i.e. $\|\psi_u(\omega)\| = 1$, $\forall \omega \in \Omega$.

By the martingale property of the square norm of the solution process and by Assumption 3.1 we get for the $\mathbb{Q}$-expectation:

$$\mathbb{E}_{\mathbb{Q}}\Big[\|\psi(t)\|^2\Big] \equiv \mathbb{E}_{\mathbb{Q}}\Big[\|\psi_u\|^2\Big] = 1\,. \tag{3.19}$$

So far, we have considered the stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{Q}\big)$ and the time interval $[0,T]$: let us regard the random variable $\|\psi(T)\|^2$ as a probability density and introduce on the measurable space $(\Omega, \mathcal{F}_T)$ a new probability measure equivalent to $\mathbb{Q}_T$, where $\mathbb{Q}_T$ is the restriction of $\mathbb{Q}$ to $\mathcal{F}_T \subset \mathcal{F}$, or $\mathbb{Q}_T := \mathbb{Q}\Big|_{\mathcal{F}_T}$. We define this probability measure as follows

$$\mathbb{P}^T_{\psi_u}(F) := \int_F \|\psi(T,\omega)\|^2\, \mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_F \|\psi(T)\|^2\right], \qquad F \in \mathcal{F}_T\,. \tag{3.20}$$

Then we write $\mathbb{E}^T_{\psi_u}$ for the expectation with respect to $\mathbb{P}^T_{\psi_u}$. We observe that the probability family $\left\{\mathbb{P}^T_{\psi_u},\ T > 0\right\}$ is consistent in this sense:

$$0 < S < T,\ F \in \mathcal{F}_S \ \Rightarrow\ \mathbb{P}^T_{\psi_u}(F) = \mathbb{P}^S_{\psi_u}(F)\,.$$

This is a straightforward consequence of the martingale property of the process $\|\psi(t)\|^2$.

The following result is an immediate consequence of the Girsanov Theorem.

Theorem 3.7 ([3, Theorem 2.14, p. 17])**.** *The process $\widehat{W} := (\widehat{W}_1, \ldots, \widehat{W}_d)$ defined by*

$$\widehat{W}_j(t) := W_j(t) - \int_0^t m_j(s)\mathrm{d}s, \qquad j = 1, \ldots, d, \quad t \in [0,T], \tag{3.21}$$

is a d-dimensional, standard Wiener process with respect to the probability $\mathbb{P}^T_{\psi_u}$ and the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$.

Furthermore for all process set $\{G_j\}_{j=1}^d \subset \Lambda^2$, *the Itô's integrals* $\sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}\widehat{W}_j(s)$ *and* $\sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}W_j(s)$ *are defined for every* $t \in [0,T]$, *under the probability laws* $\mathbb{Q}$ *and* $\mathbb{P}_{\psi_u}^T$ *respectively. Moreover the following equality holds, almost surely under* $\mathbb{Q}$ *and* $\mathbb{P}_{\psi_u}^T$

$$\sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}\widehat{W}_j(s) = \sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}W_j(s) - \sum_{j=1}^d \int_0^t G_j(s)m_j(s)\mathrm{d}s \quad \forall t \in [0,T]\,. \tag{3.22}$$

3.4 The nonlinear equation

We give now the evolution equation of the state process $\hat{\psi} = \{\hat{\psi}(t)\}_{t\in[u,T]}$ with respect to the new probability measure $\mathbb{P}_{\psi_u}^T$. Let us introduce the following quantities

$$n_j(t,x) := \langle x \big| R_j(t)x\rangle \quad \forall t \in [u,+\infty)\,, j = 1,\dots,d\,, \quad x \in \mathcal{H}\,. \tag{3.23}$$

We observe that

$$m_j(t) = 2\mathrm{Re}(n_j(t,\hat{\psi}(t))).$$

Proposition 3.8 ([3, Proposition 2.20, p. 24]). *The stochastic differential of the process* $\hat{\psi}(t)$, $t \in [u,T]$, *under the probability* $\mathbb{P}_{\psi_u}^T$, *is*

$$\begin{aligned}\mathrm{d}\hat{\psi}(t) &= \sum_j \left[R_j(t) - \mathrm{Re}\, n_j\left(t,\widehat{\psi}(t)\right)\right]\widehat{\psi}(t)\,\mathrm{d}\widehat{W}_j(t)\\ &+ \left[K(t) + \sum_j \left(\mathrm{Re}\, n_j\left(t,\widehat{\psi}(t)\right)\right)R_j(t) - \frac{1}{2}\sum_j \left(\mathrm{Re}\, n_j\left(t,\widehat{\psi}(t)\right)\right)^2\right]\widehat{\psi}(t)\,\mathrm{d}t\,. \end{aligned}\tag{3.24}$$

Proof. By Itô formula for products

$$\mathrm{d}\hat{\psi}(t) = \mathrm{d}(\|\psi(t)\|^{-1}\psi(t)) = \|\psi(t)\|^{-1}(\mathrm{d}\psi(t)) + (\mathrm{d}\|\psi(t)\|^{-1})\psi(t) + (\mathrm{d}\|\psi(t)\|^{-1})(\mathrm{d}\psi(t)).$$

Then, by Assumption 3.1,

$$\|\psi(t)\|^2 = \exp\left\{\sum_{j=1}^d \int_u^t m_j(q)\mathrm{d}W_j(q) - \frac{1}{2}\int_u^t m_j(q)^2\mathrm{d}q\right\},$$

and, so,

$$\|\psi(t)\|^{-1} = \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^d \int_u^t m_j(q)\mathrm{d}W_j(q) - \frac{1}{2}\int_u^t m_j(q)^2\mathrm{d}q\right]\right\},$$

that, under the law $\mathbb{P}_{\psi_u}^T$, is

$$\|\psi(t)\|^{-1} = \exp\left\{-\frac{1}{2}\left[\sum_{j=1}^d \int_u^t m_j(q)\mathrm{d}\widehat{W}_j(q) + \frac{1}{2}\int_u^t m_j^2(q)\mathrm{d}q\right]\right\}.$$

In conclusion, with respect to $\mathbb{P}^T_{\psi_u}$, it comes out

$$\mathrm{d}\,\|\psi(t)\|^{-1} = \|\psi(t)\|^{-1}\left\{-\frac{1}{2}\sum_{j=1}^{d}\left[m_j(t)\mathrm{d}\widehat{W}_j(t) + \frac{1}{2}\,m_j(t)^2\mathrm{d}t\right] + \frac{1}{8}\sum_{j=1}^{d} m_j(t)^2\mathrm{d}t\right\}$$
$$= -\frac{1}{2}\,\|\psi(t)\|^{-1}\sum_{j=1}^{d}\left[m_j(t)\mathrm{d}\widehat{W}_j(t) + \frac{1}{4}\,m_j(t)^2\mathrm{d}t\right].$$

The differential of $\{\psi(t)\}_{t\in[u,T]}$ under the law $\mathbb{P}^T_{\psi_u}$ is

$$\mathrm{d}\psi(t) = K(t)\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t) = \left\{K(t) + \sum_{j=1}^{d} m_j(t)R_j(t)\right\}\psi(t)\mathrm{d}t$$
$$+ \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}\widehat{W}_j(t).$$

The thesis is obtained by replacing the expression for the stochastic differentials under the probability $\mathbb{P}^T_{\psi_u}$ in the Itô formula for a product and applying the relations (3.23). $\square$

4 The stochastic master equation

So far, we have supposed that the initial state of the quantum system is a random vector $\psi_u \in \mathcal{H}$. This situation can be generalised to the context in which the initial state is a mixture of random vectors in $\mathcal{H}$. In this way, we can introduce a further uncertainty on the initial state of the system, which can be generated, for example, by a preparation procedure on the system itself. To reach this goal, it is useful to formulate the description of quantum mechanics in the language of the *statistical operators*. In this section we refer to [3, Appendix B]. Moreover, this generalisation is very suited to treat open systems and dissipative dynamics.

4.1 Statistical operators

First of all, let us recall the definition of positive operator:

$$B \geq 0 \Longleftrightarrow \langle \phi | B\phi \rangle \geq 0 \,, \quad \forall \phi \in \mathcal{H} \,. \tag{4.1}$$

In what follows, we call *statistical operator* an operator of $M_n(\mathbb{C})$ such that

$$\rho \geq 0 \,, \quad \rho = \rho^* \,, \quad \mathrm{Tr}\{\rho\} = 1 \,. \tag{4.2}$$

Then we denote by $S(\mathcal{H})$ the space of the statistical operators.

It is well known that the statistical operators are a basis for the linear space $M_n(\mathbb{C})$, in particular it is possible to write any $M_n(\mathbb{C})$ matrix as an algebraic sum of four statistical operators:

$$\forall \tau \in M_n(\mathbb{C}) \;\; \exists \{\rho_i\}_{i=1}^4 \subset S(\mathcal{H}) : \tau = \lambda_1 \rho_1 - \lambda_2 \rho_2 + \mathrm{i}(\lambda_3 \rho_3 - \lambda_4 \rho_4) \,. \tag{4.3}$$

Of course we have

$$\mathrm{Tr}\{\tau\} = \lambda_1 - \lambda_2 + \mathrm{i}(\lambda_3 - \lambda_4) \,, \quad \forall \tau \in M_n(\mathbb{C}) \,.$$

In the special case of a positive $M_n(\mathbb{C})$ operator it comes out

$$\forall \tau \in M_n(\mathbb{C}) : \tau \geq 0 \quad \exists \rho \in S(\mathcal{H}) : \tau = \mathrm{Tr}\{\tau\}\rho \,. \tag{4.4}$$

The space $S(\mathcal{H})$ is a convex set. Indeed, if we consider the statistical operators ρ_1, ρ_2 and the scalar number $\lambda \in [0,1]$ then $\rho := \lambda\rho_1 + (1 - \lambda\rho_2)$ is a statistical operator too. The convexity of $S(\mathcal{H})$ is an important fact. Indeed, if we consider the probability space $(A, \mathcal{A}, \mu)$ and a μ-measurable family of statistical operators $\{\rho(\alpha)\}_\alpha \subset S(\mathcal{H})$, then the operator

$$\rho := \int_A \rho(\alpha)\mu(\mathrm{d}\alpha)\,, \quad \forall A \in \mathcal{A},$$

is again a statistical operator. The pair $\{\mu\,, \rho(\cdot)\}$ is called a *demixture* of ρ and describes a possible uncertainty on ρ that comes out from the mechanism which produces the state ρ itself.

Another important property of the set $S(\mathcal{H})$ is that this is a closed set: the limit of a convergent sequence of state is a state itself. We shall use this property in Chapter 7 when we shall introduce the equilibrium state for a quantum dynamical.

In conclusion we observe that it is possible to interpret a system state $\psi \in \mathcal{H}$, $\|\psi\| = 1$, as in Eq. (2.1), in terms of statistical operators: such a state corresponds with the one-dimensional projector $|\psi\rangle\langle\psi|$. The one-dimensional projectors are said to be *pure states* because they do not admit a non trivial demixture.

4.2 The stochastic master equation

Let us consider the family $\{\psi^\beta\}_\beta \subset L^2(\Omega, \mathcal{F}_u; \mathcal{H})$ of random variables such that

$$\sum_\beta \|\psi^\beta\|^2 = 1 \tag{4.5}$$

and define the random operator

$$\rho_0 := \sum_\beta |\psi^\beta\rangle\langle\psi^\beta|. \tag{4.6}$$

Then ρ_0 is a random statistical operator. Viceversa, any random statistical operator ρ_0 can be written in the form (4.5), (4.6). Indeed, because of the finite dimension of the Hilbert space $\mathcal{H}$, we can give Eq. (4.6), at least, with respect to a basis of one-dimensional and mutually orthogonal, orthogonal projectors: the number of these operators, in a finite dimensional Hilbert space, equals the dimension of the space itself.

Assumption 4.1. The initial state of the system is the *random statistical operator* ρ_0.

Let $\{\psi^\beta(t)\}_{t\in[u,T]}$ be the solution of the stochastic differential equation (2.1) when the initial condition is ψ^β, i.e. $\psi_u = \psi^\beta$. Then we define the process $\sigma = \{\sigma(t)\}_{t\in[u,T]}$ as

$$\sigma(t) := \sum_\beta |\psi^\beta(t)\rangle\langle\psi^\beta(t)|. \tag{4.7}$$

Proposition 4.1. *The process σ belongs to $\mathcal{M}^2([0,T];M_n(\mathbb{C}))$ and its trace is an $(\mathcal{F}_t)$-martingale with mean one, with respect to the probability measure $\mathbb{Q}$.*

Proof. By the definition of the propagator of the linear equation (2.1), given in Chapter 2 we have

$$\sigma(t) = \sum_\beta |\psi^\beta(t)\rangle\langle\psi^\beta(t)| = \sum_\beta A(t,u)|\psi^\beta\rangle\langle\psi^\beta|A^*(t,u)$$
$$= A(t,u)\left[\sum_\beta |\psi^\beta\rangle\langle\psi^\beta|\right]A^*(t,u) = A(t,u)\rho_0 A^*(t,u). \quad (4.8)$$

In Chapter 1 we mentioned that $\|\rho_0\|_2 \leq \|\rho_0\|_1 = 1$. By applying the L^p bound for the propagator, obtained in Theorem 3.1 when $p=4$, we get:

$$\mathbb{E}_\mathbb{Q}\Big[\|\sigma(t)\|_2^2\Big] = \mathbb{E}_\mathbb{Q}\Big[\|A(t,u)\rho_0 A^*(t,u)\|_2^2\Big] \leq \mathbb{E}_\mathbb{Q}\Big[\|A(t,u)\|_2^2\,\|A^*(t,u)\|_2^2\Big]$$
$$= \mathbb{E}_\mathbb{Q}\Big[\|A(t,u)\|_2^4\Big] \leq \mathbb{E}_\mathbb{Q}\left[\sup_{t\in[u,T]}\|A(t,u)\|_2^4\right] \leq 27n^2\exp\{TP(T)\} < \infty\,, \quad \forall t\in[u,T].$$

This gives

$$\mathbb{E}_\mathbb{Q}\left[\sup_{t\in[u,T]}\|\sigma(t)\|_2^2\right] < \infty.$$

By using the definition of the norm $\|(\cdot)\|_2$ for an element of $M_n(\mathbb{C})$, the last relation is

$$\mathbb{E}_\mathbb{Q}\left[\sup_{t\in[u,T]}\sum_{i,j}|\sigma_{ij}(t)|^2\right] < \infty\,,$$

and this involves that

$$\mathbb{E}_\mathbb{Q}\left[\sup_{t\in[u,T]}|\sigma_{ij}(t)|^2\right] < \infty\,, \quad \forall i,j = 1,\ldots,n.$$

By recalling Definition 1.1 and the properties of the integral, we can claim that $\sigma \in \mathcal{M}^2([0,T];M_n(\mathbb{C}))$.

We prove now the statements about the expectation. First of all we have

$$\mathrm{Tr}\big\{\sigma(t)\big\} = \sum_\beta \mathrm{Tr}\Big\{|\psi^\beta(t)\rangle\langle\psi^\beta(t)|\Big\} = \sum_\beta\langle\psi^\beta(t)\big|\psi^\beta(t)\rangle = \sum_\beta\|\psi^\beta(t)\|^2.$$

By Assumption 3.1 and Theorem 3.6, $\{\|\psi^\beta(t)\|^2\}_{t\in[u,T]}$ is a mean-one $\mathbb{Q}$-martingale and, so, $\big\{\mathrm{Tr}\{\sigma(t)\}\big\}_{t\in[u,T]}$ is a $\mathbb{Q}$-martingale. By relation (4.5) we can conclude that

$$\mathbb{E}_\mathbb{Q}\Big[\mathrm{Tr}\big\{\sigma(t)\big\}\Big] = \mathbb{E}_\mathbb{Q}\left[\sum_\beta\|\psi^\beta\|^2\right] = 1.$$

□

As in the case of initial state in $\mathcal{H}$, thanks to the previous proposition, we can use the process $\big\{\mathrm{Tr}\{\sigma(t)\}\big\}_{t\in[u,T]}$ as a probability density: we define the consistent probability family $\big\{\mathbb{P}^t_{\rho_0}\,, t>0\big\}$ by

$$\mathbb{P}^t_{\rho_0} := \mathbb{E}_{\mathbb{Q}}\Big[1_F\,\mathrm{Tr}\big\{\sigma(t)\big\}\Big]\,, \quad F\in\mathcal{F}_t\,, \forall t\in[u,T]\,. \tag{4.9}$$

Exactly as in the case of initial state in $\mathcal{H}$, the consistence property is a straightforward consequence of the martingale property. Let us write $\mathbb{E}^t_{\rho_0}$ for the expectation with respect to the probability measure $\mathbb{P}^t_{\rho_0}$.

We want to state a stochastic differential equation for the evolution of the process σ. First of all, we compute its stochastic differential and we show that this process satisfies a closed SDE. Then, we shall show that this equation admits a pathwise unique solution.

By construction, the equation we shall obtain preserves pure states. We shall see in the following that more general situations are possible.

First of all, we recall that, if two operators $B,C\in M_n(\mathbb{C})$ are given, the *commutator* of B and C is defined by

$$\big[B,C\big] := BC-CB\,, \tag{4.10}$$

and their *anticommutator* by

$$\{B,C\} = BC+CB\,. \tag{4.11}$$

Proposition 4.2. *Under Assumption 2.3, the stochastic differential of the process* $\sigma=\{\sigma(t)\}_{t\in[u,T]}$ *is*

$$\mathrm{d}\sigma(t) = \mathcal{L}(t)\big[\sigma(t)\big]\mathrm{d}t + \sum_{j=1}^{d}\Big[R_j(t)\sigma(t)+\sigma(t)R_j^*(t)\Big]\mathrm{d}W_j(t)\,, \tag{4.12}$$

where $\mathcal{L}(t)[\cdot]$ *is the linear map of* $M_n(\mathbb{C})$ *into itself defined by*

$$\begin{aligned}\mathcal{L}(t)[\tau] &:= K(t)\tau+\tau K^*(t)+\sum_{j=1}^{d}R_j(t)\tau R_j^*(t)\\ &= -\mathrm{i}[H(t),\tau]+\frac{1}{2}\sum_{j=1}^{d}\big([R_j(t)\tau,R_j^*(t)]+[R_j(t),\tau R_j^*(t)]\big)\\ &= -\mathrm{i}[H(t),\tau]+\sum_{j=1}^{d}\left(R_j(t)\tau R_j^*(t)-\frac{1}{2}\{R_j^*(t)R_j(t),\tau\}\right)\,, \quad \forall\tau\in M_n(\mathbb{C})\end{aligned} \tag{4.13}$$

Proof. By Eq. (4.8), Itô formula for the products and the definition of the effective

Hamiltonian of the system in Assumption 2.2 we get

$$
\begin{aligned}
\mathrm{d}\sigma(t) &= \mathrm{d}\Big(A(t,u)\rho_0 A^*(t,u)\Big) = \mathrm{d}\Big(A(t,u)\rho_0\Big)A^*(t,u) + \Big(A(t,u)\rho_0\Big)\mathrm{d}A^*(t,u) \\
&+ \mathrm{d}A(t,u)\rho_0\mathrm{d}A^*(t,u) = \left[K(t)A(t,u)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)A(t,u)\mathrm{d}W_j(t)\right]\rho_0 A^*(t,u) \\
&+ A(t,u)\rho_0\left[A^*(t,u)K^*(t)\mathrm{d}t + \sum_{j=1}^{d} A^*(t,u)R_j^*(t)\mathrm{d}W_j(t)\right] \\
&+ \sum_{j=1}^{d} R_j(t)A(t,u)\rho_0 A^*(t,u)R_j^*(t)\mathrm{d}t \\
&= \left[K(t)\sigma(t) + \sigma(t)K^*(t) + \sum_{j=1}^{d} R_j(t)\sigma(t)R_j^*(t)\right]\mathrm{d}t \\
&+ \sum_{j=1}^{d}\Big[R_j(t)\sigma(t) + \sigma(t)R_j^*(t)\Big]\mathrm{d}W_j(t)\,.
\end{aligned}
$$

□

The equation

$$
\begin{cases}
\mathrm{d}\sigma(t) = \mathcal{L}(t)\big[\sigma(t)\big]\mathrm{d}t + \displaystyle\sum_{j=1}^{d}\Big[R_j(t)\sigma(t) + \sigma(t)R_j^*(t)\Big]\mathrm{d}W_j(t)\,, \qquad t \geq u \geq 0 \\[2ex]
\sigma(u) = \rho_0
\end{cases}
\tag{4.14}
$$

can be called *linear stochastic master equation.*

By Proposition 4.2 we can straightly conclude that there exists a solution of Eq. (4.14). Let us point out that we take $\sigma_u = \rho_0$. Then, this equation is the system evolution equation when the initial condition is a random statistical operator, or rather a mixture of pure states.

The following proposition holds.

Proposition 4.3. *Under Assumption 2.3 the solution of Eq.* (4.14) *is pathwise unique.*

Proof. To show this claim we shall use the propagator properties. Let $\zeta = \{\zeta(t)\}_{t\in[u,T]}$ be another solution of Eq. (4.14), with the same initial condition. If we set $\zeta(u) = \rho_0$ then

$$
\begin{aligned}
\mathrm{d}\zeta(t) = &\left[K(t)\zeta(t) + \zeta(t)K^*(t) + \sum_{j=1}^{d} R_j(t)\zeta(t)R_j^*(t)\right]\mathrm{d}t \\
&+ \sum_{j=1}^{d}\Big[R_j(t)\zeta(t) + \zeta(t)R_j^*(t)\Big]\mathrm{d}W_j(t)\,.
\end{aligned}
$$

The propagator $A(t,u)$ is almost surely invertible and so, by previous proposition, the process $\{A^{-1}(u,t)\zeta(t)\}_{t\in[u,T]}$ is almost surely well defined: let us compute its stochastic differential.

$$\begin{aligned}
\mathrm{d}\big(A^{-1}(u,t)\zeta(t)\big) &= (\mathrm{d}A^{-1}(u,t))\zeta(t) + A^{-1}(u,t)\mathrm{d}\zeta(t) + (\mathrm{d}A^{-1}(u,t))\mathrm{d}\zeta(t)\\
&= A^{-1}(t,u)\left[\sum_{j=1}^{d}\zeta(t)R_j(t)^2 - \zeta(t)K(t)\right]\mathrm{d}t - A^{-1}(t,u)\sum_{j=1}^{d}\zeta(t)R_j(t)\mathrm{d}W_j(t)\\
&\quad + A^{-1}(t,u)\left[K(t)\zeta(t) + \zeta(t)K^*(t) + \sum_{j=1}^{d}R_j(t)\zeta(t)R_j^*(t)\right]\mathrm{d}t\\
&\quad + A^{-1}(t,u)\sum_{j=1}^{d}\Big[R_j(t)\zeta(t) + \zeta(t)R_j^*(t)\Big]\mathrm{d}W_j(t)\\
&\quad - A^{-1}(t,u)\sum_{j=1}^{d}\Big[R_j(t)^2\zeta(t) + R_j(t)\zeta(t)R_j^*(t)\Big]\mathrm{d}t\\
&= A^{-1}(t,u)\zeta(t)K^*(t)\mathrm{d}t + A^{-1}(t,u)\sum_{j=1}^{d}\zeta(t)R_j^*(t)\mathrm{d}W_j(t).
\end{aligned}$$

The stochastic differential equation fulfilled by the inverse of the adjoint of the propagator has been given in Eq. (3.11). Then, the stochastic differential of the almost surely defined process $\{A^{-1}(u,t)\zeta(t)A^{-1^*}(t,u)\}_{t\in[u,T]}$ is

$$\begin{aligned}
\mathrm{d}\Big(A^{-1}(u,t)\zeta(t)A^{-1^*}(t,u)\Big) &= \Big(\mathrm{d}A^{-1}(u,t)\zeta(t)\Big)A^{-1^*}(t,u)\\
&\quad + \Big(A^{-1}(u,t)\zeta(t)\Big)\mathrm{d}A^{-1^*}(t,u) + \Big(\mathrm{d}A^{-1}(u,t)\zeta(t)\Big)\mathrm{d}A^{-1^*}(t,u)\\
&= A^{-1}(t,u)\left\{\zeta(t)K^*(t)\mathrm{d}t + \sum_{j=1}^{d}\zeta(t)R_j^*(t)\mathrm{d}W_j(t)\right\}A^{-1^*}(t,u) + A^{-1}(t,u)\\
&\quad\times\left\{\left[\sum_{j=1}^{d}\zeta(t)R_j^*(t)^2 - \zeta(t)K^*(t)\right]\mathrm{d}t - \sum_{j=1}^{d}\zeta(t)R_j^*(t)\mathrm{d}W_j(t)\right\}A^{-1^*}(t,u)\\
&\quad - A^{-1}(t,u)\sum_{j=1}^{d}\zeta(t)R_j^*(t)^2A^{-1^*}(t,u)\mathrm{d}t = 0.
\end{aligned}$$

So, we have obtained the relation

$$A^{-1}(t,u)\zeta(t)A^{-1^*}(t,u) \equiv \mathrm{const}.$$

By setting the initial conditions for the processes involved in this relation we have

$$A^{-1}(u,u)\zeta(u)A^{-1^*}(u,u) = \mathbb{1}\rho_0\mathbb{1} = \rho_0.$$

We can conclude that

$$A^{-1}(t,u)\zeta(t)A^{-1^*}(t,u) \equiv \rho_0.$$

By multiplying both sides of this equation, on the right hand side by $A^*(t,u)$ and on the other by $A(t,u)$, we obtain

$$\zeta(t) \equiv A(t,u)\rho_0 A^*(t,u) = \sigma(t)\,, \quad \mathbb{Q}\text{-a.s.}\,, \forall t \in [u,T]\,.$$

□

We said above that the stochastic differential equation (4.14) has the interpretation of evolution equation when the initial condition is a random statistical operator. By construction this equation admits a solution. Proposition 4.3 states that this solution is pathwise unique. Then, we have that Eq. (4.14) is a closed SDE for the process σ, regardless of the particular decomposition of ρ_0 in the form (4.6), with a pathwise unique solution. Therefore, ρ_0 acquires the meaning of initial state of the system: the construction of Chapters 2 and 3 can be reformulated in the language of statistical operators. On the other hand, we can consider a mixed initial state by replacing Assumption 3.1 with Assumption 4.1. We point out that all these considerations do not depend on the particular decomposition of the statistical operator ρ_0, but only on ρ_0 itself.

4.3 The mean equation

By Propositions 4.2 and 4.3 we have

$$\sigma(t) = \rho_0 + \int_u^t \mathcal{L}(s)[\sigma(s)]\mathrm{d}s + \sum_{j=1}^{d} \int_u^t \big[R_j(s)\sigma(s) + \sigma(s)R_j^*(s)\big]\mathrm{d}W_j(s). \tag{4.15}$$

Moreover, from Assumption 2.3 and Proposition 4.1 the integrating process in the stochastic integral (4.15) belongs to $\mathcal{M}^2$; then, this integral is a mean zero $(\mathcal{F}_t)$-martingale. If we define

$$\eta(t) := \mathbb{E}_{\mathbb{Q}}\big[\sigma(t)\big] \tag{4.16}$$
$$\eta_0 := \mathbb{E}_{\mathbb{Q}}\big[\rho_0\big] \tag{4.17}$$

and we calculate the expectation, with respect to the probability measure $\mathbb{Q}$, in (4.15) we obtain

$$\eta(t) = \eta_0 + \int_u^t \mathbb{E}_{\mathbb{Q}}\Big[\mathcal{L}(s)[\sigma(s)]\Big]\mathrm{d}s. \tag{4.18}$$

Remark 4.1. Let us observe that this equation is not closed for η. This is because of the randomness of $\mathcal{L}$, which comes out from the randomness of the coefficient processes K and $\{R_j\}_{j=1}^d$. When $\mathcal{L}$ is a deterministic operator, and the randomness of our system comes just from that one of the Wiener process or of the initial condition ρ_0, Eq. (4.18) becomes

$$\eta(t) = \eta_0 + \int_u^t \mathcal{L}(s)[\eta(s)]\mathrm{d}s. \tag{4.19}$$

The latter equation is known as *master equation*.

5

The stochastic master equation: an incomplete observation

5.1 The reduced observation

For simplicity, we assume now that the starting time is 0: we shall indicate with s a successive arbitrary time in $[0,T]$ and, as usual, the running time with t. In the following treatment the initial state is the statistical operator ρ_0 and it is considered at the time $t=0$.

In the statistical formulation it is possible to choose to observe not the whole output W but only some of its components, say the first m. We can mathematically model this situation by introducing the augmented natural filtration of the increments of the first m components of the Wiener process, that is the two-time filtration $\{\mathcal{E}_t^s\}_{0\le s\le t}$ such that

$$\mathcal{E}_t^s := \sigma\Big\{W_j(r)-W_j(s),\, r\in[s,t]\,,\ j=1,\dots,m\Big\}\vee\mathcal{N}\,; \tag{5.1}$$

$$\mathcal{E}^s := \bigvee_{t\ge s}\mathcal{E}_t^s\,;\quad \mathcal{E}:=\mathcal{E}^0\,. \tag{5.2}$$

Let us stress an important property of the filtration $\{\mathcal{E}_t^0\}_{t\ge0}$.

Proposition 5.1. *Let $(\Omega,\mathcal{F}_t,\mathbb{Q})$ be a probability space, where $\mathcal{F}_t$ is the element at time t of the reference filtration $\{\mathcal{F}_t\}_{t\ge0}$. Let X be an $\mathcal{F}_t$-measurable random variable, then*

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{E}_t^0]=\mathbb{E}_{\mathbb{Q}}[X|\mathcal{E}]\,. \tag{5.3}$$

Proof. In [2, Ex. 3.2, p. 60 and p. 285] it is shown that if Y is a random variable in a generic probability space $(\Omega,\mathcal{Z},\mathbb{P})$, $\mathcal{G}\subset\mathcal{Z}$ and $\mathcal{D}\subset\mathcal{Z}$ are other σ-algebras such that $\mathcal{D}$ is independent of $\sigma(X)\vee\mathcal{G}$, then

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{D}\vee\mathcal{G}]=\mathbb{E}_{\mathbb{Q}}[X|\mathcal{G}]\,.$$

By this result, it is enough to set $\mathcal{D}:=\mathcal{E}^t$ and $\mathcal{G}:=\mathcal{E}_t^0$ to obtain the statement. Indeed, by the property of the independence of the increments of the Wiener process,

$\mathcal{E}^t$ is independent of $\mathcal{F}_t$ and of $\mathcal{E}_t^0$. By the $\mathcal{F}_t$-measurability of X we can say that $\sigma(X) \subset \mathcal{F}_t$ and, so, that $\mathcal{E}^t$ is independent of $\sigma(X) \vee \mathcal{E}_t^0$. The thesis is obtained by observing that $\mathcal{E} = \mathcal{E}^t \vee \mathcal{E}_t^0$. □

To give the physical probabilities in this context it is sufficient to restrict the reference probability $\mathbb{Q}$ to the new filtration, i.e. we shall work in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q}|_{\mathcal{E}})$.

We define the process $\varrho = \{\varrho(t)\}_{t\geq 0}$ as

$$\varrho(t) := \mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{E}_t^0], \quad \forall t \in [0,T], \; \forall T > 0. \tag{5.4}$$

where $\{\sigma(t)\}_{t\geq 0}$ is the solution process of the stochastic master equation (4.14) when the starting time is $u = 0$. Then the following result holds.

Proposition 5.2. *The trace of the processes σ and ϱ is almost surely non zero, when $\rho_0(\omega) \in S(\mathcal{H})$, $\forall \omega \in \Omega$.*

Proof. The definition of σ through the propagator of the linear stochastic Schrödinger equation, starting from 0, is $\sigma(t) = A(t,0)\rho_0 A^*(t,0)$, $\rho_0(\omega) \in S(\mathcal{H})$. Then $\sigma(t,\omega)$ is a symmetric positive definite matrix. Furthermore one has

$$\mathrm{Tr}\{\sigma(t)\} = \mathrm{Tr}\{A(t,0)\rho_0 A^*(t,0)\}. \tag{5.5}$$

We saw in Chapter 1 that the norm-one of a positive definite operator equals its trace and, so,

$$\mathrm{Tr}\{\sigma(t)\} = \mathrm{Tr}\{A(t,0)\rho_0 A^*(t,0)\} = \|A(t,0)\rho_0 A^*(t,0)\|_1$$

which implies $\mathrm{Tr}\{A(t,0)\rho_0 A^*(t,0)\} = 0 \Leftrightarrow A(t,0)\rho_0 A^*(t,0) = 0$. By the propagator invertibility, we can say that this relation is fulfilled if and only if ρ_0 is the null matrix, which is not a statistical operator. So we have

$$\mathrm{Tr}\{A(t,0)\rho_0 A^*(t,0)\} \neq 0, \quad \forall t \in [0,T], \quad \mathbb{Q}\text{-a.s.}$$

Then, by Definition 5.4, we can conclude also that $\mathrm{Tr}\{\varrho(t)\}$ is almost surely non zero, because it turns out to be the conditional expectation of $\mathrm{Tr}\{\sigma(t)\}$ with respect to $\mathcal{E}_t^0$. □

By Proposition 5.2 we can define the quantity

$$\hat{\varrho}(t) := \frac{\varrho(t)}{\mathrm{Tr}\{\varrho(t)\}}, \quad \forall t \in [0,T], \; \mathbb{Q}\text{-a.s.} \tag{5.6}$$

Let us observe that Eq. (5.6) almost surely defines an $S(\mathcal{H})$-valued process. The random state $\hat{\varrho}(t)$ will represent the state of the system at time t, given the observation of the output up to this time. It is called *a posteriori state.*

5.2 The stochastic master equation

In this section we shall get a closed linear equation for the process ϱ. However, to reach this goal, we need some assumptions more for the $M_n(\mathbb{C})$-valued processes which are the coefficients of the equations we are discussing.

Under these further assumptions we shall obtain a linear equation for ϱ, in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q}\big|_{\mathcal{E}})$.

Assumption 5.1 ([4, Assumption p. 311])**.** The process $\{H(t)\}_{t\geq 0}$ and the processes $\{R_j(t)\}_{t\geq 0}$, $j = 1, \ldots, d$ are adapted to the filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$.

Let us define the process $\mathcal{R}_j = \{\mathcal{R}_j(t)\}_{t\geq 0}$, $j = 1, \ldots, d$, as

$$\mathcal{R}_j(t)[\tau] = R_j(t)\tau + \tau R_j^*(t)\,, \quad t \in [0, T]\,, j = 1, \ldots, d\,, \quad \tau \in M_n(\mathbb{C})\,. \tag{5.7}$$

We point out that the state space of these processes is the space of the linear maps acting on $M_n(\mathbb{C})$. With this notation the integral form of the stochastic master equation (4.14) becomes

$$\sigma(t) = \rho_0 + \int_0^t \mathcal{L}(s)[\sigma(s)]\mathrm{d}s + \sum_{j=1}^{d} \int_0^t \mathcal{R}_j(s)[\sigma(s)]\mathrm{d}W_j(s)\,.$$

5.2.1 The reduced linear stochastic master equation

We can show now that $\varrho(t)$ satisfies a closed equation.

Proposition 5.3. *Under Assumptions 2.3, 4.1, 5.1, the process $\{\varrho(t)\}_{t\geq 0}$, in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q}\big|_{\mathcal{E}})$, fulfills the linear stochastic equation*

$$\begin{cases} \mathrm{d}\varrho(t) = \mathcal{L}(t)[\varrho(t)]\mathrm{d}t + \displaystyle\sum_{j=1}^{m} \mathcal{R}_j(t)[\varrho(t)]\mathrm{d}W_j(t)\,, & t \geq 0 \\ \varrho(0) = \eta_0 := \mathbb{E}_{\mathbb{Q}}[\rho_0]. \end{cases} \tag{5.8}$$

Proof. Let us rewrite Eq. (4.14) as

$$\sigma(t) = \rho_0 + \int_0^t \mathcal{L}(q)[\sigma(q)]\mathrm{d}q + \sum_{j=1}^{m} \int_0^t \mathcal{R}_j(q)[\sigma(q)]\mathrm{d}W_j(q) + \sum_{j=m+1}^{d} \int_0^t \mathcal{R}_j(q)[\sigma(q)]\mathrm{d}W_j(q).$$

By Assumption 5.1, $\mathcal{L}(t)$ and $\mathcal{R}_j(t)$ are $\mathcal{E}_t^0$-measurable. By recalling the definition of $\varrho(t)$, the componentwise independence of the Wiener process, and Proposition 5.1

we have

$$
\begin{aligned}
\varrho(t) = \mathbb{E}_{\mathbb{Q}}[\sigma(t)\big|\mathcal{E}_t^0] &= \mathbb{E}_{\mathbb{Q}}[\rho_0\big|\mathcal{E}_t^0] \\
&+ \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \mathcal{L}(q)[\sigma(q)]\mathrm{d}q\bigg|\mathcal{E}_t^0\right] + \sum_{j=1}^m \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \mathcal{R}_j(q)[\sigma(q)]\mathrm{d}W_j(q)\bigg|\mathcal{E}_t^0\right] \\
&\qquad + \sum_{j=m+1}^d \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \mathcal{R}_j(q)[\sigma(q)]\mathrm{d}W_j(q)\bigg|\mathcal{E}_t^0\right] \\
&= \mathbb{E}_{\mathbb{Q}}[\rho_0] + \int_0^t \mathcal{L}(q)\left[\mathbb{E}_{\mathbb{Q}}[\sigma(q)\big|\mathcal{E}_q^0]\right]\mathrm{d}q + \sum_{j=1}^m \int_0^t \mathcal{R}_j(q)\left[\mathbb{E}_{\mathbb{Q}}[\sigma(q)\big|\mathcal{E}_q^0]\right]\mathrm{d}W_j(q) \\
&= \eta_0 + \int_0^t \mathcal{L}(q)\left[\varrho(q)\right]\mathrm{d}q + \sum_{j=1}^m \int_0^t \mathcal{R}_j(q)\left[\varrho(q)\right]\mathrm{d}W_j(q).
\end{aligned}
$$

To conclude it is enough to rewrite this relation in differential form. □

Proposition 5.4 ([3, Theorem 3.4, p. 55]). *Under Assumptions 2.3, 4.1 and 5.1, Eq. (5.8) admits a pathwise unique solution in the stochastic basis* $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q}\big|_{\mathcal{E}})$.

Proof. It is enough to prove that the processes $\mathcal{L}$ and $\{\mathcal{R}_j\}_j$ are uniformly bounded in t and ω, then the statement will come out of Theorem 2.3, paying attention to the dimension of this problem which now is $n \times n$ instead of n.

From the definition of $\mathcal{L}$ stated in (4.13) we have, $\forall \tau$ in $M_n(\mathbb{C})$,

$$
\begin{aligned}
\mathcal{L}(t)[\tau] = &-\mathrm{i}H(t)\tau - \mathrm{i}\tau H(t) \\
&+ \frac{1}{2}\sum_{j=1}^d \Big(2R_j(t)\tau R_j^*(t) - R_j(t)R_j^*(t)\tau - \tau R_j(t)R_j^*(t)\Big).
\end{aligned}
$$

Then, by using the properties of the norms which we have introduced in Chapter 1 and Assumption 2.3, we obtain

$$
\begin{aligned}
\|\mathcal{L}(t)[\tau]\|_2 &\leq 2\|H(t)\tau\|_2 + \Big\|\sum_{j=1}^d R_j(t)\tau R_j^*(t)\Big\|_2 + \Big\|\sum_{j=1}^d R_j(t)R^*(t)\tau\Big\|_2 \\
&\leq 2\|H(t)\tau\|_2 + \sum_{j=1}^d \|R_j(t)\tau R_j^*(t)\|_2 + \Big\|\sum_{j=1}^d R_j(t)R_j^*(t)\tau\Big\|_2 \\
&\leq 2\|H(t)\tau\|_2 + \sum_{j=1}^d \|R_j(t)\tau R_j^*(t)\|_2 + \Big\|\sum_{j=1}^d R_j(t)R_j^*(t)\tau\Big\|_2 \\
&\leq \left[2\|H(t)\| + \sum_{j=1}^d \|R_j(t)R_j^*(t)\| + \Big\|\sum_{j=1}^d R_j(t)R_j^*(t)\Big\|\right]\|\tau\|_2
\end{aligned}
$$

$$\leq \left[2\|H(t)\| + \sum_{j=1}^{d} \|R_j(t)\|^2 + \left\| \sum_{j=1}^{d} R_j(t) R_j^*(t) \right\| \right] \|\tau\|_2$$
$$\leq \big[2M(T) + (1+d)L(T)\big] \|\tau\|_2 .$$

If we set

$$\ell_T := \max\{M(T), L(T)\}$$

we obtain

$$\|\mathcal{L}(t)[\tau]\|_2 \leq (3+d)\ell_T \|\tau\|_2 , \quad \forall \omega \in \Omega,\ t \in [0,T],\ \tau \in M_n(\mathbb{C}). \tag{5.9}$$

In a similar way, we have

$$\sum_{j=1}^{m} \|\mathcal{R}_j(t)[\tau]\|_2^2 = \sum_{j=1}^{m} \|R_j(t)\tau + \tau R_j^*(t)\|_2^2 \leq 2 \sum_{j=1}^{m} \|R_j(t)\tau\|_2^2 + \|\tau R_j^*(t)\|_2^2$$
$$= 4 \sum_{j=1}^{m} \|R_j(t)\tau\|_2^2 \leq 4 \sum_{j=1}^{m} \|R_j(t)\| \|\tau\|_2^2 \leq 4m\ell_T \|\tau\|_2^2$$

and, so,

$$\sum_{j=1}^{m} \|\mathcal{R}_j(t)[\tau]\|_2^2 \leq 4m\ell_T \|\tau\|_2 , \quad \forall \omega \in \Omega,\ t \in [0,T],\ \tau \in M_n(\mathbb{C}). \tag{5.10}$$

□

Equation (5.8) is still called *linear stochastic master equation.*

5.2.2 The process trace-of-the-solution

We showed that in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0]}, \mathbb{Q}\big|_{\mathcal{E}})$, the linear stochastic master equation (5.8) admits a pathwise unique solution; we are going to show now that in the same stochastic basis the trace process $\big\{\mathrm{Tr}\{\varrho(t)\}\big\}_{t\geq 0}$ is a mean one martingale.

First we introduce the quantities

$$v_j(t) := \mathrm{Tr}\{(R_j(t) + R_j^*(t))\hat{\varrho}(t)\} = 2\,\mathrm{Re}\,\mathrm{Tr}\{R_j(t)\hat{\varrho}(t)\}\,, \quad j = 1, \ldots, m\,, \tag{5.11}$$

where the a posteriori state $\hat{\varrho}(t)$ is defined in Eq. (5.6)

Remark 5.1. Let us observe that

1. $\mathrm{Tr}\{\mathcal{L}(t)[\tau]\} \equiv 0\,, \forall t \in [0,T],\ \forall \tau \in M_n(\mathbb{C})$. This statement follows easily from the structure of $\mathcal{L}(t)$ and the cyclic property of the trace.

2. $v_j(t) \leq 2\sqrt{L(T)}$. Indeed, by reminding Assumption 2.3 and that $\hat{\varrho}(t,\omega) \in S(\mathcal{H})$ a.s., we have

$$v_j(t) = \mathrm{Tr}\{(R_j(t) + R_j^*(t))\hat{\varrho}(t)\} \leq \big|\mathrm{Tr}\{(R_j(t) + R_j^*(t))\hat{\varrho}(t)\}\big|$$
$$\leq \|R_j(t) + R_j^*(t)\| \|\hat{\varrho}(t)\|_1$$
$$\leq 2\sqrt{L(T)} \|\hat{\varrho}(t)\|_1 = 2\sqrt{L(T)}\,\mathrm{Tr}\{\hat{\varrho}(t)\} = 2\sqrt{L(T)}.$$

We can state the results about $\{\mathrm{Tr}\{\varrho(t)\}\}_{t\geq 0}$.

Proposition 5.5 ([3, Theorem 3.4, p. 55], [4, p. 311-312])**.** *Let* $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q}\big|_{\mathcal{E}})$ *a stochastic basis and suppose that Assumptions 2.3, 4.1 and 5.1 hold. Then, the trace process* $\{\mathrm{Tr}\{\varrho(t)\}\}_{t\geq 0}$ *is a positive mean-one square integrable martingale and it fulfils the Doléans equation*

$$\begin{cases} \mathrm{d}\,\mathrm{Tr}\{\varrho(t)\} = \mathrm{Tr}\{\varrho(t)\} \displaystyle\sum_{j=1}^{m} v_j(t)\mathrm{d}W_j(t), \quad t \geq 0, \\ \\ \mathrm{Tr}\{\varrho(0)\} = 1\,, \end{cases} \tag{5.12}$$

which gives

$$\mathrm{Tr}\{\varrho(t)\} = \exp\left\{ \sum_{j=1}^{m} \int_0^t v_j(q)\mathrm{d}W_j(q) - \frac{1}{2}\int_0^t v_j(q)^2 \mathrm{d}q \right\}. \tag{5.13}$$

Proof. First, we have

$$\begin{aligned}
\mathrm{Tr}\{\varrho(t)\} &= \mathrm{Tr}\left\{ \eta_0 + \int_0^t \mathcal{L}(q)\,[\varrho(q)]\,\mathrm{d}q + \sum_{j=1}^{m}\int_0^t \mathcal{R}_j(q)\,[\varrho(q)]\,\mathrm{d}W_j(q) \right\} \\
&= 1 + \sum_{j=1}^{m}\int_0^t \mathrm{Tr}\left\{\mathcal{R}_j(q)\,[\varrho(q)]\right\}\mathrm{d}W_j(q) \\
&= 1 + \sum_{j=1}^{m}\int_0^t \mathrm{Tr}\left\{(R_j(q) + R_j^*(q))\varrho(q)\right\}\mathrm{d}W_j(q) \\
&= 1 + \sum_{j=1}^{m}\int_0^t \mathrm{Tr}\{\varrho(q)\}\,\mathrm{Tr}\left\{(R_j(q) + R_j^*(q))\frac{\varrho(q)}{\mathrm{Tr}\{\varrho(q)\}}\right\}\mathrm{d}W_j(q) \\
&= 1 + \sum_{j=1}^{m}\int_0^t \mathrm{Tr}\{\varrho(q)\}v_j(q)\mathrm{d}W_j(q).
\end{aligned}$$

We can rewrite this relation in the differential form (5.12). Then, we have a Doléans equation for the trace process whose solution is well known and of the form (5.13). We end up with the thesis by applying Theorem 3.5. □

We are allowed to use the process $\mathrm{Tr}\{\varrho\}$ as a probability density to define the following probability measure equivalent to $\mathbb{Q}\big|_{\mathcal{E}_t^0}$:

$$\mathbb{P}_{\eta_0}^t(E) := \mathbb{E}_{\mathbb{Q}}\left[1_E\,\mathrm{Tr}\{\varrho(t)\}\right], \quad \forall E \in \mathcal{E}_t^0. \tag{5.14}$$

We write $\mathbb{E}_{\eta_0}^t$ for the expectation with respect this probability measure. The probability family $\left\{\mathbb{P}_{\eta_0}^t,\ t\in[0,T]\right\}$ is consistent in the following sense

$$0 < s < t,\ E \in \mathcal{E}_s^0 \ \Rightarrow\ \mathbb{P}_{\eta_0}^t(E) = \mathbb{P}_{\eta_0}^s(E)\,;$$

also in this case, this is a straightforward consequence of the martingale property of the trace process.

Let $\hat{\sigma}$ be the normalization of the solution of the linear stochastic master equation (4.14) with respect to its trace, or in other words the almost surely defined process

$$\hat{\sigma}(t) := \frac{\sigma(t)}{\mathrm{Tr}\{\sigma(t)\}}\,, \quad t \in [0,T]\,,\ \mathbb{Q}\text{-a.s.} \tag{5.15}$$

It is possible to prove a relation between the process $\hat{\varrho}$ and $\hat{\sigma}$ as given here below.

Proposition 5.6 ([4, p. 310])**.** *Given $\hat{\varrho}(t)$ and $\hat{\sigma}(t)$ as defined in Eqs.* (5.6) *and* (5.15) *respectively, we have*

$$\hat{\varrho}(t) = \mathbb{E}^t_{\eta_0}[\hat{\sigma}(t)\big|\mathcal{E}^0_t].$$

Proof. Let X be a $\mathcal{E}^0_t$-measurable random variable. By Eq. (5.14) and the properties of the conditional expectations we have

$$\begin{aligned}\mathbb{E}^t_{\eta_0}[X\,\hat{\varrho}(t)] &= \mathbb{E}_{\mathbb{Q}}[\mathrm{Tr}\{\varrho(t)\}X\,\hat{\varrho}(t)]\\ &= \mathbb{E}_{\mathbb{Q}}[X\,\varrho(t)] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\sigma(t)\big|\mathcal{E}^0_t]X] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[X\sigma(t)\big|\mathcal{E}^0_t]] = \mathbb{E}_{\mathbb{Q}}[X\sigma(t)]\\ &= \mathbb{E}_{\mathbb{Q}}\left[X\,\mathrm{Tr}\{\sigma(t)\}\frac{\sigma(t)}{\mathrm{Tr}\{\sigma(t)\}}\right] = \mathbb{E}^t_{\eta_0}[X\hat{\sigma}(t)].\end{aligned}$$

□

Remark 5.2. Let us point out that in the proof of the previous result we have used the equality

$$\mathbb{E}_{\mathbb{Q}}[1_E\sigma(t)] = \mathbb{E}_{\mathbb{Q}}[1_E\varrho(t)]\,, \quad \forall E \in \mathcal{E}^0_t\,.$$

This relation is a direct consequence of Eq. (5.4) and of the properties of the conditional expectations.

We end this section with the following observation. A direct consequence of Theorem 3.7 is that the process $\widehat{W} := (\widehat{W}_1, \ldots, \widehat{W}_m)$, componentwise defined by setting

$$\widehat{W}_j(t) := W_j(t) - \int_0^t v_j(q)\mathrm{d}q\,, \quad j = 1, \ldots, m\,, \tag{5.16}$$

is a Wiener process in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}^0_t\}_{t\geq 0}, \mathbb{Q}\big|_{\mathcal{E}})$.

5.3 The non linear equation for $\hat{\varrho}$

In this section we want to give the stochastic differential equation fulfilled by the process $\hat{\varrho}$: this is a non linear equation and it is given in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}^0_t\}_{t\geq 0}, \mathbb{P}^T_{\eta_0})$.

Proposition 5.7. *Let Assumptions 2.3, 4.1 and 5.1 hold. Then, in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}^0_t\}_{t\geq 0}, \mathbb{P}^T_{\eta_0})$, the process $\{\hat{\varrho}(t)\}_{t\geq 0}$ satisfies the stochastic differential equation*

$$\begin{cases}\mathrm{d}\hat{\varrho}(t) = \mathcal{L}(t)[\hat{\varrho}(t)]\mathrm{d}t + \displaystyle\sum_{j=1}^m \{\mathcal{R}_j(t)[\hat{\varrho}(t)] - v_j(t)\hat{\varrho}(t)\}\,\mathrm{d}\widehat{W}_j(t)\,, & t \geq 0\\ \hat{\varrho}(0) = \eta_0\,, & \end{cases} \tag{5.17}$$

where the processes $v_j(t)$, $j = 1, \ldots, m$, *depend on* $\hat{\varrho}(t)$ *and are defined by Eq.* (5.11).

Proof. By Eq. (5.16), we can write the component W_j of the process $W = (W_1, \ldots, W_m)$ as

$$W_j(t) = \widehat{W}_j(t) + \int_0^t v_j(q)\mathrm{d}s\,, \quad j = 1, \ldots, m\,,$$

where $\widehat{W}$ is a Wiener process in the basis $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{P}_{\eta_0}^T)$. In this basis the stochastic differential of ϱ is obtained by replacing the expression of W with respect to $\widehat{W}$ in Eq. (5.8), that is

$$\mathrm{d}\varrho(t) = \left\{\mathcal{L}(t)[\varrho(t)] + \sum_{j=1}^m \mathcal{R}_j(t)[\varrho(t)]\right\}\mathrm{d}t + \sum_{j=1}^m \mathcal{R}_j(t)[\varrho(t)]\mathrm{d}\widehat{W}_j(t). \tag{5.18}$$

By Eq. (5.13) we have

$$\mathrm{Tr}\{\varrho(t)\}^{-1} = \exp\left\{-\frac{1}{2}\sum_{j=1}^m \int_0^t v_j(q)^2\mathrm{d}q - \sum_{j=1}^m \int_u^t v_j(q)\mathrm{d}\widehat{W}_j(q)\right\}. \tag{5.19}$$

We calculate the stochastic differential of $\mathrm{Tr}\{\varrho\}^{-1}$ from Eq. (5.19):

$$\mathrm{d}\,\mathrm{Tr}\{\varrho(t)\}^{-1} = -\,\mathrm{Tr}\{\varrho(t)\}^{-1}\sum_{j=1}^m v_j(t)\mathrm{d}\widehat{W}_j(t).$$

We end up with the differential of $\hat{\varrho}$ by applying the Itô formula for products:

$$\begin{aligned}
\mathrm{d}\hat{\varrho}(t) &= \mathrm{d}\left[\mathrm{Tr}\{\varrho(t)\}^{-1}\varrho(t)\right] \\
&= \left(\mathrm{d}\,\mathrm{Tr}\{\varrho(t)\}^{-1}\right)\varrho(t) + \mathrm{Tr}\{\varrho(t)\}^{-1}\left(\mathrm{d}\varrho(t)\right) + \left(\mathrm{d}\,\mathrm{Tr}\{\varrho(t)\}^{-1}\right)\left(\mathrm{d}\varrho(t)\right) \\
&= -\hat{\varrho}(t)\sum_{j=1}^m v_j(t)\mathrm{d}\widehat{W}_j(t) + \mathcal{L}(t)[\hat{\varrho}(t)]\mathrm{d}t + \sum_{j=1}^m v_j(t)\mathcal{R}_j[\hat{\varrho}(t)]\mathrm{d}t \\
&\qquad - \sum_{j=1}^m v_j(t)\mathcal{R}_j[\hat{\varrho}(t)]\mathrm{d}t + \sum_{j=1}^m \mathcal{R}_j[\hat{\varrho}(t)]\mathrm{d}t \\
&= \mathcal{L}(t)[\hat{\varrho}(t)]\mathrm{d}t + \sum_{j=1}^m \left\{\mathcal{R}_j(t)[\hat{\varrho}(t)] - v_j(t)\hat{\varrho}(t)\right\}\mathrm{d}\widehat{W}_j(t)\,.
\end{aligned}$$

The statement is obtained by observing that

$$\hat{\varrho}(0) = \frac{\varrho(0)}{\mathrm{Tr}\{\varrho(0)\}} = \frac{\eta_0}{\mathrm{Tr}\{\eta_0\}} = \eta_0\,.$$

□

5.4 The propagator of the linear stochastic master equation

In Section 5.2.1 we showed that the process ϱ satisfies the linear stochastic differential equation (5.8). As in the case of the Hilbert space and of the linear stochastic Schrödinger equation, we want to introduce the propagator related to equation (5.8). It is useful to introduce the propagator by setting $s \geq 0$ as the initial time of the system. In this context the propagator which we are going to define is a process whose states are linear maps of $M_n(\mathbb{C})$ into itself.

Let us consider the linear stochastic differential equation for a process whose states are linear maps of $M_n(\mathbb{C})$ into itself

$$\begin{cases} \Lambda(t,s) = \mathcal{L}(t) \circ \Lambda(t,s)\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t) \circ \Lambda(t,s)\mathrm{d}W_j(t)\,, \quad t \geq s \geq 0 \\ \\ \Lambda(s,s) = \mathrm{Id}_n\,, \end{cases} \tag{5.20}$$

where Id_n is the identity map on $M_n(\mathbb{C})$, i.e.

$$\mathrm{Id}_n[\tau] = \tau\,, \quad \forall \tau \in M_n(\mathbb{C})\,.$$

We can state the following result.

Theorem 5.8. *Let Assumptions 2.3, 4.1 and 5.1 hold. Then, Eq.* (5.20) *has a pathwise unique solution in the stochastic basis* $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q}|_{\mathcal{E}})$.

Proof. The proof of this result is similar to that one of Theorem 3.1, but now we are working with a map-valued process instead of a matrix-valued one.

Let $\{e_i\}_{i=1}^n \subseteq \mathcal{H}$ be the canonical basis in $\mathcal{H}$, that is $e_{ik} = \delta_{ik}\,,\ i = 1, \ldots, n\,;\ k = 1, \ldots, n$, where δ_{ik} is the Kronecker symbol. It is easy to show that $\{|e_i\rangle\langle e_k|\}_{i,k=1,\ldots,n}$ is a basis for $M_n(\mathbb{C})$. From Proposition 5.4 the SDE

$$\begin{cases} \mathrm{d}\varrho^{ik}(t) = \mathcal{L}(t)\big[\varrho^{ik}(t)\big]\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)[\varrho^{ik}(t)]\mathrm{d}W_j(t)\,, \qquad t \geq s \geq 0 \\ \varrho^{ik}(s) = |e_i\rangle\langle e_k| \end{cases} \tag{5.21}$$

has a pathwise unique solution; then, we can define, in an unique way, the process

$$\Lambda(t,s)\big[|e_i\rangle\langle e_k|\big] := \varrho^{ik}(t)\,, \quad \forall i,k = 1, \ldots, n\,, \quad 0 \leq s \leq t\,, \tag{5.22}$$

where ϱ^{ik} is the solution of Eq. (5.21), starting from $s \geq 0$. Then, we have

$$\Lambda(t,s)\big[|e_i\rangle\langle e_k|\big] = |e_i\rangle\langle e_k| + \int_s^t \mathcal{L}(q)\big[\varrho^{ik}(q)\big]\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q)\big[\varrho^{ik}(q)\big]\mathrm{d}W_j(q)$$

$$= |e_i\rangle\langle e_k| + \int_s^t \mathcal{L}(q)\Big[\Lambda(q,s)\big[|e_i\rangle\langle e_k|\big]\Big]\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q)\Big[\Lambda(q,s)\big[|e_i\rangle\langle e_k|\big]\Big]\mathrm{d}W_j(q)$$

$$= |e_i\rangle\langle e_k| + \int_s^t \mathcal{L}(q)\circ\Lambda(q,s)\big[|e_i\rangle\langle e_k|\big]\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q)\circ\Lambda(q,s)\big[|e_i\rangle\langle e_k|\big]\mathrm{d}W_j(q)$$

$$= \left(\mathrm{Id}_n + \int_s^t \mathcal{L}(q)\circ\Lambda(q,s)\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q)\circ\Lambda(q,s)\mathrm{d}W_j(q)\right)\big[|e_i\rangle\langle e_k|\big].$$

The uniqueness is a trivial consequence of Eq. (5.22): $\Lambda(t,s)$ is defined in an unique way on a basis of $M_n(\mathbb{C})$. □

Thanks to the previous result, we can give the following definition.

Definition 5.1. We call *propagator* of the linear stochastic master equation (5.8) the two-time process $\Lambda = \{\Lambda(t,s)\}$, $0 \leq s \leq t$.

Remark 5.3. Of course, $\Lambda(t,0)$ represents the linear application $\eta_0 \mapsto \varrho(t)$ and it is well defined because of the linear structure of Eq. (5.8), independently of the existence and uniqueness of the solution of Eq. (5.20).

We gather in the following proposition the main properties of the process Λ.

Proposition 5.9. *Let Assumptions 2.3, 4.1 and 5.1 hold and Λ be the solution of Eq.* (5.20)*. We have*

1. $\varrho(t) = \Lambda(t,0)[\eta_0]$;
2. $\Lambda(t,r) = \Lambda(t,s)\circ\Lambda(s,r)\,,\quad t \geq s \geq r \geq 0\,;$
3. $\Lambda(t,s)[\tau] = \mathbb{E}_{\mathbb{Q}}\big[A(t,s)\tau A^*(t,s)\big|\mathcal{E}_t^0\big]$, $\forall \tau \in M_n(\mathbb{C})$, *$\mathbb{Q}$-a.s., where $A(t,s)$ is the propagator of the linear stochastic Schrödinger equation* (2.1)*. Moreover $\Lambda(t,s)$ is $\mathcal{E}_t^0$-measurable.*

Proof.

1. We have

$$\Lambda(t,0)[\eta_0] = \eta_0 + \int_0^t \mathcal{L}(q)\Big[\Lambda(q,0)[\rho_0]\Big]\mathrm{d}q + \sum_{j=1}^m \int_0^t \mathcal{R}_j(q)\Big[\Lambda(q,0)[\rho_0]\Big]\mathrm{d}W_j(q).$$

This is the same equation fulfilled by ϱ: then we obtain the thesis by the uniqueness of the solution of this equation.

2. The proof of this claim is the same of that of Proposition 3.2.
   We choose $\tau \in M_n(\mathbb{C})$ and we define the process
$$\mathcal{B}(t)[\tau] := \begin{cases} \Lambda(t,s)[\tau]\,, & s \le t < r \\ \Lambda(t,r) \circ \Lambda(r,s)[\tau]\,, & t \ge r > s\,. \end{cases}$$
   We state now $t < r$.
$$\mathcal{B}(t)[\tau] = \Lambda(t,s)[\tau] = \tau + \int_s^t \mathcal{L}(q) \circ \Lambda(q,s)[\tau]\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q) \circ \Lambda(q,s)[\tau]\mathrm{d}W_j(q)$$
$$= \tau + \int_s^t \mathcal{L}(q) \circ \mathcal{B}(q)[\tau]\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q) \circ \mathcal{B}(q)[\tau]\mathrm{d}W_j(q)\,.$$
   We state now $t \ge r$.
$$\mathcal{B}(t)[\tau] = \Lambda(t,r) \circ \Lambda(r,s)[\tau] = \Lambda(t,r)\Big[\Lambda(r,s)[\tau]\Big]$$
$$= \Lambda(r,s)[\tau] + \int_r^t \mathcal{L}(q) \circ \Lambda(q,r)\Big[\Lambda(r,s)[\tau]\Big]\mathrm{d}q$$
$$+ \sum_{j=1}^m \int_r^t \mathcal{R}_j(q) \circ \Lambda(q,r)\Big[\Lambda(r,s)[\tau]\Big]\mathrm{d}W_j(q)$$
$$= \tau + \int_s^r \mathcal{L}(q) \circ \Lambda(q,s)[\tau]\mathrm{d}q + \sum_{j=1}^m \int_s^r \mathcal{R}_j(q) \circ \Lambda(q,s)[\tau]\mathrm{d}W_j(q)$$
$$+ \int_r^t \mathcal{L}(q) \circ \Lambda(q,r) \circ \Lambda(r,s)[\tau]\mathrm{d}q + \sum_{j=1}^m \int_r^t \mathcal{R}_j(q) \circ \Lambda(q,r) \circ \Lambda(r,s)[\tau]\mathrm{d}W_j(q)$$
$$= \tau + \int_s^t \mathcal{L}(q) \circ \mathcal{B}(q)[\tau]\mathrm{d}q + \sum_{j=1}^m \int_s^t \mathcal{R}_j(q) \circ \mathcal{B}(q)[\tau]\mathrm{d}W_j(q)\,.$$
   Then $\mathcal{B}(t)$ and $\Lambda(t,s)$ satisfy the same equation for all t: by uniqueness they are equal almost surely.

3. By Proposition 5.1, the Itô formula for products, Eqs. (3.1) and (3.12) and Assumption 5.1 we have
$$\mathrm{d}\left(\mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}_t^0]\right) = \mathrm{d}\left(\mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}]\right)$$
$$= \mathbb{E}_{\mathbb{Q}}[\mathrm{d}A(t,s)(\tau A^*(t,s))|\mathcal{E}] + \mathbb{E}_{\mathbb{Q}}[(A(t,s)\tau)\mathrm{d}A^*(t,s)|\mathcal{E}] + \mathbb{E}_{\mathbb{Q}}[\mathrm{d}A(t,s)\tau\mathrm{d}A^*(t,s)|\mathcal{E}]$$
$$= \mathbb{E}_{\mathbb{Q}}\Big[\mathcal{L}(t)[A(t,s)\tau A^*(t,s)]\Big|\mathcal{E}\Big]\mathrm{d}t + \sum_j \mathbb{E}_{\mathbb{Q}}\Big[\mathcal{R}_j(t)[A(t,s)\tau A^*(t,s)]\Big|\mathcal{E}\Big]\mathrm{d}W_j(t)$$
$$= \mathcal{L}(t) \circ \mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}]\mathrm{d}t + \sum_j \mathcal{R}_j(t) \circ \mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}]\mathrm{d}W_j(t)$$
$$= \mathcal{L}(t) \circ \mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}_t^0]\mathrm{d}t + \sum_j \mathcal{R}_j(t) \circ \mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}_t^0]\mathrm{d}W_j(t)\,.$$

Then the processes $\mathbb{E}_{\mathbb{Q}}[A(t,s)\tau A^*(t,s)|\mathcal{E}_t^0]$ and $\Lambda(t,s)[\tau]$ fulfill the same equation, $\forall\tau \in M_n(\mathbb{C})$. Finally, the statement on the measurability is a direct consequence of the properties of conditional expectations.

$\square$

6

Continuous measurement and instruments

6.1 Physical interpretation

Let us consider a quantum system on which we want to perform some measuring procedures. In Chapter 4 we assumed that the initial state is a statistical operator: this hypothesis allows us to mathematically model the physical situation in which there is uncertainty on the initial state of the system, because of some preparing procedure of the quantum system itself. Then, we obtained the linear stochastic master equation (4.18) and the non random master equation .

By using the formalism of statistical operators, we can insert dissipative effects in the model. The considerations about this phenomenon have been carried out in Chapter 5, where we have chosen the form of the observed process by assuming that the output of the measurement is a function of the first $m \leq d$ components of the Wiener process W, and not of the other $m-d$ ones. Then, the $m-d$ ignored components are those representing dissipative phenomena, due to the interaction of the system with the external world, which are not observed.

To mathematically cope with this situation we introduced the filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$, the natural augmented filtration of the first m components of the Wiener process W. Then, by considering the conditional expectation of the process σ, defined in Eq. (4.7), with respect to this filtration we introduced the process ϱ in Eq. (5.4). Moreover, we introduced the process $\hat{\varrho}$ as the normalisation of the process ϱ with respect to its trace. The state space of the latter process is the space $S(\mathcal{H})$ of the statistical operators: we can think to $\hat{\varrho}$ as the process of the *a posteriori* states of the system, that is $\hat{\varrho}(t)$ is the state of the system once the measuring experiment has been carried out on it, supposing that the trajectory of the output up to the time t has been observed.

By the way, if we want to obtain a closed stochastic differential equation for the process ϱ, as we did for the process σ, we must state some further assumptions. Indeed, the introduced quantities feel the effect of the randomness of the coefficients H and $\{R_j\}_{j=1}^d$. These are physical assumptions and they consist of the hypothesis that the dissipative phenomena do not directly influence the output of the measurement and that do not introduce memory effects, that is they are of Markov type. This physical hypothesis is inserted in the mathematical model by means of Assumption

5.1, that is by requiring that the coefficients of the SDEs we are considering are adapted to the filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$. This further assumption enables us to state the closed equation (5.8), which can be again interpreted as a linear stochastic master equation. Let us stress that the main difference between the two linear stochastic master equations we have derived is that while Eq. (4.14) preserves pure states, Eq. (5.8) does not have this property.

Furthermore, we showed that the process $\mathrm{Tr}\{\varrho\}$ is a positive martingale with mean one, with respect to the filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$. Then, this process can be used as a density process with respect to the reference probability. The new probability is defined in Eq. (5.14) and we interpret it as the physical probability of the events which are determined by the output of the system.

By means of Eq. (5.16), we observed that the component W_j, $j \leq m$, of the Wiener process decomposes, with respect to the physical probabilities, in two different terms: the first one is completely unpredictable (the Wiener process $\widehat{W_j}$), the second one is a bounded trajectory stochastic process, depending on $\hat{\varrho}$ (the process $\int_0^t v_j(s)\mathrm{d}s$). Although it would be natural to interpret this decomposition of the signal W_j, under the physical probabilities, as an observed process plus a measuring noise, typically the two terms are not independent. Let us observe that, under the physical probability, the process W_j is not a Wiener process, because of the presence of the therm v_j.

Finally, we stated a non-linear closed stochastic differential equation for the process $\hat{\varrho}$, which can be interpreted as the evolution of the a posteriori state of the system under the physical probability.

6.2 Instruments

6.2.1 Introduction

In this section, in which we shall refer to [3, Appendix B] and to [4], we want to give the interpretation of the *a priori* state (the state of the system in the case of a measuring procedure in which the outcome of the experiment is not observed) and of the *a posteriori* state (the state that one can assign to the system as soon after the measuring procedure, once the outcome of the experiment has been observed).

The *instruments* are the mathematical objects which allow us to represent a measurement on a quantum physical system. If the initial state of the system is known, the instruments permit to give the probabilities for the possible outcome of the observation and the state of the system conditioned by the result of the measurement itself. Moreover, instruments can represent instantaneous observations or observations which have some temporal duration.

It is well known that every measuring procedure carried out on a quantum system causes a change of its state. For this reason, if we want to put into effect another measuring procedure after the first one, we must know the state of the system after the first measurement, conditioned by its outcome. In other words, it is necessary to give the transformation of the *pre-measurement* state into the *post-measurement* one, conditioned by an arbitrary event which can occur in the experiment, in both the cases in which the occurrence is observed or not.

Before to formally introduce the instruments, we give some preliminary notions.

Definition 6.1. Let $\mathcal{A}$ be a linear map of $M_n(\mathbb{C})$ into itself. We shall say that $\mathcal{A}$ is *completely positive* if, for every natural number m and for every choice of the vectors $\{\psi_i\}_{i=1}^m$ in $\mathcal{H}$ and of the matrices $\{\tau_i\}_{i=1}^m$ in $M_n(\mathbb{C})$, one has

$$\sum_{i,j=1}^{m} \langle \psi_i | \mathcal{A}[\tau_i^* \tau_j] \psi_j \rangle \geq 0 . \tag{6.1}$$

Definition 6.2. Let $\mathcal{A}$ be a linear map of $M_n(\mathbb{C})$ into itself. The *adjoint map* of $\mathcal{A}$ is the linear map $\mathcal{A}^*$ such that

$$\mathrm{Tr}\big\{\mathcal{A}[A]B\big\} = \mathrm{Tr}\big\{A\mathcal{A}^*[B]\big\} , \quad \forall A , B \in M_n(\mathbb{C}) . \tag{6.2}$$

6.2.2 Observables

A measurement on a quantum system can produce different results with some probability distribution, depending on the state ρ of the system itself. Generally, the possible outcomes of an experiment are represented by a set Ω and the possible events are the elements of a σ-algebra $\mathcal{F}$ of subsets of Ω. We can also think the random outcomes of the experiment as the values of an $(\Omega, \mathcal{F})$-valued observable. The observables of a quantum system are represented by positive operator-valued measures on some measurable space $(\Omega, \mathcal{F})$, the *value space*.

Definition 6.3. Let $(\Omega, \mathcal{F})$ be a measurable space; a *positive operator-valued measure* with value space $(\Omega, \mathcal{F})$ is a map E from $\mathcal{F}$ into the set of positive operators such that it is normalised and σ-additive, i.e.

1. $E(F) \geq 0 , \quad \forall F \in \mathcal{F}$;
2. $E(\Omega) = \mathbb{1}$;
3. $E\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} E\left(F_k\right) , \quad \forall \{F_i\}_{i\in\mathbb{N}} \subset \mathcal{F} : \; F_i \cap F_j = \emptyset , \; \forall i \neq j .$

Let us point out that if we consider a quantum system in the state $\rho \in S(\mathcal{H})$, the conditional probabilities that the observed physical quantity takes values into $F \in \mathcal{F}$ is

$$\mathbb{P}_\rho(F) = \mathrm{Tr}\{\rho E(F)\} . \tag{6.3}$$

6.2.3 Observables and Instruments

If we want to perform more than one measurement on a quantum system, its initial state ρ and the POM associated to the first measurement alone do not give a description exhaustive enough of the first measurement: they give the probability distribution of the first outcome, but not the system state after the first measurement, conditioned on the response, which is needed to evaluate the conditional probability distribution of the outcome of a second arbitrary measurement. If we want the system state after the measurement of an observable of the quantum system, then the notion of *instrument* enters into play. An instrument gives both the probability distribution of the observable and the state change due to the measurement.

Definition 6.4 (Instrument)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. An *instrument* $\mathcal{I}$, with value space $(\Omega, \mathcal{F})$, is a normalised completely positive-linear-map-valued measure, i.e.

1. $\mathcal{I}(F)$ is a completely positive linear map from $M_n(\mathbb{C})$ into itself, $\forall F \in \mathcal{F}$;
2. (normalisation) $\mathrm{Tr}\{\mathcal{I}(\Omega)[\tau]\} = \mathrm{Tr}\{\tau\}$, $\forall \tau \in M_n(\mathbb{C})$;
3. (σ-additivity) for every countable family $\{F_i\}_{i\in\mathbb{N}}$ of disjoint sets in $\mathcal{F}$

$$\mathcal{I}\left(\bigcup_i F_i\right) = \sum_i \mathcal{I}(F_i)\,.$$

Remark 6.1. From Definitions 6.4 and 6.3, the map

$$E_{\mathcal{I}}(F) := \mathcal{I}^*(F)[\mathbb{1}] \tag{6.4}$$

turns out to be a POM, whose interpretation is that of the observable associated with the instrument $\mathcal{I}$.

Let ρ be the pre-measurement state of the system. The probability that the observable associated to the instrument $\mathcal{I}$ takes values in $F \in \mathcal{F}$ is

$$\mathbb{P}_\rho(F) = \mathrm{Tr}\{\mathcal{I}(F)[\rho]\} = \mathrm{Tr}\{\rho\mathcal{I}^*(F)[\mathbb{1}]\} = \mathrm{Tr}\{\rho E_{\mathcal{I}}(F)\}\,. \tag{6.5}$$

The post-measurement state, conditioned by the event that the observable associated with $\mathcal{I}$ takes values in $F \in \mathcal{F}$ is

$$\varrho(F) = \frac{\mathcal{I}(F)[\rho]}{\mathbb{P}_\rho(F)}\,, \tag{6.6}$$

which is a statistical operator, obviously defined only if $\mathbb{P}_\rho(F) > 0$.

6.2.4 Definition of *a priori* and *a posteriori* states

If in Eq. (6.5) we set $F = \Omega$, we have $\mathbb{P}_\rho(\Omega) = 1$. This implies that the conditional state (6.6) is

$$\varrho(\Omega) = \mathcal{I}(\Omega)[\rho]\,. \tag{6.7}$$

We interpret (6.7) as the *a priori state* of the system: if we know the pre-measurement state ρ and the instrument $\mathcal{I}$, $\varrho(\Omega)$ is the state that we can “a priori” assign to the system, soon after the measurement, if its outcome is not observed.

Let us suppose now to shrink F around an element $\omega \in \Omega$, until we obtain an “infinitesimally small” set $\mathrm{d}\omega$: the quantity $\varrho(\omega) = \mathcal{I}(\mathrm{d}\omega)[\rho]/\mathbb{P}_\rho(\mathrm{d}\omega)$ represents the state of the system conditioned on the outcome $\mathrm{d}\omega$. In other terms, we can name $\varrho(\omega)$ a posteriori state of the system because we can interpret it as the state we can assign to those systems in which we observe the outcome ω in the measuring procedure. The formal definition of the a posteriori state has been given by Ozawa in [14].

Definition 6.5 (A posteriori states)**.** The family $\{\rho(\omega),\, \omega \in \Omega\}$ is said to be a family of a posteriori states for the pre-measurement state ρ and the $(\Omega, \mathcal{F})$-valued instrument $\mathcal{I}$, if the function $\omega \mapsto \varrho(\omega) \in \mathcal{S}(\mathcal{H})$ is measurable and

$$\mathcal{I}(F)[\rho] = \int_F \varrho(\omega)\mathbb{P}_\rho(\mathrm{d}\omega). \tag{6.8}$$

Mathematically speaking the function $\varrho : \Omega \to \mathcal{S}(\mathcal{H})$ is a random variable, the a posteriori random state that is not known before the experiment because it depends on the possible outcome ω, which is distributed with probability $\mathbb{P}_\rho$.

Let us observe that, thanks to Eq. (6.8), we can give a very significant interpretation of the a priori state (6.7): this is the mean of the a posteriori state with respect to the physical probability $\mathbb{P}_\rho$. Indeed, we have

$$\mathcal{I}(\Omega)[\rho] = \int_\Omega \varrho(\omega)\mathbb{P}_\rho(\mathrm{d}\omega) = \mathbb{E}_\rho[\varrho]\,.$$

By using the language of statistical operators, we can say that $\{\varrho(\cdot), \mathbb{P}_\rho\}$ is a demixture of the a priori state $\mathcal{I}(\Omega)[\rho]$.

Let us conclude by observing that Eq. (6.8) defines the a posteriori state of the system, once the instrument $\mathcal{I}$ and the pre-measuring state ρ are known. On the other hand, if we assign $\varrho(\omega)$ and $\mathbb{P}_\rho(\omega)$ for every pre-measuring state ρ and we know that they come from an instrument, then, Eq. (6.8) enable us to reconstruct the instrument $\mathcal{I}$.

6.3 Construction of the instruments for the continuous observation

After the previous short general introduction we want to define the instruments of our own case.

6.3.1 Non random instruments

Let us consider the linear stochastic master equation (5.8), starting with the initial time $t = 0$:

$$\varrho(t) = \eta_0 + \int_0^t \mathcal{L}(s)[\varrho(s)]\mathrm{d}s + \sum_{j=1}^m \int_0^t \mathcal{R}_j(s)[\varrho(s)]\mathrm{d}W_j(s)\,. \tag{6.9}$$

The reference filtration is that one generated by the first m components of the Wiener process, that is $\{\mathcal{E}_t^0\}_{t\geq 0]}$. The following proposition holds.

Proposition 6.1 ([4, p. 309])**.** *Let E be an event in $\mathcal{E}_t^0$. Then,*

$$\mathcal{I}_t(E)[\tau] := \mathbb{E}_\mathbb{Q}[1_E A(t,0)\tau A^*(t,0)] \tag{6.10}$$

defines an instrument on the measurable space $(\Omega, \mathcal{E}_t^0)$.

Proof. We have to show that $\mathcal{I}_t(E)$ is a completely positive linear map, for all $t \geq 0$, the σ-additivity property and the trace normalisation.

Obviously (6.10) defines a linear map of $M_n(\mathbb{C})$ into itself. We show that it is completely positive. Let k be a natural number:

$$\sum_{i,j=1}^{k} \langle \varphi_i | \mathcal{I}_t(E)[\tau_i^* \tau_j] \varphi_j \rangle = \sum_{i,j=1}^{k} \mathbb{E}_{\mathbb{Q}} \big[1_E \langle \varphi_i | A(t,0) \tau_i^* \tau_j A^*(t,0) \varphi_j \rangle \big]$$
$$= \mathbb{E}_{\mathbb{Q}} \left[1_E \left\langle \sum_{i=1}^{k} \tau_i A^*(t,0) \varphi_i \Big| \sum_{j=1}^{k} \tau_j A^*(t,0) \varphi_j \right\rangle \right] = \mathbb{E}_{\mathbb{Q}} \left[1_E \left\| \sum_{i=1}^{k} \tau_i A^*(t,0) \varphi_i \right\|^2 \right] \geq 0\,.$$

To prove the normalisation property it is enough to show that it is satisfied for positive operators. Then, we remind the property (4.4) of the statistical operators and Proposition 4.1 and we obtain

$$\begin{aligned} \operatorname{Tr}\{\mathcal{I}_t(\Omega)[\tau]\} &= \mathbb{E}_{\mathbb{Q}}[\operatorname{Tr}\{A(t,0)\tau A^*(t,0)\}] = \operatorname{Tr}\{\tau\}\mathbb{E}_{\mathbb{Q}}[\operatorname{Tr}\{A(t,0)\rho A^*(t,0)\}] \\ &= \operatorname{Tr}\{\tau\}\mathbb{E}_{\mathbb{Q}}[\operatorname{Tr}\{\sigma(t)\}] = \operatorname{Tr}\{\tau\}\,, \quad \forall \tau \in M_n(\mathbb{C}):\ \tau \geq 0\,. \end{aligned}$$

The σ-additivity property comes out from the σ-additivity of the integrals

$$\mathcal{I}_t(E)[\tau] = \int_E A(t,0;\omega) \tau A^*(t,0;\omega) \mathbb{Q}(\mathrm{d}\omega).$$

□

Remark 6.2 ([4, Remark 4.2, p. 309]). The restriction on $\mathcal{E}_t^0$ of the physical probability $\mathbb{P}_{\eta_0}^t$, defined in (4.9) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$, can be immediately obtained from the instruments. Indeed, we have

$$\operatorname{Tr}\{\mathcal{I}_t(E)[\eta_0]\} = \mathbb{E}_{\mathbb{Q}}[1_E \operatorname{Tr}\{\sigma(t)\}]\,, \quad \forall E \in \mathcal{E}_t^0\,. \tag{6.11}$$

Furthermore, because of the fact that $\{\mathcal{E}_t^0\}_{t\geq 0}$ is the natural augmented filtration of the output process, we can claim that $\mathcal{I}_t$ describes the measuring procedure and gives also the physical probabilities $\mathbb{P}_{\eta_0}^t\big|_{\mathcal{E}_t^0}$.

6.3.2 A priori and a posteriori states

Generally, instruments give the a priori state, the a posteriori state and the physical probabilities: let us concretise this statement to η_0 and $\mathcal{I}_t$.

The instruments introduced in (6.10) permit to interpret the process $\hat{\varrho}$, defined in Eq. (5.6), as the a posteriori state process of the system, that is the state of the system soon after the measurement. For simplicity, to prove this claim, we assume that the a priori state of the system is $\eta_0 \in S(\mathcal{H})$, i.e. a *non random* statistical operator. From Eq. (5.4) we get

$$\operatorname{Tr}\{\varrho(t)\} = \mathbb{E}_{\mathbb{Q}}[\operatorname{Tr}\{\sigma(t)\}|\mathcal{E}_t^0];$$

then, we can claim that $\mathrm{Tr}\{\varrho(t)\}$ is the probability density of the measure $\mathbb{P}^t_{\eta_0}\big|_{\mathcal{E}^0_t}$ with respect to $\mathbb{Q}\big|_{\mathcal{E}^0_t}$ [4, p. 310]. Let us consider now $E \in \mathcal{E}^0_t$; we have

$$\begin{aligned}\mathcal{I}_t(E)[\eta_0] &= \mathbb{E}_{\mathbb{Q}}[1_E\sigma(t)] = \mathbb{E}_{\mathbb{Q}}\big[1_E\mathbb{E}[\sigma(t)|\mathcal{E}^0_t]\big] = \mathbb{E}_{\mathbb{Q}}[1_E\varrho(t)] \\ &= \mathbb{E}^t_{\eta_0}[1_E\hat{\varrho}(t)] = \int_E \hat{\varrho}(t,\omega)\mathbb{P}^t_{\eta_0}(\mathrm{d}\omega)\,. \qquad (6.12)\end{aligned}$$

By reminding the definition of a posteriori state given in (6.8), this interpretation is immediate. Indeed, the process $\hat{\varrho}$ is such that $\hat{\varrho}(t,\omega)$ is a statistical operator and it is the state of the system at the time t if the trajectory $\{W_j(s,\omega)\,,\ s\in[0,t]\,,\ j=1,\dots,m\}$ is observed, in a measurement experiment on a quantum system prepared in the state $\eta_0 \in S(\mathcal{H})$.

By using the instrument $\mathcal{I}_t$, we can also obtain the probability of an outcome of the measurement with respect to the initial state:

$$\begin{aligned}\mathrm{Tr}\{\mathcal{I}_t(E)[\eta_0]\} &= \mathrm{Tr}\{\mathbb{E}_{\mathbb{Q}}[1_E\sigma(t)]\} = \mathbb{E}_{\mathbb{Q}}\big[1_E\mathbb{E}[\mathrm{Tr}\{\sigma(t)\}|\mathcal{E}^0_t]\big] \\ &= \mathbb{E}_{\mathbb{Q}}[1_E\,\mathrm{Tr}\{\varrho(t)\}] = \mathbb{P}^t_{\eta_0}(E)\,, \quad \forall E\in\mathcal{E}^0_t\,. \qquad (6.13)\end{aligned}$$

Let us conclude this section by observing that the non random instruments just defined, permit to define the a priori state of the system, that is the state that one can assign to the system itself, once the measurement has been carried out, but the result has not been taken into account or it is not known. Indeed, by Remark 5.2 we get

$$\begin{aligned}\mathcal{I}_t(\Omega)[\eta_0] &= \mathbb{E}_{\mathbb{Q}}[\sigma(t)] = \mathbb{E}_{\mathbb{Q}}[\varrho(t)] \\ &= \mathbb{E}^t_{\eta_0}[\hat{\varrho}(t)] = \int_\Omega \hat{\varrho}(t,\omega)\mathbb{P}^t_{\eta_0}(\mathrm{d}\omega) = \int_\Omega \hat{\varrho}(t,\omega)\,\mathrm{Tr}\{\mathcal{I}_t(\mathrm{d}\omega)[\eta_0]\}\,.\end{aligned}$$

6.3.3 Random instruments

Now we want to show that the observation in $[0,t]$ can be analysed as a sequence of measurements performed in a sequence of subintervals, say in $[0,s]$ and $[s,t]$. This allows us to show that in some sense the arrow of time is respected. To do this we have to introduce the random instruments.

Let us suppose that Assumption 5.1 holds and consider the natural filtration of the increments of the first m components of the Wiener process, $\{\mathcal{E}^s_t\}_{t\in[s,T]}$. Then we define the following quantities

$$\mathcal{I}^s_t(E)[\tau] := \mathbb{E}_{\mathbb{Q}}\big[1_E\Lambda(t,s)[\tau]\big|\mathcal{E}^0_s\big]\,, \quad \forall E\in\mathcal{E}^s_t\,, \qquad (6.14)$$

for which we have the following result.

Proposition 6.2. *The object* $\mathcal{I}^s_t(\omega)$ *defined in* (6.14) *is an instrument on the measurable space* $(\Omega,\mathcal{E}^s_t)$.

Proof. The proof is similar to that one we gave for the non random instruments. The σ-additivity property is a trivial consequence of the σ-additivity of the integral in $\mathbb{Q}$ involved in the definition of $\mathcal{I}_t^s$ given in Eq. (6.14). Furthermore Eq. (6.14) define a linear map of $M_n(\mathbb{C})$ into itself and this map is completely positive. Indeed, if $E \in \mathcal{E}_t^s$ then

$$\begin{aligned}\sum_{i,j=1}^{k} \langle \varphi_i | \mathcal{I}_t^s(E)[\tau_i^* \tau_j] \varphi_j \rangle &= \sum_{i,j=1}^{k} \langle \varphi_i | \mathbb{E}_{\mathbb{Q}}[\mathbb{1}(E) A(t,s) \tau_i^* \tau_j A^*(t,s) | \mathcal{E}_s^0] \varphi_j \rangle \\ &= \sum_{i,j=1}^{k} \mathbb{E}_{\mathbb{Q}}\big[1_E \langle \varphi_i | A(t,s) \tau_i^* \tau_j A^*(t,s) \varphi_j \rangle \big| \mathcal{E}_s^0\big] \\ &= \mathbb{E}_{\mathbb{Q}}\left[1_E \left\langle \sum_{i=1}^{k} \tau_i A^*(t,s) \varphi_i \Big| \sum_{j=1}^{k} \tau_j A^*(t,s) \varphi_j \right\rangle \middle| \mathcal{E}_s^0 \right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[1_E \left\| \sum_{i=1}^{k} \tau_i A^*(t,s) \varphi_i \right\|^2 \middle| \mathcal{E}_s^0 \right] \geq 0\,.\end{aligned}$$

To prove the normalisation property it is enough to show it for any positive operator. By recalling Eqs. (4.4), (5.20), (5.10), Remark 5.1 and the independence property of the increments of the Wiener process, we have, $\forall \tau \in M_n(\mathbb{C}):\ \tau \geq 0$,

$$\begin{aligned}&\mathrm{Tr}\{\mathcal{I}_t^s(\Omega)[\tau]\} = \mathrm{Tr}\Big\{\mathbb{E}_{\mathbb{Q}}\big[\Lambda(t,s)[\tau] \big| \mathcal{E}_s^0\big]\Big\} \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{ \tau + \int_s^t \mathcal{L}(q)\big[\Lambda(q,s)[\tau]\big]\mathrm{d}q + \sum_{j=1}^{m} \int_s^t \mathcal{R}_j\big[\Lambda(q,s)[\tau]\big]\mathrm{d}W_j(q) \right\} \middle| \mathcal{E}_s^0 \right] \\ &\qquad = \mathrm{Tr}\{\tau\} + \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{ \sum_{j=1}^{m} \int_s^t \mathcal{R}_j\big[\Lambda(q,s)[\rho_0]\big]\mathrm{d}W_j(q) \right\} \middle| \mathcal{E}_s^0 \right] = \mathrm{Tr}\{\tau\}.\end{aligned}$$

□

6.3.4 The composition law of the random instruments

Although in the previous paragraph we said that the random instruments allow us to make a sequence of measurements, starting from the origin time $t = 0$, we need to establish a *composition law* of this objects to give a well-defined interpretation of random instruments as representing a measuring sequence. We set below this result before we give a physical interpretation of it explaining why we can associate a family of instruments to a measuring sequence, thanks to the composition law.

Proposition 6.3. *For all τ in $M_n(\mathbb{C})$ the following relation results*

$$\mathcal{I}_t^0(E \cap F)[\tau] = \int_E \mathcal{I}_t^s(F;\omega) \circ \mathcal{I}_s^0(\mathrm{d}\omega)[\tau]\,, \quad \forall E \in \mathcal{E}_s^0\,,\ F \in \mathcal{E}_t^s\,,\ t \geq 0\,. \tag{6.15}$$

Proof. Let us consider the events $E \in \mathcal{E}_s^0$ and $F \in \mathcal{E}_t^s$. Then, Proposition 5.9, the

$\mathcal{E}_s^0$-measurability and the properties of the conditional expectation involve

$$\begin{aligned}\mathcal{I}_t^0(E\cap F)[\tau] &= \mathcal{I}_t(E\cap F)[\tau] = \mathbb{E}_{\mathbb{Q}}\big[1_{E\cap F}\Lambda(t,0)[\tau]\big] = \mathbb{E}_{\mathbb{Q}}\big[1_E 1_F \Lambda(t,0)[\tau]\big] \\ &= \mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}[1_E 1_F \Lambda(t,s)\circ\Lambda(s,0)[\tau]|\mathcal{E}_s^0]\big] = \mathbb{E}_{\mathbb{Q}}\big[1_E \mathbb{E}_{\mathbb{Q}}[1_F\Lambda(t,s)|\mathcal{E}_s^0]\circ\Lambda(s,0)[\tau]\big] \\ &= \mathbb{E}_{\mathbb{Q}}\big[1_E \mathcal{I}_t^s(F)\circ\Lambda(s,0)[\tau]\big] = \int_E \mathcal{I}_t^s(F;\omega)\circ\Lambda(s,0;\omega)[\tau]\mathbb{Q}(\mathrm{d}\omega)\,.\end{aligned}$$

For any $\mathcal{E}_s^0$-measurable function $g(\cdot)$ we have

$$\int_E g(\omega)\mathcal{I}_s^0(\mathrm{d}\omega)[\tau] = \int_E g(\omega)\Lambda(s,0)[\tau]\mathbb{Q}(\mathrm{d}\omega)\,.$$

Then, by setting $g(\omega) := \mathcal{I}_t^s(F;\omega)[\tau]$ we reach the thesis. □

The composition law guarantees that in a sequence of measurements on a quantum system, each one represented by an instrument, the temporal order is respected. If we make a measuring experiment on the system, starting from $t = 0$, and than another one following the first, the mathematical objects which model the situation gather in the right temporal way: we can say that they respect the temporal causality principle.

Furthermore, because of the composition law, we can claim that the whole measuring procedure is represented by a single instrument instead of an uneven string of them. In other words, we can analyse a measuring procedure in $[0, T]$ by dividing it in a sequence of measurement experiments, on ordered disjoint temporal subintervals, each one represented by a random instrument. If we compose the random instruments representing the disjoint measurements, in the ordered sense specified by the composition law (6.15), then we obtain the non-random instrument representing the whole measuring process.

Let us point out that we are implicitly assuming that what happens to the system before the origin-time $t = 0$ *does not influence* it in the temporal window $[0, T]$. For this reason the first measurement is not random and, so, we represent this experiment by a non-random instrument of the form (6.10). By the way, when we start with a measuring procedure on the quantum system in $t = 0$, we do not know what will be the result of the experiment in $s > 0$. In other words, we are saying that what happens to the system in the temporal window $[0, s]$ *does influence* the system in $(s, T]$. Then, if we want to put into effect another measuring procedure in this second temporal interval, we have to decide the observable of the system which we shall measure in $(s, T]$, on the basis of the outcome of experiment that we observed in $[0, s]$. The random instruments represent successive measurements: the randomness of these objects comes out from the fact that they are defined as a conditional expectation in Eq. (6.14) and the conditioning σ-algebra is that one generated by the increments of the output during the measurement starting in $t = 0$ up to $t = s$. In conclusion, we can interpret a sequence of measurements on our system as a control procedure on the system itself. We are actually saying that we are allowed to decide how to choose the instrument in a second measurement, according to the information we get from the previous one. This is, we establish what we shall measure in a successive experiment on the basis of the observed occurrence in the previous one.

6.4 The observables of the continuous measurement

The choice of the σ-algebra $\mathcal{E}_t^0$ reflects the fact that we consider as events which can be observed in the time interval $[0,t]$ only events related to the Wiener process W. In an heuristical interpretation we can say that instantly the observed signal is the singular process $\dot{W}$, i.e. the time derivative of the Wiener process, which "morally" assumes the meaning of output of the system. However, $\dot{W}$, at a fixed time t, is not well defined because the typical trajectories of the Wiener process are not differentiable.

By the way it is possible to give a mathematical meaning to $\dot{W}(t)$ as *generalised stochastic process*. A generalised stochastic process is a linear random functional on a suitable space of test functions. We want to interpret $\dot{W}$ as the first (generalised) derivative of a well behaving process (the process W under $\mathbb{Q}$).

Test functions

Let us introduce $L^2_{\text{loc}} := L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^m)$, the space of the locally square integrable functions k from $\mathbb{R}_+$ into $\mathbb{R}^m$; $k \in L^2_{\text{loc}}$ means that, for all finite $t > 0$, we have $1_{(0,t)}k \in L^2((0,t);\mathbb{R}^m) =: L^2_t$. From now on a *test function* is any element of $L^2_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^m)$.

6.4.1 Observables

Let us take $k \in L^2_{\text{loc}}$ and $0 \leq s < t \leq T$. We define the random variables

$$X_t^s(k) := \sum_{j=1}^m \int_s^t k_j(r)\,\mathrm{d}W_j(r). \tag{6.16}$$

Let us note that $X_t^s(k)$ is $\mathcal{E}_t^s$-measurable and that

$$X_t^s(k) = X_T^0(1_{(s,t)}k).$$

Either under $\mathbb{Q}$, either under $\mathbb{P}_{\eta_0}^T$, $X_t^s(\cdot)$ is a linear random functional on L^2_{loc} and, so, it is a generalised stochastic process.

Heuristically, Eq. (6.16) can be written as

$$X_t^s(k) = \sum_{j=1}^m \int_s^t k_j(r)\,\dot{W}_j(r)\mathrm{d}r$$

and the generalised process $X_t^s(\cdot)$ is what gives a rigorous meaning to $\dot{W}$.

6.4.2 Finite dimensional laws

Under the reference probability $\mathbb{Q}$, the real random variable $X_t^s(k)$ is independent of $\mathcal{E}_s^0$ and is normally distributed with zero mean and variance $\sum_{j=1}^m \int_s^t |k_j(r)^2|\mathrm{d}r$ (it can be easily obtained by the Itô isometry). More generally, the random vector $\big(X_T^0(k^{(1)}),\ldots,\, X_T^0(k^{(q)})\big)$ is Gaussian with zero means and covariance matrix

$$\operatorname{Cov}_{\mathbb{Q}}[X_T^0(k^{(i)}),\, X_T^0(k^{(j)})] = \langle k^{(i)}|k^{(j)}\rangle_{L^2_T}\,. \tag{6.17}$$

When the test functions $k^{(1)}, \dots, k^{(q)}$ are linearly independent, the covariance matrix with elements (6.17) is not singular and the distribution of the vector $\big(X_T^0(k^{(1)}), \dots, X_T^0(k^{(q)})\big)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^q$.

However, we are interested in the random variables (6.16) under the physical probability $\mathbb{P}_{\eta_0}^T$. Let us consider a single variable $X_t^s(k)$; by the definition of the instruments in Eq. (6.10) and Eq. (6.13), its cumulative distribution function, under the physical probabilities is given by

$$\mathbb{P}_{\eta_0}^T \left[X_t^s(k) \le x\right] = \mathrm{Tr}\left\{\mathcal{I}_T^0\left(X_t^s(k) \le x\right)[\eta_0]\right\}. \tag{6.18}$$

This distribution is diffuse on the whole real line and a similar statement also holds for $\big(X_T^0(k^{(1)}), \dots, X_T^0(k^{(q)})\big)$, as the following proposition says.

Proposition 6.4. *For $0 \le t \le T$ and $k \neq 0$, the distribution of $X_t^0(k)$ under $\mathbb{P}_{\eta_0}^T$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and its closed support is $\mathbb{R}$, i.e. there exists a density $f_{X_t^s(k)}(x) > 0$, $\forall x \in \mathbb{R}$, such that*

$$\mathbb{P}_{\eta_0}^T \left[X_t^s(k) \le x\right] = \int_{-\infty}^{x} f_{X_t^s(k)}(y)\mathrm{d}y\,, \qquad \forall x \in \mathbb{R}.$$

If $k^{(1)}, \dots, k^{(q)}$ are linearly independent elements of L_T^2, the distribution of $\big(X_T^0(k^{(1)}), \dots, X_T^0(k^{(q)})\big)$ under $\mathbb{P}_{\eta_0}^T$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^q$ and its density can be taken strictly positive $\forall x \in \mathbb{R}^q$.

Proof. Let us introduce the laws of $X_t^s(k)$ under $\mathbb{Q}$ and $\mathbb{P}_{\eta_0}^T$:

$$\mathbb{Q}_X(A) := \mathbb{Q}[X_t^s(k) \in A], \qquad \mathbb{P}_X(A) := \mathbb{P}_{\eta_0}^T[X_t^s(k) \in A], \qquad \forall A \in \mathcal{B}(\mathbb{R}).$$

The probability measures $\mathbb{Q}$ and $\mathbb{P}_{\eta_0}^T$ are equivalent, because of Proposition 5.5, and this implies the equivalence of the two laws; indeed, we have

$$0 = \mathbb{Q}_X(N) = \mathbb{Q}[X_t^s(k) \in N] \ \Leftrightarrow\ 0 = \mathbb{P}_{\eta_0}^T[X_t^s(k) \in N] = \mathbb{P}_X(N).$$

The measure $\mathbb{Q}_X$ is a normal distribution on $\mathbb{R}$ and, so, it is equivalent to the Lebesgue measure; then, also $\mathbb{P}_X$ is equivalent to the Lebesgue measure and has a density with respect to it.

By recalling that the closed support of a probability is the smallest closed set with probability one, we have that the support of $\mathbb{P}_X$ is the whole real line as for the normal distribution $\mathbb{Q}_X$. Then, the density of $\mathbb{P}_X$ with respect to the Lebesgue measure can be taken strictly positive everywhere.

The proof of the second part of the Proposition follows exactly the same steps as the first one. □

Remark 6.3. The distribution of the random vector $\big(X_T^0(k^{(1)}), \dots, X_T^0(k^{(q)})\big)$ under $\mathbb{P}_{\eta_0}^T$ is said to be a *finite dimensional law* of the generalised stochastic process $\{X_T^0(k),\ k \in L_T^2,\ T \ge 0\}$.

6.5 Characteristic functional and finite dimensional laws

In this section we want to introduce a fundamental object, the *characteristic functional*, which allow us to compute the moments of the observables of the quantum system. We can say that the characteristic functional is the generalisation of the characteristic function for a random variable in the context of process and generalised process. For this reason, before to introduce it, we recall the definition and the main properties of a characteristic function of a random variable.

6.5.1 Characteristic function of a distribution

The characteristic function of a joint distribution on $\mathbb{R}^q$ is its Fourier transform; so, if $\mathbb{P}$ is a probability measure on $\big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q)\big)$, the characteristic function of $\mathbb{P}$ is

$$g(k) = \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}k\cdot x}\, \mathbb{P}(\mathrm{d}x), \qquad k \in \mathbb{R}^q. \tag{6.19}$$

Another important property is that $g \in L^1\big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q), \mathrm{d}x\big)$ if and only if the probability is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$: $\mathbb{P}(\mathrm{d}x) = f(x)\mathrm{d}x$. In this case $f \in L^1\big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q), \mathrm{d}x\big)$ and

$$g(k) = \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}k\cdot x} f(x)\mathrm{d}x\,, \qquad f(x) = \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}k\cdot x} g(k)\mathrm{d}k\,. \tag{6.20}$$

6.5.2 Characteristic function of a random vector

The characteristic function of a random vector is simply the characteristic function of its distribution. Let X be a q-dimensional random vector: $X : \big(\Omega, \mathcal{F}, \mathbb{P}\big) \to \big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q)\big)$. The characteristic function ϕ_X of X is defined as the characteristic function of $\mathbb{P}_X$, i.e. $\phi_X(k) = \mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\mathrm{i}k\cdot X}\right]$. Obviously, from ϕ_X one can reobtain only $\mathbb{P}_X$, by anti-Fourier transform: from the characteristic function of a random variable is not possible to reconstruct the probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ and the random variable X itself, as a function on Ω.

When the moments of order r of X exist, then its characteristic function is differentiable up to order r and for $1 \leq m \leq r$

$$\mathbb{E}_{\mathbb{P}}[X_{j_1} \cdots X_{j_m}] = (-\mathrm{i})^m \frac{\partial^m \phi_X(k)}{\partial k_{j_1} \cdots \partial k_{j_m}}\bigg|_{k=0}. \tag{6.21}$$

6.5.3 Characteristic functional

The theoretical treatment of the characteristic functional in much more difficult that one of the characteristic function. Here we introduce the various notions and results only in the case of our interest.

Let us start by considering the finite dimensional distributions discussed in the previous section. We fix the test functions $k^{(j)} \in L^2_T$, $j = 1, \dots, q$, with $k^{(1)}, \dots, k^{(q)}$ linearly independent, and we denote by $\lambda = (\lambda_1, \dots, \lambda_q)$ the indeterminate in $\mathbb{R}^q$. By setting $Y_j := X_T^0(k^{(j)})$ we obtain a random vector $Y = (Y_1, \ldots, Y_q)$ in the probability space $\left(\Omega, \mathcal{E}_T^0, \mathbb{P}_{\eta_0}^T\right)$, whose characteristic function is

$$\phi_Y(\lambda) = \mathbb{E}_{\eta_0}^T \left[\mathrm{e}^{\mathrm{i}\lambda\cdot Y} \right]. \tag{6.22}$$

By Proposition 6.4 and Eq. (6.20), the probability distribution of Y has a density, given by

$$f_Y(y) = \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}y\cdot\lambda} \phi_Y(\lambda)\, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_q \,. \tag{6.23}$$

Now, we have that the observables (6.16) are linear in the test function and we have $\lambda\cdot Y = \sum_{j=1}^q \lambda_j X_T^0(k^{(j)}) = X_T^0\left(\sum_{j=1}^q \lambda_j k^{(j)}\right)$. So, if we know the characteristic function of the random variable $X_T^0(k)$ as a functional of k, we know in principle all the finite dimensional distributions; what we obtain in this way is the notion of characteristic functional.

We define the *characteristic functional* of the generalised process $\{\dot{W}_j,\, j = 1, \ldots, m\}$ under the probability $\mathbb{P}_{\eta_0}^T$ by

$$\Phi_t(k|\eta_0) := \mathbb{E}_{\eta_0}^T \left[\exp\left\{ \mathrm{i} \sum_{j=1}^m \int_0^t k_j(s) \mathrm{d}W_j(s) \right\} \right], \qquad T \geq t \geq 0, \tag{6.24}$$

or, by using the observables (6.16),

$$\Phi_t(k|\eta_0) = \mathbb{E}_{\eta_0}^T \left[\mathrm{e}^{\mathrm{i}X_t^0(k)} \right]. \tag{6.25}$$

We are considering $\Phi_t(k|\eta_0)$ as a functional of the test function $k \in L^2_{\mathrm{loc}}$; obviously, it is a function also of $t \geq 0$ and of $\eta_0 \in S(\mathcal{H})$, but not of T by the consistency of the probabilities.

Then, the characteristic function of the random vector Y given above is

$$\phi_Y(\lambda) = \Phi_T\left(\textstyle\sum_{j=1}^q \lambda_j k^{(j)} \middle| \eta_0 \right) \tag{6.26}$$

and the characteristic functional gives all the finite dimensional distributions of $X_T^0(\cdot)$ introduced in Remark 6.3.

6.6 Characteristic operator

We want to give now an object whose definition is strictly parallel to that one of characteristic functional, this is the characteristic operator. So far, the notions of observables of a quantum system and of characteristic functional are the same that were stated in [3, Chap.4], but for the characteristic operator we have some differences. While in [3] a closed equation for the characteristic operator has been obtained, now it is impossible to state an analogous result and this because of the randomness of the coefficients of our stochastic differential equations.

Definition 6.6. For $k \in L^2_{\mathrm{loc}}$ we define the *characteristic operator* $\mathcal{G}(t;k)$ of the measurement defined by the *non random* instrument $\mathcal{I}_t$ and by the output $\{\dot{W}(r),\ r \in [0,t]\}$, $\forall \tau \in M_n(\mathbb{C})$, as

$$\mathcal{G}(t;k)[\tau] := \int_\Omega \exp\left\{ \mathrm{i} \sum_{j=1}^m \left(\int_0^t k_j(s)\mathrm{d}W_j(s) \right)(\omega) \right\} \mathcal{I}_t(\mathrm{d}\omega)[\tau] \,. \tag{6.27}$$

6.6.1 Some properties of the characteristic operator

Let us recall that, by Eq. (6.10), we have

$$\mathcal{I}_t(E)[\eta_0] = \mathbb{E}_\mathbb{Q}[1_E \sigma(t)] \,, \quad \forall E \in \mathcal{E}_t^0 \,.$$

Then, by Eq. (5.4), reminding the properties of conditional expectations and the measurability of the stochastic integral appearing in the definition of $\mathcal{G}(t;k)$ with respect to $\mathcal{E}_t^0$, we have

$$\begin{aligned}\mathcal{G}(t;k)[\eta_0] &= \mathbb{E}_\mathbb{Q}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^m \int_0^t k_j(s)\mathrm{d}W_j(s) \right\} \sigma(t) \right] \\ &= \mathbb{E}_\mathbb{Q}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^m \int_0^t k_j(s)\mathrm{d}W_j(s) \right\} \varrho(t) \right] .\end{aligned} \tag{6.28}$$

First of all we note that the characteristic operator is strictly connected to the characteristic functional. Indeed, by the fact that the trace of the process ϱ is the density of the physical probability, we have

$$\begin{aligned}\mathrm{Tr}\,\{\mathcal{G}(t;k)[\eta_0]\} &= \mathbb{E}_\mathbb{Q}\left[\exp\left\{ \sum_{j=1}^m \int_0^t k_j(s)\mathrm{d}W_j(s) \right\} \mathrm{Tr}\,\varrho(t) \right] \\ &= \mathbb{E}^T_{\eta_0}\left[\exp\left\{ \sum_{j=1}^m \int_0^t k_j(s)\mathrm{d}W_j(s) \right\} \right] = \Phi_t(k|\eta_0) \,.\end{aligned} \tag{6.29}$$

Moreover we have

$$\mathcal{G}(t;0)[\eta_0] = \mathbb{E}_\mathbb{Q}[\sigma(t)] = \mathbb{E}_\mathbb{Q}[\varrho(t)] = \eta(t) \,.$$

Let us stress that the consistency property of the physical probabilities involves the consistency of the characteristic operator. Indeed, if we define the test functions

$$k_{(a,b)}(t) = 1_{(a,b)}(t)k(t),$$

by the composition law of the propagator of the linear stochastic master equation stated in Proposition 5.9, we have

$$
\begin{aligned}
\mathcal{G}\left(T;k_{(0,t)}\right)[\eta_0] &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}W_j(s)\right\}\varrho(T)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}W_j(s)\right\}\Lambda(T,0)[\eta_0]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}W_j(s)\right\}\Lambda(T,t)\circ\Lambda(t,0)[\eta_0]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}W_j(s)\right\}\Lambda(T,t)[\varrho(t)]\right].
\end{aligned}
$$

Furthermore, reminding the definition of observable of the quantum system, given in Eq. 6.16, we have that the random variable X_t^0 is $\mathcal{E}_t^0$-measurable and, by using the martingale property of the trace of the process ϱ with respect to the filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$, we have

$$
\begin{aligned}
\Phi_T(k_{(0,t)}|\eta_0) &= \mathrm{Tr}\left\{\mathcal{G}\left(T;k_{(0,t)}\right)[\eta_0]\right\} \\
&= \mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\varrho(T)\right]\right\} = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\,\mathrm{Tr}\{\varrho(T)\}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\mathbb{E}_{\mathbb{Q}}[\mathrm{Tr}\{\varrho(T)\}|\mathcal{E}_t^0]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\,\mathrm{Tr}\{\varrho(t)\}\right] \\
&= \mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\varrho(t)\right]\right\} = \mathrm{Tr}\left\{\mathcal{G}(t;k)[\eta_0]\right\} = \Phi_t(k|\eta_0).
\end{aligned}
$$

6.6.2 An useful form of the characteristic operator

In this section we want to give an explicit form of the characteristic operator. We proceed in a heuristic way, by using *time ordered products.*

Recalling the SDE of the propagator of the linear stochastic master equation (5.8), we have

$$
\Lambda(t+\mathrm{d}t,t) = \mathrm{Id}_n + \mathcal{L}(t)\mathrm{d}t + \sum_{j=1}^{m}\mathcal{R}_j(t)\mathrm{d}W_j(t)\,.
$$

On the other hand it results

$$
\mathrm{e}^{\mathrm{i}\sum_{j=1}^{m}k_j(t)\mathrm{d}W_j(t)} = 1 + \mathrm{i}\sum_{j=1}^{m}k_j(t)\mathrm{d}W_j(t) - \frac{1}{2}\sum_{j=1}^{m}k_j(t)^2\mathrm{d}t
$$

and, so,

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}\sum_{j=1}^m k_j(t)\mathrm{d}W_j(t)}\Lambda(t+\mathrm{d}t,t)\\
&=\mathrm{Id}_n+\sum_{j=1}^m\left(\mathcal{R}_j(t)+\mathrm{i}k_j(t)\mathrm{Id}_n\right)\mathrm{d}W_j(t)-\frac{1}{2}\sum_{j=1}^m k_j(t)^2\mathrm{Id}_n\mathrm{d}t+\mathcal{L}(t)\mathrm{d}t+\mathrm{i}\sum_{j=1}^m k_j(t)\mathcal{R}_j(t)\mathrm{d}t\\
&\qquad=\exp\left\{\sum_{j=1}^m\left(\mathcal{R}_j(t)+\mathrm{i}k_j(t)\mathrm{Id}_n\right)\mathrm{d}W_j(t)+\mathcal{L}(t)\mathrm{d}t-\frac{1}{2}\sum_{j=1}^m\mathcal{R}_j(t)^2\mathrm{d}t\right\} \qquad (6.30)
\end{aligned}
$$

where the last equality follows by applying the Taylor expansion and the Itô rules.

By using the composition law of Λ we can gather all the contributions of the infinitesimal time intervals, respecting the ordering due to the composition law. In other words, using the concept of time ordered products we end up with the formula

$$
\begin{aligned}
\mathcal{G}(T;k)=\mathbb{E}_{\mathbb{Q}}\Bigg[\overleftarrow{\mathrm{T}}\exp\Bigg\{\int_0^T\Bigg[\left(\mathcal{L}(t)-\frac{1}{2}\sum_{j=1}^m\mathcal{R}_j(t)^2\right)\mathrm{d}t\\
+\sum_{j=1}^m\big(\mathcal{R}_j(t)+\mathrm{i}k_j(t)\big)\mathrm{d}W_j(t)\Bigg]\Bigg\}\Bigg]; \qquad (6.31)
\end{aligned}
$$

we have suppressed the identity operator by identifying Id_n and 1.

As for a priori states, we have not a closed evolution equation for the characteristic operator, but, at least, the expression (6.31) is useful to obtain suitable formulas for the moments.

6.7 Moments

In this section we want to obtain formulas to compute the moments of the output of our measurement.

The mean

By Eq. (5.16) we have

$$W_j(t)=\widehat{W}_j(t)+\int_0^t v_j(s)\mathrm{d}s,$$

where $\widehat{W}_j$ is a Wiener process under the physical probability $\mathbb{P}^t_{\eta_0}$ and $v_j(t)=\mathrm{Tr}\{\mathcal{R}_j(t)\hat{\varrho}(t)\}$. By taking the expectation with respect to the physical probability, we have

$$
\begin{aligned}
\mathbb{E}^T_{\eta_0}[W_j(t)]=\int_0^t\mathbb{E}^T_{\eta_0}[v_j(s)]\mathrm{d}s=\int_0^t\mathbb{E}^T_{\eta_0}\left[\mathrm{Tr}\{\mathcal{R}_j(s)\hat{\varrho}(s)\}\right]\mathrm{d}s\\
=\int_0^t\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\{\mathcal{R}_j(s)\varrho(s)\}\right]\mathrm{d}s.
\end{aligned}
$$

Then, we are allowed to write

$$\mathbb{E}^T_{\eta_0}\left[\dot{W}_j(t)\right]=\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_j(t)[\varrho(t)]\right\}\right]. \qquad (6.32)$$

Higher moments

The best way to obtain the higher moments of the output of the system is to go through the characteristic functional and to get them by functional differentiation:

$$\mathbb{E}^T_{\eta_0}[\dot{W}_{j_1}(t_1)\dot{W}_{j_2}(t_2)\dots\dot{W}_{j_q}(t_q)] = (-\mathrm{i})^q \frac{\delta^q \Phi_T(k|\eta_0)}{\delta k_{j_1}(t_1)\delta k_{j_2}(t_2)\cdots\delta k_{j_q}(t_q)}\,. \tag{6.33}$$

The explicit computation of these derivatives is a consequence Eq. (6.31) and of the concept of functional derivative of time ordered product. Then, we have for the second moments, in which we are interested,

$$\begin{aligned}\mathbb{E}^T_{\eta_0}\left[\dot{W}_j(t)\dot{W}_i(s)\right] &= \delta_{ij}\delta(t-s)\\ &\quad + 1_{(0,+\infty)}(t-s)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\Lambda(t,s)\circ\mathcal{R}_i(s)[\varrho(s)]\right\}\right]\\ &\quad + 1_{(0,+\infty)}(s-t)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_i(s)\circ\Lambda(s,t)\circ\mathcal{R}_j(t)[\varrho(t)]\right\}\right]\end{aligned} \tag{6.34}$$

Using Eq. (6.34) it is possible to obtain the second moments of an observable of the quantum system. Indeed, we recall that an observable $X^0_T(k)$ of the form (6.16) is the mathematical tool which gives a rigorous meaning to the output $\dot{W}$. The heuristic form of $X^0_T(k)$ is, for $k \in L^2_{\mathrm{loc}}$,

$$X^0_T(k) = \sum_{j=1}^m \int_0^T k_j(r)\dot{W}_j(r)\mathrm{d}r\,.$$

Let us consider another test function $h \in L^2_{\mathrm{loc}}$: by the previous formula and Eq. (6.34) we get

$$\begin{aligned}\mathbb{E}^T_{\eta_0}[X^0_T(k)X^0_T(h)] &= \mathbb{E}^T_{\eta_0}\left[\left(\sum_{j=1}^m\int_0^T \mathrm{d}t\; k_j(t)\dot{W}_j(t)\right)\left(\sum_{i=1}^m\int_0^T \mathrm{d}s\; h_i(s)\dot{W}_i(s)\right)\right]\\ &= \mathbb{E}^T_{\eta_0}\left[\sum_{j=1}^m\sum_{i=1}^m\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\; k_j(t)h_i(s)\dot{W}_j(t)\dot{W}_i(s)\right]\\ &= \sum_{j=1}^m\sum_{i=1}^m\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\; k_j(t)h_i(s)\mathbb{E}^T_{\eta_0}\left[\dot{W}_j(t)\dot{W}_i(s)\right]\\ &= \sum_{j=1}^m\int_0^T \mathrm{d}s\; k_j(s)h_j(s)\\ &\quad + \sum_{j=1}^m\sum_{i=1}^m\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\; k_j(t)h_i(s)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\Lambda(t,s)\circ\mathcal{R}_i(s)[\varrho(s)]\right\}\right]\\ &\quad + \sum_{j=1}^m\sum_{i=1}^m\int_0^T \mathrm{d}t\int_t^T \mathrm{d}s\; k_j(t)h_i(s)\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_i(s)\circ\Lambda(s,t)\circ\mathcal{R}_j(t)[\varrho(t)]\right\}\right]\,.\end{aligned} \tag{6.35}$$

7 Quantum optical systems

This chapter shows how to apply the general theory to a concrete physical system. The model is a two-level atom which absorbs and emits light and it is affected by dissipative effects of thermal and dephasing type. The atom is stimulated by a coloured laser. The fluorescence light is monitored by detectors of heterodyne/homodyne type.

7.1 Introduction

In Chapter 4 we obtained Eq. (4.14), in the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$, for the process σ, that we had introduced in Eq. (4.7), and we called this equation linear stochastic master equation. Then, in Chapter 5, we restricted our σ-algebra choosing to observe not all the components of the Wiener process but just the first $m \leq d$: we used the remaining $d - m$ components to introduce some dissipative effects, due to the interaction of the system with the external world, which is not observed. Furthermore, we have implicitly assumed that these dissipative phenomena are of Markov type, that is they do not introduce any memory effect in the observed output (the lost light for example).

The mathematical modelling of the reduced observation has been done through Assumption 5.1 where we have stated that the coefficients of the linear stochastic master equation are adapted with respect to the σ-algebra generated by the first m components of the Wiener process. Under this assumption we obtained a closed SDE for the process ϱ, introduced in Eq. (5.4), which still has the interpretation of linear stochastic master equation.

By the way we can develop our situation introducing some memory effects in the observation. An easy mode to do this is to introduce another natural number, say $\bar{m} \leq m$, and to assume that the observed components are those from 1 to $\bar{m}$. The components from $\bar{m}+1$ to m are not observed, but it is impossible to eliminate them from the description, as we did with the components from $m+1$ to d. Then, we use the channels from $\bar{m}+1$ to m to introduce dissipative effects with memory and/or randomness in the external world, into the stimulating laser, in the detection apparatus...

For some $k \in [\bar{m}+1, m] \cap \mathbb{N}$ one can have that the corresponding coefficient is identically equal to zero, i.e. $R_k(t) \equiv 0$: this means that the related component W_k of the Wiener process is used to introduce randomness in some components of the physical system, but not to introduce dissipative effects. The various coefficients $\{R_j(t)\}_{j=\bar{m}+1}^m$ and $H(t)$ can depend also on $W_{\bar{m}+1}(s), \ldots, W_m(s)$, $s \in [0,t]$: this generate memory and represent random environment. Also the dependence on $W_1(s), \ldots, W_{\bar{m}}(s)$, $s \in [0,t]$, generates memory, but now it can be interpreted also as feedback.

We can schematise all the previous considerations in the following table

Components	**Meaning**
1,...,d	Components of the Wiener process W.
1,...,m	Components which can not be ignored in the description.
1,...,$\bar{m}$	Components which are observed.
$m+1$,...,d	Ignored components, for dissipative effects with no memory.
$\bar{m}+1$,...,m	Not observed components, for dissipative effects and/or randomness.

Then, we can split the coefficients R_j as follows

$$\{R_j\}_{j=1}^d = \Big\{\{R_k\}_{k=1}^{\bar{m}} \cup \{R_j\}_{j=\bar{m}+1}^m \cup \{R_j\}_{m+1=1}^d\Big\}$$

Let us introduce the augmented filtration generated by the increments of the first $\bar{m}$ components of the Wiener process

$$\mathcal{D}_t^s := \sigma\Big\{W_j(r) - W_j(s),\, r \in [s,t]\,,\; j = 1, \ldots, \bar{m}\Big\} \vee \mathcal{N}\,; \tag{7.1}$$

$$\mathcal{D}^s := \bigvee_{t \geq s} \mathcal{D}_t^s\,; \quad \mathcal{D} := \mathcal{D}^0\,. \tag{7.2}$$

Let us observe that the following inclusions hold

$$\mathcal{D}_t^s \subset \mathcal{E}_t^s \subset \mathcal{F}_t\,, \quad \forall t \in [s,T]..\tag{7.3}$$

We remind that the filtrations appearing in the formula above are the following

$\{\mathcal{F}_t\}_{t\geq 0}$: reference filtration ;

$\{\mathcal{E}_t^s\}_{t\geq s}$: two times filtration of the increments of the components $1, \ldots, m$;

$\{\mathcal{D}_t^s\}_{t\geq s}$: two times filtration of the increments of the observed components $1, \ldots, \bar{m}$.

Let us stress that we do not require that the processes $\{R_j\}_{j=1}^m$ are $\mathcal{D}_t^0$-adapted. So, if we proceed in a similar way to Chapter 5 and we introduce the process $\zeta = \{\zeta(t)\}_{t\geq 0}$ by

$$\zeta(t) := \mathbb{E}_{\mathbb{Q}}\left[\sigma(t)|\mathcal{D}_t^0\right]\,, \quad \forall t \geq 0\,, \tag{7.4}$$

it is impossible to end up with a closed SDE for ζ in the stochastic basis $(\Omega, \mathcal{D}, \{\mathcal{D}_t^0\}_{t\geq 0}, \mathbb{Q})$, as we did for the process ϱ, for which we obtained Eq. (5.8) in the stochastic basis $(\Omega, \mathcal{E}, \{\mathcal{E}_t^0\}_{t\geq 0}, \mathbb{Q})$. In other words, we can say that in this context it is no possible to derive a closed equation for the state of the system only with respect to the observation, because of the randomness of other components not directly connected with the observation itself, but with the detecting apparatus or with the stimulating laser or because of some memory effects.

If we want to model the measuring experiment the true instruments are not those introduced by Eqs. (6.10) and (6.14), but we have to restrict them to the augmented natural filtration of the observation. Then, we have that the "true" instruments are $\mathcal{J}_t := \mathcal{I}_t\Big|_{\mathcal{D}_t^0}$ and $\mathcal{J}_t^s := \mathcal{I}_t^s\Big|_{\mathcal{D}_t^s}$, that is $\forall \tau \in M_n(\mathbb{C})$,

$$\mathcal{J}_t(D)[\tau] = \mathbb{E}_{\mathbb{Q}}\left[1_D A(t,0)\tau A^*(t,0)\right] , \quad \forall D \in \mathcal{D}_t^0 ; \tag{7.5}$$

$$\mathcal{J}_t^s(D)[\tau] = \mathbb{E}_{\mathbb{Q}}\left[1_D \Lambda(t,s)[\tau]\big|\mathcal{E}_s^0\right] , \quad \forall D \in \mathcal{D}_t^s . \tag{7.6}$$

By construction, the temporal order of the measurement is respected also by the new instruments, but it cannot be formulated in the form expressed by Proposition 6.3, which does not hold in this form.

7.2 The physical model

Up to here we have considered the abstract theory which involves several operators acting on the Hilbert space $\mathcal{H}$: to fix the physical model we have to fix these operators.

7.2.1 The general structure of the system operators

It is clear that the mathematical model is completely determined when the operators R_j and H are chosen: to fix them we introduce the following operators.

- D_k, system operators responsible of the emission of quanta of the field;
- S_{kj}, system operators such that $\sum_k S_{kj}^* S_{ki} = \sum_k S_{jk} S_{ik}^* = \delta_{ij}\mathbb{1}$, where δ_{ij} is the Kroneker symbol; S is a unitary matrix of system operators and is involved in terms responsible of scattering of quanta;
- H_0, $H_0 = H_0^*$, the free Hamiltonian of the system;
- $f_k(t)$, a stochastic process describing some nearly coherent external stimulation ("the state of the external field");
- $h_k(t)$, with $|h_k(t)| = 1$ a.s., a stochastic process appearing when the measuring apparatus uses some interference mechanism, as in the so called "heterodyne" and "homodyne" photon-detection techniques.

The final result for the operators characterising the reduced description in terms of SDEs is the following structure for $R_k(t)$ and $H(t)$

$$R_k(t) = \overline{h_k(t)} \left[D_k + \sum_{j=1}^{d} S_{kj} f_j(t) \right], \tag{7.7}$$

$$H(t) = H_0 + \frac{1}{2} \sum_{k=1}^{d} \sum_{j=1}^{d} \left[\mathrm{i}\overline{f_k(t)} S^*_{jk} D_j - \mathrm{i} D^*_j S_{jk} f_k(t) \right]. \tag{7.8}$$

Of course this is the general structure of our operators: to completely determine the model we have to establish the form of h_k, f_j, D_k, S_{kj}, H_0. We conclude this section rewriting the Liouville operator, defined by Eq. (4.13), in a suitable form. Let τ be in $M_n(\mathbb{C})$,

$$\mathcal{L}(t)[\tau] = -\mathrm{i}[H(t), \tau] + \sum_{k=1}^{d} \left(R_k(t) \tau R^*_k(t) - \frac{1}{2} \{ R^*_k(t) R_k(t), \tau \} \right). \tag{7.9}$$

We observe that the Liouville operator can be thought as divided in two different terms: the first one is related to the Hamiltonian part of the dynamic and to any dissipative effect not directly connected with the observation, and the other term tied to the random phenomena of the system. In other words we can name with $\mathcal{L}_0$ the Hamiltonian and dissipative part of the Liouvillian and with $\mathcal{L}_1$ the other one, and we can write

$$\mathcal{L}(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t),$$

where, $\forall \tau \in M_n(\mathbb{C})$,

$$\mathcal{L}_0(t)[\tau] := -\mathrm{i}[H(t), \tau] + \sum_{j=\bar{m}+1}^{d} \left(R_j(t) \tau R^*_j(t) - \frac{1}{2} \{ R^*_j(t) R_j(t), \tau \} \right), \tag{7.10}$$

$$\mathcal{L}_1(t)[\tau] := \sum_{k=1}^{\bar{m}} \left(R_k(t) \tau R^*_k(t) - \frac{1}{2} \{ R^*_k(t) R_k(t), \tau \} \right). \tag{7.11}$$

Case $S_{ij} = \delta_{ij} \mathbb{1}$. This case is particularly important and allows for some simplification: by inserting the explicit form of the operators $R_j(t)$, we have

$$\mathcal{L}(t)[\tau] = -\mathrm{i}[H(t), \tau] + \sum_{j=1}^{d} \left(D_j \tau D^*_j - \frac{1}{2} \{ D^*_j D_j, \tau \} \right), \tag{7.12}$$

$$H(t) = H_0 + \mathrm{i} \sum_{j=1}^{d} \left[\overline{f_j(t)} D_j - f_j(t) D^*_j \right]. \tag{7.13}$$

7.2.2 Homodyne and heterodyne detection

In the previous section we stated a general structure for the operators of the quantum system: if we want to completely specify the model we have to choose the operators h_k, f_j, D_k, S_{kj}, H_0. In order to justify the choice of these operators one has to study the physical properties of the system and the physical consequence of the model; this would eventually provide a phenomenological justification.

The measurement scheme

We shall follow [3, Chap.7] where the following measuring scheme has been stated.

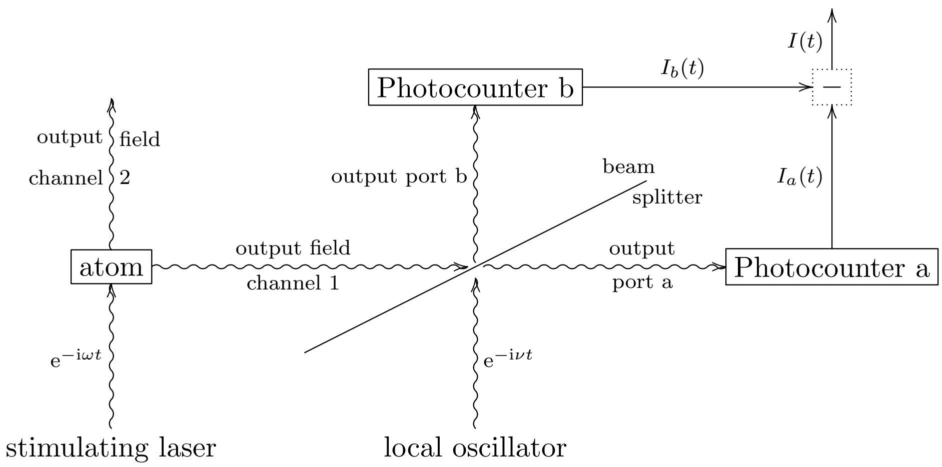

Figure 7.1: Balanced heterodyne/homodyne detection

The atom is stimulated by a laser, of carrier frequency ω, and then it emits fluorescence light. The emitted light is made to interfere with a hight intensity laser with carrier frequency ν (*local oscillator*). The interference between fluorescence light and the local oscillator is obtained by means of a beam splitter which is a half transparent mirror. Then the light coming out from the splitter is detected by photoelectron counters. The scheme in Figure 7.1 is called *balanced homodyne detection*: each photo-counter receives the light coming out from one of the two output ports of the beam splitter and the two currents are then subtracted. This set up with two photo-counters reduces the noise in the final current.

Part of the fluorescence light is lost in the surrounding world (dissipative effects) and part reaches the beam splitter. We shall assume that the lost light does not reach neither the beam splitter nor the photo-counter. Furthermore, we shall suppose that the stimulating laser is directed in such a way that its light does not impinge directly on the beam splitter. With this assumption, we can say that the lost light is described by the *forward channel* (channel 2) and that the detected light by the *side channel* (channel 1).

In our model (no feedback) we consider an atom and two detection tools: we can depict the situation in the following figure We want to observe the electrical current I_1 coming out from the detector 1: in other words, using the terminology introduced in Chapter 6, the current I_1 will be the observable of our quantum system, i.e. the output of the measurement experiment. We assume the following functional form for the output current

$$I_1(t) := \int_0^t F(t-s)\mathrm{d}W_1(s)\,, \tag{7.14}$$

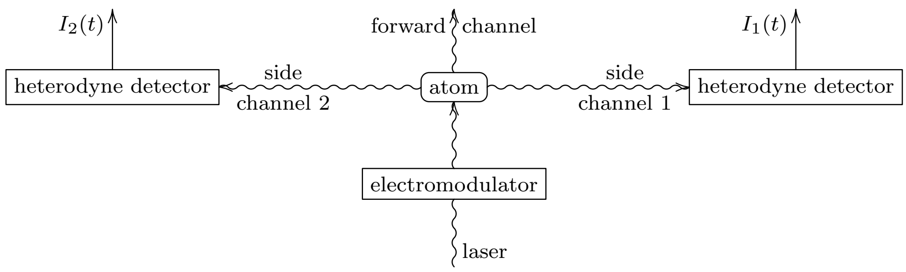

Figure 7.2: Channel 0: forward channel with laser; channel 1, channel 2: side channels without feedback.

where F is a *detector response function*, say

$$F(t) := k_1 \sqrt{\frac{\varkappa}{4\pi}} \exp\left\{-\frac{\varkappa}{2} t\right\}, \quad \varkappa > 0\,, \quad k_1 \neq 0\,. \tag{7.15}$$

Let us stress that, by recalling Eq. (6.16), the current I_1 has the form of an observable of the system. The constant k_1 and $\varkappa$ depend on the measuring apparatus; k_1 has the dimensions of a current and $\frac{1}{\varkappa}$ the dimensions of a time. The constant $\varkappa$ controls the time resolution: for $\varkappa \to +\infty$ the current $I_1(t)$ becomes formally proportional to the singular process $\dot{W}_1(t)$, and the past time are not involved: in other words, a big value of $\varkappa$ gives a good time resolution.

The electrical power carried out by the current I_1 is proportional to its square, say

$$P_1(t) = k_2 I_1(t)^2\,, \tag{7.16}$$

where $k_2 > 0$ has the dimensions of a resistance.

We underline that the detected output current is I_1. We shall use the side channel 2 to give some proposal for the feedback case: thus the output current of the channel 2 is I_2 and it is not detected but it will be signal used for the feedback.

Heterodyne and homodyne detection

The terms heterodyning and homodyning come from radio technique. The term homodyne detection is reserved for the case in which the local oscillator is in resonance with the carrier frequency of the field reaching the measuring apparatus. When the local oscillator is out of resonance the term heterodyne detection is used.

In our case we can speak of homodyne detection when $\nu = \omega$ and the stimulating light and the local oscillator are produced by the same source, because only in this case the phase is maintained in the course of time.

7.3 A two-level atom stimulated by a laser

Before to build up the physical model we introduce some useful mathematical objects. Let us remind that the physical model that we would give is a two level atom stimulated by a non perfectly monochromatic laser and with dissipative and (stochastic) dephasing effects.

The basic assumption is that the frequencies, the polarisations and the energies involved are such that a good description of the dynamics of the atom can be given by using only two non-degenerate levels. This fixes the Hilbert space:

$$\mathcal{H} := \mathbb{C}^2 \,. \tag{7.17}$$

The Pauli matrices and Bloch representation

A convenient way to treat with operators on $\mathbb{C}^2$ is to use the so called *Pauli matrices*:

$$\sigma_x \equiv \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y \equiv \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z \equiv \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Other useful matrices are the "lowering" operator σ_-, the "rising" operator σ_+, the projection on the "up state" P_+ and the projection on the "down state" P_-:

$$\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad P_+ := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us note that

$$\sigma_+ = \sigma_-^* \,, \qquad P_+ = \sigma_+\sigma_- \,, \qquad P_- = \sigma_-\sigma_+ \,,$$
$$2\sigma_\pm = \sigma_x \pm \mathrm{i}\sigma_y \,, \qquad 2P_\pm = \mathbb{1} \pm \sigma_z \,.$$

The matrices $\mathbb{1}$, σ_x, σ_y, σ_z are linearly independent and form a basis in M_2, called the Pauli basis. Then, any $\tau \in M_2$ can be written as

$$\tau = \frac{1}{2}\left(c_0\mathbb{1} + \vec{d}\cdot\vec{\sigma}\right), \quad c_0 = \mathrm{Tr}\{\tau\} \in \mathbb{C}, \quad \vec{d} = \mathrm{Tr}\left\{\vec{\sigma}\tau\right\} \in \mathbb{C}^3 \,, \tag{7.18}$$

where

$$\vec{\sigma} := \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.$$

As can be easily checked, every positive definite, trace one, 2×2 complex matrix ρ (a statistical operator) can be represented as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{pmatrix} = \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right). \tag{7.19}$$

This formula defines a statistical operator if and only if $\vec{x} \in \mathbb{R}^3$ and $|\vec{x}| \le 1$. The state ρ is pure ($\rho^2 = \rho$) if and only if $|\vec{x}| = 1$.

So, the statistical operators are represented by the points in the unit sphere in the 3-dimensional real space, which takes the name of *Bloch sphere*, and the pure states are represented by the points in the surface of this sphere. Given ρ, the *Bloch vector* $\vec{x}$ is obtained by

$$x_i = \mathrm{Tr}\{\sigma_i\,\rho\}, \qquad i = 1,2,3. \tag{7.20}$$

By using the Bloch representation the distance in trace norm between two statistical operators $\rho^{(i)} = \frac{1}{2}\left(\mathbb{1} + \vec{x}^{(i)} \cdot \vec{\sigma}\right)$ becomes very simple and significant and one can obtain

$$\left\|\rho^{(1)} - \rho^{(2)}\right\|_1 = \left|\vec{x}^{(1)} - \vec{x}^{(2)}\right|. \tag{7.21}$$

7.4 The phase diffusion model of a laser

To describe a not perfectly coherent laser we add a random phase to a plane wave. The *phase diffusion model* for a laser is to describe it by a wave of the form $\lambda \exp\{-\mathrm{i}\omega t + \mathrm{i}\varepsilon B(t)/2\}$, where $B(t)$ is a standard Wiener process. Note that its spectral density is given by a Lorentzian with width $\delta = \varepsilon^2/4$:

$$\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[\left|\int_0^T \lambda \exp\{-\mathrm{i}\omega t + \mathrm{i}\varepsilon B(t)/2\}\,\mathrm{d}t\right|^2\right]$$
$$= \lim_{T\to+\infty} \frac{2\,|\lambda|^2}{T}\,\mathrm{Re}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\omega(t-s)}\mathbb{E}\left[\mathrm{e}^{\frac{\mathrm{i}\varepsilon}{2}(B(t)-B(s))}\right] = \frac{|\lambda|^2\,\delta}{\omega^2 + \delta^2/4}.$$

Moreover, in the heterodyne detection scheme we have that the stimulating laser and the local oscillators are not in resonance and come out from different sources, that is they have different stochastic phases.

7.4.1 The operators of the system

As we mentioned, to completely determine the model we need to concretely choose h_k, f_j, D_k, S_{kj}, H_0.

The free Hamiltonian

The free Hamiltonian of the atom can be only a selfadjoint operator with two distinct eigenvalues; the traditional choice is to take it proportional to σ_z.

- The Hamiltonian of the free atom is

$$H_0 = \frac{1}{2}\,\omega_0\,\sigma_z\,, \qquad \omega_0 > 0\,. \tag{7.22}$$

The electromagnetic channels

The second step in constructing the model is to consider the interaction of the atom with the electromagnetic field. The operator matrix S can be used to describe some kind of scattering of light but, when the predominant effect responsible of the scattering of light is only emission/absorption, the operator matrix S can be taken to be the identity. Here we consider only this simplified situation; so, we take

- $S_{kj} = \delta_{kj}\mathbb{1}, \quad \forall k, j,$

where δ_{ij} is the Kronecker symbol.

When stimulated, the atom emits light in the whole solid angle, but it is enough to distinguish three channels for the outgoing light as schematised in Figure 7.1: one for the light which reaches each measuring apparatus (the beam splitter and the detectors), say channels 1 and 2, and one for the light which is lost in the surrounding free space, say the forward channel (we remind that according with Figure 7.2 we have two side channels, one for each measuring apparatus, and one forward channel). Usually the laser is well collimated and it involves a small solid angle in the forward direction, but as we have already said, in this channel we include also all other "forward" or "lateral" directions along which the light does not reach the detection apparatus. Instead, the side channels are made up of all the light rays which reach each detector, eventually after some focussing by lenses and mirrors.

In the so called "electric dipole and rotating wave approximations", when a photon is emitted into the external electromagnetic field, the atom makes a transition from the upper to the lower level, which means to take the emission operators proportional to σ_-.

- The "emission operators", in the dipole and rotating wave approximation, are given by

$$D_k = \sqrt{\gamma}\,\alpha_k\sigma_- \,, \qquad \sum_{k=1}^{3}|\alpha_k|^2 = 1\,, \quad \alpha_k \in \mathbb{C}, \quad k = 1,2,3\,.$$

  The quantity γ has the dimensions of 1/time and represents the *natural linewidth* of the atom. The quantities $|\alpha_1|^2$, $|\alpha_2|^2$ and $|\alpha_3|^2$ are the proportions of light in the side channels and in the forward channel, respectively.

Assumption 7.1. We assume that the physical parameter of the system γ is chosen such that $\gamma > 0$.

No feedback case

As sketched in Figure 7.2, the stimulating laser acts only in the forward channel; so we take:

- The stimulating laser is described by a phase diffusion model with carrier frequency $\omega > 0$:

$$f_k(t) = \delta_{3k}\lambda \exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon_3}{2}B_3(t)\right\}\,, \quad \lambda \in \mathbb{C}\,, \tag{7.23}$$

where, δ_{ij} is, as usual, the Kroneker symbol, ε_3 is an arbitrary real number and B_3 is a Brownian motion. Let us stress that the unique nonzero f_k is f_3.

The laser is said to be in resonance with the atom when $\omega = \omega_0$ and out of resonance or "detuned" when $\omega \neq \omega_0$. The intensity of the laser is proportional to $|\lambda|^2$, while the phase of λ gives the phase of the laser at time 0 in the location of the atom.

- The following quantity $\Delta\omega$ is called *detuning*:

$$\Delta\omega := \omega_0 - \omega\,. \tag{7.24}$$

Heterodyne detection

As already said, the phase factors $h_1(t)$ and $h_2(t)$ are produced by the interference with probing lasers, the local oscillators, which we describe again by phase diffusion models, while in channel 3 there is no local oscillator. So, we take

$$h_k(t) := \exp\left\{\mathrm{i}\nu t - \mathrm{i}\frac{\varepsilon_k}{2}B_k(t)\right\}\,, \quad k = 1, 2\,, \tag{7.25}$$

$$h_3(t) \equiv 1\,, \tag{7.26}$$

where ε_1 , ε_2 are two arbitrary real numbers and ν is the carrier frequency.

All the Wiener processes are taken to be independent. So, we can say that we have a multidimensional standard Wiener process with W_1, W_2, W_3 associated with the three electromagnetic channels and $W_4 = B_1$, $W_5 = B_2$, $W_6 = B_3$. The channels 4, 5, 6 do not contribute to the dissipative part of the dynamics and we take $R_4 = R_5 = R_6 = 0$.

Other dissipative effects

As we said in the introduction, the unobserved and eliminable channels can be used to introduce in the model some Markovian dissipative effects. In our model of the two level atom we shall introduce effects due to the interaction with a "thermal bath" and the dephasing effect. In these terms there is no contribution like f_k and they appear only in the dissipative part of the Liouville operator, i.e. in $\mathcal{L}_0$ defined in Eq. (7.10); so, the coefficients can be taken positive.

- Dephasing term:
$$D_7 = \sqrt{\gamma k_{\mathrm{d}}}\sigma_z\,, \quad k_{\mathrm{d}} \geq 0\,. \tag{7.27}$$
- Terms simulating a thermal bath
$$D_8 = \sqrt{\gamma\bar{n}}\sigma_-\,; \quad D_9 = \sqrt{\gamma\bar{n}}\sigma_+\,, \qquad \bar{n} \geq 0. \tag{7.28}$$

The coefficient $\sqrt{\gamma}$ has been introduced in all the terms just by dimensional reasons; in this way k_{d} and $\bar{n}$ result to be dimensionless. As discussed, the phase factors related to the unobserved channels do not influence the unobserved channels themselves and, so, they can be chosen arbitrarily. We take $h_7(t) = h_8(t) = h_9(t) = 1$.

To be precise the interaction with a thermal bath does not vanish for zero temperature ($\overline{n} = 0$), but introduces also a term of emission type, which can be absorbed in a redefinition of γ. So, the term with the lost light contains also what is lost in the thermal bath.

7.4.2 The final model for the heterodyne detection

By recalling Eqs. (7.7), (7.22), (7.23), (7.25), (7.26), (7.27), (7.28), we have that the final model for the heterodyne detection of a two level atom stimulated by a non-monochromatic laser is

$$R_1(t) = \exp\left\{\mathrm{i}\nu t - \mathrm{i}\frac{\varepsilon_1}{2}B_1(t)\right\}\sqrt{\gamma}\alpha_1\sigma_-\,; \tag{7.29a}$$

$$R_2(t) = \exp\left\{\mathrm{i}\nu t - \mathrm{i}\frac{\varepsilon_2}{2}B_2(t)\right\}\sqrt{\gamma}\alpha_2\sigma_-\,; \tag{7.29b}$$

$$R_3(t) = \sqrt{\gamma}\alpha_3\sigma_- + \lambda\exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon_3}{2}B_3(t)\right\}\mathbb{1}\,; \tag{7.29c}$$

$$\textstyle\sum_{i=1}^{3}|\alpha_i|^2 = 1\,, \quad \alpha_1\,,\ \alpha_2\,,\ \alpha_3 \in \mathbb{C}\,; \tag{7.29d}$$

$$R_4 = R_5 = R_6 = 0\,; \tag{7.29e}$$

$$R_7(t) \equiv R_7 = \sqrt{\gamma k_{\mathrm{d}}}\sigma_z\,; \tag{7.29f}$$

$$R_8(t) \equiv R_8 = \sqrt{\gamma\bar{n}}\sigma_-\,, \quad R_9(t) \equiv R_9 = \sqrt{\gamma\bar{n}}\sigma_+\,; \tag{7.29g}$$

$$H_0 = \frac{\omega_0}{2}\sigma_z\,; \tag{7.29h}$$

$$B_1 = W_4\,,\ B_2 = W_5\,,\ B_3 = W_6\,. \tag{7.29i}$$

In [3] the case $\varepsilon_1 = \varepsilon_3 = 0$ has been studied: in this work we shall consider the case under the following assumption

Assumption 7.2. In the following treatment, we assume

$$\varepsilon_1^2 + \varepsilon_3^2 > 0\,.$$

The generator of the reduced dynamic

By using the definition stated in Eqs. (7.29) and Eq. (7.13) and replacing them in Eq. (7.9), by direct computation, we obtain the following form for the *stochastic Liouville operator* of the reduced dynamic

$$\begin{aligned}\mathcal{L}(t)[\tau] = {} & -\frac{\mathrm{i}}{2}\omega_0[\sigma_z,\tau] + \gamma k_{\mathrm{d}}\left(\sigma_z\tau\sigma_z - \tau\right) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\left\{P_+,\tau\right\}\right)\\ & + \gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\left\{P_-,\tau\right\}\right) + \sqrt{\gamma}\alpha_3\overline{\lambda}\exp\left\{\mathrm{i}\omega t - \mathrm{i}\frac{\varepsilon_3}{2}B_3(t)\right\}[\sigma_-,\tau]\\ & - \sqrt{\gamma}\overline{\alpha}_3\lambda\exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon_3}{2}B_3(t)\right\}[\sigma_+,\tau]\,.\end{aligned}$$

Let us stress that the Liouvillian is random because it depends on the one-dimensional Wiener process B_3 and not on other stochastic terms.

The Rabi frequency

The following quantity Ω is called *Rabi frequency*:

$$\Omega := 2\sqrt{\gamma}\,|\lambda|\,|\alpha_3|\,. \tag{7.30}$$

Let us stress that of the parameters Ω, $|\lambda|$, $|\alpha_3|$, the physical ones are Ω and $|\alpha_3|$. Indeed, as we shall see, in the mean dynamics only the parameter Ω appears, not $|\lambda|$ and $|\alpha_3|$ by themselves. We already said that $|\alpha_3|^2$ is the percentage of the lost light. If the percentage $|\alpha_3|^2$ of lost light is changed by changing the measuring apparatus, the laser/atom interaction does not change and the mean dynamics cannot change; so, $|\lambda|$ has to be changed in order to maintain Ω constant.

About the phases of λ and α_3, let us note that in Dirac notation we can write $\sigma_- = |\mathrm{down}\rangle\langle\mathrm{up}|$ and by redefining the relative phase of the two states we can absorb into σ_- the constant phase of the factor multiplying it. This means that, without loss of generality, we can assume

$$\arg\left(\mathrm{i}\overline{\lambda}\,\alpha_3\right) = 0. \tag{7.31}$$

Now we have no more freedom in changing phases. We observe now that with this choice of the argument of $\alpha_3\overline{\lambda}$ we have that $\mathrm{Re}(\alpha_3\overline{\lambda}) = 0$. Indeed,

$$\alpha_3\overline{\lambda} = -\mathrm{i}(\mathrm{i}\alpha_3\overline{\lambda}) = -\mathrm{i}|\lambda||\alpha_3|\mathrm{e}^{\mathrm{i}\arg(\mathrm{i}\alpha_3\overline{\lambda})} = -\mathrm{i}\frac{\Omega}{2\sqrt{\gamma}}\,.$$

Furthermore we have

$$\mathrm{Im}(\alpha_3\overline{\lambda}) = -\frac{\Omega}{2\sqrt{\gamma}}\,.$$

In conclusion, now the Liouville operator becomes

$$\begin{aligned}\mathcal{L}(t)[\tau] = -\frac{\mathrm{i}}{2}\,\omega_0[\sigma_z,\tau] + \gamma k_{\mathrm{d}}\left(\sigma_z\tau\sigma_z - \tau\right) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\left\{P_+,\tau\right\}\right) \\ + \gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\left\{P_-,\tau\right\}\right) - \mathrm{i}\frac{\Omega}{2}\exp\left\{\mathrm{i}\omega t - \mathrm{i}\frac{\varepsilon_3}{2}B_3(t)\right\}[\sigma_+,\tau] \\ - \mathrm{i}\frac{\Omega}{2}\exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon_3}{2}B_3(t)\right\}[\sigma_-,\tau]\,.\end{aligned} \tag{7.32}$$

Summary of the involved parameters and quantities in the heterodyne model. For simplicity, we gather in this paragraph the parameters and the most important quantities involved in our model.

- Side channel 1: the channel of the detected light.
- Side channel 2: the channel for the feedback proposals.
- Forward chanel: the channel of the lost light and of the stimulating laser.
- Components of the Wiener process used to introduce a heterodyne random phase in the two local oscillators: $W_4 \equiv B_1$ and $W_5 \equiv B_2$.
- Components of the Wiener process used to introduce random phase in the stimulating laser: $W_6 \equiv B_3$.
- Carrier frequency of the two local oscillators: $\nu > 0$.
- Carrier frequency of the stimulating laser: $\omega > 0$.
- Intensity of the stimulating laser: λ.
- Parameters to control the intensity of the randomness in the phase of the local oscillators and of the stimulating laser, respectively: ε_1, ε_2 and ε_3.

- Natural linewidth of the atom: $\gamma > 0$.
- Control parameter in the thermal bath: $\bar{n} \geq 0$.
- Control parameter in the dephasing: $k_d \geq 0$.
- Proportionality constant of the free hamiltonian of the system: $\omega_0 > 0$.
- Detuning: $\Delta\omega = \omega - \omega_0$.
- Proportions of the light in the side channels and in the forward channel respectively: α_1, α_2 and α_2 such that $\sum_{k=1}^{3} |\alpha_k|^2 = 1$.
- Detection random operator: $R_1(t) = \gamma\alpha_1 \mathrm{e}^{\mathrm{i}(\nu-\omega)-\mathrm{i}\frac{\varepsilon_1}{2}B_1(t)}\sigma_-$.
- Rabi frequency: $\Omega = 2|\lambda||\alpha_3|$. We assume $\arg(\mathrm{i}\alpha_3\overline{\lambda}) = 0$.
- Output current of the channel 1: $I_1(t) = \int_0^t F(t-s)\mathrm{d}W_1(s)$.
- Detector response function: $F(t) = k_1\sqrt{\frac{\varkappa}{4\pi}} \exp\left\{-\frac{\varkappa}{2}t\right\}$, $\quad \varkappa > 0\,, \quad k_1 \neq 0$.
- Electrical power carried out by the current $I_1(t)$: $P_1(t) = k_2 I_1(t)^2\,, \quad k_2 > 0$.

The reduced linear stochastic master equation

Now we want to give the reduced linear stochastic master equation for the reduced model. Recalling the notations that we stated in the introduction to this chapter we have:

$$d = 9\,; \quad m = 6\,; \quad \bar{m} = 2\,. \tag{7.33}$$

Let us stress that we are assuming to not observe the output of channel 3 (the forward channel) so, we shall take the mean also on the component W_3 of the Wiener process. The filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$ is the augmented filtration generated by the first $m = 6$ components of the Wiener process, W_3 excluded, that is

$$\mathcal{E}_t^0 = \sigma\Big\{W_1(r),\ W_2(r),\ W_4(r),\ W_5(r),\ W_6(r)\,; \quad r \in [0,t]\Big\}\vee\mathcal{N}\,, \quad \forall t \geq 0\,. \tag{7.34}$$

The definition of the process ϱ is that one stated in Eq. (5.4), but now the conditioning filtration $\mathcal{E}_t^0$ is given in Eq. (7.34). To obtain the linear reduced stochastic master equation for the final model, it is enough to go back to Proposition 5.3 using the notations just introduced in Eqs. (7.33) and (7.34). Then, we end up with the following reduced linear stochastic master equation for the final model given by Eqs. (7.29)

$$\varrho(t) = \eta_0 + \int_0^t \mathcal{L}(s)[\varrho(s)]\mathrm{d}s + \sum_{j=1}^{2}\int_0^t \mathcal{R}_j(s)[\varrho(s)]\mathrm{d}W_j(s)\,, \tag{7.35}$$

and the consistent family of the physical probabilities is given by

$$\mathbb{P}_{\eta_0}^t(E) := \mathbb{E}_{\mathbb{Q}}\left[1_E \operatorname{Tr}\{\varrho(t)\}\right]\,, \quad \forall E \in \mathcal{E}_t^0\,. \tag{7.36}$$

Remark 7.1. In the introduction to this chapter we noted that if we consider the filtration generated only by the output of the system it is impossible to obtain a closed equation for the process obtained conditioning the process σ with respect to this filtration. Now the situation is very clear: using the same notations of Eq. (7.2), we have

$$\mathcal{D}_t^0 = \sigma\Big\{W_1(r),\ W_2(r);\quad r\in[0,t]\Big\}\vee\mathcal{N}\,,\quad \forall t\ge 0\,.$$

Then, the process

$$\zeta(t) = \mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{D}_t^0]$$

does not satisfy a closed stochastic differential equation in the basis $(\Omega,\mathcal{D},\{\mathcal{D}_t^0\}_{t\ge0})$ because the stochastic Liouvillian given in Eq. (7.32) depends on $B_3 = W_6$, which is adapted with respect to $\mathcal{E}_t^0$, but not with respect to $\mathcal{D}_t^0$.

Rotating frame

In the next section we shall compute some moments of the output process, in other words we have to study Eqs. (6.32) and (6.34). To obtain these objects we have to consider a unitary linear transformation of the process ϱ: we introduce the process $\breve\varrho$, which is ϱ in a rotating frame, as

$$\breve\varrho(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\varrho(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\,. \tag{7.37}$$

The equation in the rotating frame. To obtain the stochastic differential of the process ϱ in the rotating frame we apply the Itô formula for products to Eq. (7.37) and we take into account that $\mathrm{d}\rho$ does not contain $\mathrm{d}B_3$. We obtain

$$\begin{aligned}
\mathrm{d}\breve\varrho(t) = \mathrm{d}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\varrho(t)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\right)&\\
=\left(\mathrm{d}\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\right)\,\varrho(t)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}&\\
+\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\left(\mathrm{d}\varrho(t)\right)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}&\\
+\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\varrho(t)\left(\mathrm{d}\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\right)&\\
+\left(\mathrm{d}\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\right)\,\varrho(t)\,\left(\mathrm{d}\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\right)&\\
=\left\{\left[\frac{\mathrm{i}}{2}\,\omega\sigma_z-\frac{\varepsilon_3^2}{32}\,\mathbb{1}\right]\mathrm{d}t-\mathrm{i}\frac{\varepsilon_3}{4}\,\sigma_z\mathrm{d}B_3(t)\right\}\breve\varrho(t)&\\
+\breve\varrho(t)\left\{-\left[\frac{\mathrm{i}}{2}\,\omega\sigma_z+\frac{\varepsilon_3^2}{32}\,\mathbb{1}\right]\mathrm{d}t+\mathrm{i}\frac{\varepsilon_3}{4}\,\sigma_z\mathrm{d}B_3(t)\right\}&\\
+\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\mathcal{L}(t)[\varrho(t)]\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\mathrm{d}t&\\
+\sum_{j=1}^{2}\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\mathcal{R}_j(t)[\varrho(t)]\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t-\frac{\varepsilon_3}{2}B_3(t)\right)\sigma_z}\mathrm{d}W_j(t)+\frac{\varepsilon_3^2}{16}\,\sigma_z\breve\varrho(t)\sigma_z\mathrm{d}t&
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varepsilon_3^2}{16}\,\sigma_z\check{\varrho}(t)\sigma_z \mathrm{d}t + \left\{ \left[\frac{\mathrm{i}}{2}\,\omega\sigma_z - \frac{\varepsilon_3^2}{32}\,\mathbb{1}\right]\mathrm{d}t - \mathrm{i}\frac{\varepsilon_3}{4}\,\sigma_z \mathrm{d}B_3(t)\right\}\check{\varrho}(t) \\
&\qquad + \check{\varrho}(t)\left\{ - \left[\frac{\mathrm{i}}{2}\,\omega\sigma_z + \frac{\varepsilon_3^2}{32}\,\mathbb{1}\right]\mathrm{d}t + \mathrm{i}\frac{\varepsilon_3}{4}\,\sigma_z\mathrm{d}B_3(t)\right\} \\
&+ \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\mathcal{L}(t)\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\check{\varrho}(t)\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\mathrm{d}t \\
&\qquad + \sum_{j=1}^{2}\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\mathcal{R}_j(t)\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\check{\varrho}(t)\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\right] \\
&\qquad\qquad \times \mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\mathrm{d}W_j(t).
\end{aligned}
$$

We have to study now the following quantity, $\forall \tau \in M_n(\mathbb{C})$,

$$
\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\mathcal{L}(t)\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\tau\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}. \tag{7.38}
$$

By recalling the definition of $\mathcal{L}(t)$, given in Eq. (7.32) and using that

$$
\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\sigma_-\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z} = \mathrm{e}^{-\mathrm{i}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)}\sigma_-\,, \tag{7.39a}
$$

$$
\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\sigma_+\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z} = \mathrm{e}^{\mathrm{i}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)}\sigma_+\,, \tag{7.39b}
$$

we obtain, $\forall \tau \in M_n(\mathbb{C})$,

$$
\begin{aligned}
&\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\mathcal{L}(t)\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\tau\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z} \\
&\quad = -\frac{\mathrm{i}}{2}\,\omega_0[\sigma_z,\tau] + \gamma k_{\mathrm{d}}\left(\sigma_z\tau\sigma_z - \tau\right) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right) \\
&\qquad\qquad + \gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\{P_-,\tau\}\right) - \mathrm{i}\frac{\Omega}{2}[\sigma_x,\tau],
\end{aligned}
$$

which is a deterministic and time independent operator.

On the other hand, from the definition of R_j given in (7.29) and by Eqs. (7.39), we have

$$
\begin{aligned}
\check{R}_1(t) &:= \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}R_1(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z} \\
&= \exp\left\{\mathrm{i}\left(\nu-\omega\right)t - \frac{\mathrm{i}}{2}\left(\varepsilon_1 B_1(t) - \varepsilon_3 B_3(t)\right)\right\}\sqrt{\gamma}\alpha_1\sigma_-
\end{aligned} \tag{7.40}
$$

$$
\begin{aligned}
\check{R}_2(t) &:= \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z}R_2(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon_3}{2}\,B_3(t)\right)\sigma_z} \\
&= \exp\left\{\mathrm{i}\left(\nu-\omega\right)t - \frac{\mathrm{i}}{2}\left(\varepsilon_2 B_2(t) - \varepsilon_3 B_3(t)\right)\right\}\sqrt{\gamma}\alpha_2\sigma_-\,.
\end{aligned} \tag{7.41}
$$

Then, by defining

$$
\check{\mathcal{R}}_j(t)[\tau] = \check{R}_j(t)\tau + \tau\check{R}_j^*(t)\,, \quad \forall \tau \in M_n(\mathbb{C})\,, \quad j=1,2\,, \tag{7.42}
$$

we can write

$$\mathrm{d}\check{\varrho}(t) = \check{\mathcal{L}}[\check{\varrho}(t)]\mathrm{d}t + \sum_{j=1}^{2} \check{\mathcal{R}}_j(t)[\check{\varrho}(t)]\mathrm{d}W_j(t) - \frac{\mathrm{i}\varepsilon_3}{4}\left[\sigma_z, \check{\varrho}(t)\right] \mathrm{d}B_3(t)\,, \tag{7.43}$$

where

$$\begin{aligned}\check{\mathcal{L}}[\tau] := -\frac{\mathrm{i}}{2}\Delta\omega[\sigma_z,\tau] + \left(\gamma k_\mathrm{d} + \frac{{\varepsilon_3}^2}{16}\right)(\sigma_z\tau\sigma_z - \tau) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right)\\ + \gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\{P_-,\tau\}\right) - \mathrm{i}\frac{\Omega}{2}\left[\sigma_x,\,\tau\right]\end{aligned} \tag{7.44}$$

is a bona fide Liouville operator. Let us note that the phase-diffusion effect of B_3 has increased the coefficient of the dephasing term, with respect to [3].

7.5 Moments in the heterodyne case

Recalling Eqs. (6.32) and (6.34), to compute the moments of the output of channel 1 of our quantum system, in the heterodyne detection situation, we have to calculate $\mathbb{E}_\mathbb{Q}\left[\mathrm{Tr}\{\mathcal{R}_1(t)[\varrho(t)]\}\right]$ and $\mathbb{E}_\mathbb{Q}\left[\mathrm{Tr}\{\mathcal{R}_1(t)\circ\Lambda(t,s)\circ\mathcal{R}_1(s)[\varrho(s)]\}\right]$.

7.5.1 The mean

First of all we observe that

$$\mathrm{Tr}\{\mathcal{R}_1(t)[\varrho(t)]\} = \mathrm{Tr}\{\check{\mathcal{R}}_1(t)[\check{\varrho}(t)]\}\,. \tag{7.45}$$

This is an immediate consequence of the forms of the operators involved and to prove this relation it is enough to use the cyclic property of the trace.

Let us set $\{\mathcal{G}_t\}_{t\geq 0}$ for the augmented natural filtration of B_1, B_3 and define the process ξ as

$$\xi(t) = \mathbb{E}_\mathbb{Q}[\check{\varrho}(t)|\mathcal{G}_t] \tag{7.46}$$

By Eqs. (7.40) and (7.42) we have that the process $\check{\mathcal{R}}_1$ is $\{\mathcal{G}_t\}_{t\geq 0}$-measurable and then, by the properties of conditional expectations, we have

$$\mathbb{E}_\mathbb{Q}[\mathrm{Tr}\{\check{\mathcal{R}}_1(t)[\check{\varrho}(t)]\}] = \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{Q}[\mathrm{Tr}\{\check{\mathcal{R}}_1(t)[\check{\varrho}(t)]\}|\mathcal{G}_t]\right] = \mathbb{E}_\mathbb{Q}[\mathrm{Tr}\{\check{\mathcal{R}}_1(t)[\xi(t)]\}]\,. \tag{7.47}$$

Then, we can obtain the mean of the output of channel 1 by $\mathbb{E}_\mathbb{Q}[\mathrm{Tr}\{\check{\mathcal{R}}_1(t)[\xi(t)]\}]$.

Let us observe that the inclusion

$$\mathcal{G}_t \subset \mathcal{E}_t^0\,, \quad \forall t \geq 0\,,$$

holds and then, by an analogue of Proposition 5.1, we can write, for all $\mathcal{F}_t$-measurable random variables X,

$$\mathbb{E}_\mathbb{Q}[X|\mathcal{G}_t] = \mathbb{E}_\mathbb{Q}[X|\mathcal{G}]\,, \tag{7.48}$$

where

$$\mathcal{G} := \bigvee_{t\geq 0}\mathcal{G}_t\,.$$

By this property we can compute the stochastic differential of $\xi(t)$ conditioning Eq. (7.43) with respect to $\mathcal{G}_t$. Indeed, by the independence properties of the Wiener process and the properties of the conditional expectations, we have

$$\begin{aligned}\xi(t) &= \mathbb{E}_\mathbb{Q}[\check{\varrho}(t)|\mathcal{G}_t] = \mathbb{E}_\mathbb{Q}[\check{\varrho}(t)|\mathcal{G}] \\ &= \mathbb{E}_\mathbb{Q}\left[\check{\varrho}(0) + \int_0^t \check{\mathcal{L}}[\check{\varrho}(s)]\mathrm{d}s + \sum_{j=1}^2 \int_0^t \check{\mathcal{R}}_j(s)[\check{\varrho}(s)]\mathrm{d}W_j(s) - \mathrm{i}\frac{\varepsilon_3}{4}\int_0^t [\sigma_z, \check{\varrho}(s)]\mathrm{d}B_3(s)\middle|\mathcal{G}\right] \\ &= \xi(0) + \int_0^t \check{\mathcal{L}}[\xi(s)]\mathrm{d}s - \mathrm{i}\frac{\varepsilon_3}{4}\int_0^t [\sigma_z, \xi(s)]\mathrm{d}B_3(s)\end{aligned}$$

or equivalently

$$\mathrm{d}\xi(t) = \check{\mathcal{L}}[\xi(t)]\mathrm{d}t - \frac{\mathrm{i}\varepsilon_3}{4}\,[\sigma_z, \xi(t)]\,\mathrm{d}B_3(t)\,. \tag{7.49}$$

Remark 7.2. Let us underline that Eq. (7.49) is a SDE with constant coefficients. Then the uniqueness of the solution is a straightforward consequence of classical results for SDEs (see [2]).

If we define $\zeta(t)$ as the mean of $\xi(t)$,

$$\zeta(t) := \mathbb{E}_\mathbb{Q}[\xi(t)]\,, \tag{7.50}$$

from Eq. (7.49) we have

$$\mathrm{d}\zeta(t) = \check{\mathcal{L}}[\zeta(t)]\mathrm{d}t\,, \tag{7.51}$$

whose solution is

$$\zeta(t) = \mathrm{e}^{\check{\mathcal{L}}t}[\eta_0]\,. \tag{7.52}$$

As we noted, $\check{\mathcal{L}}$ is a bona fide Liouville operator: it can be written in the Lindblad form, see [11], and then it is the generator of a quantum dynamical semigroup. For this reason, Eq. (7.52) defines, for every $t \geq 0$, an element belonging to $S(\mathcal{H})$.

To compute the mean of the output of channel 1 we have to calculate the mean of

$$z_\pm(t) := \exp\left\{\pm\left[\mathrm{i}\,(\nu-\omega)\,t - \frac{\mathrm{i}}{2}\,(\varepsilon_1 B_1(t) - \varepsilon_3 B_3(t))\right]\right\}\xi(t)\,. \tag{7.53}$$

First we consider the process z_+ (z_- is its conjugate process). The stochastic differential of this process is

$$\begin{aligned}\mathrm{d}z_+(t) = &\left\{\left(\mathrm{i}(\nu-\omega) - \frac{\varepsilon_1^2}{8} - \frac{\varepsilon_3^2}{8}\right) z_+(t) + \check{\mathcal{L}}[z_+(t)] + \frac{\varepsilon_3^2}{8}\,[\sigma_z, z_+(t)]\right\}\mathrm{d}t \\ &+ \mathrm{i}\frac{\varepsilon_3}{2}\left(z_+(t) - \frac{1}{2}\,[\sigma_z, z_+(t)]\right)\mathrm{d}B_3(t) - \mathrm{i}\frac{\varepsilon_1}{2}\,z_+(t)\mathrm{d}B_1(t)\,.\end{aligned}$$

For the mean of $z_+(t)$, we write

$$\mu_+(t) := \mathbb{E}_\mathbb{Q}[z_+(t)]\,. \tag{7.54}$$

By the just derived SDE for z_+ we have

$$\mu_+(t) = \mu_+(0) + \left[\mathrm{i}(\nu-\omega) - \frac{\varepsilon_1^2}{8} - \frac{\varepsilon_3^2}{8}\right]\int_0^t \mu_+(s)\mathrm{d}s + \int_0^t \check{\mathcal{K}}[\mu_+(s)]\mathrm{d}s\,, \tag{7.55}$$

where, $\forall \tau \in M_n(\mathbb{C})$,

$$\check{\mathcal{K}}[\tau] := \check{\mathcal{L}}[\tau] + \frac{\varepsilon_3^2}{8}\,[\sigma_z, \tau]\,. \tag{7.56}$$

Then, we have that $\check{\mathcal{K}}$ is the infinitesimal generator of a semigroup and we can write the solution of Eq. (7.55) as

$$\mu_+(t) = \mathrm{e}^{\left\{\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{8}-\frac{\varepsilon_3^2}{8}\right\}t}\,\mathrm{e}^{\check{\mathcal{K}}t}[\mu_+(0)]\,.$$

Let us stress that the following chain of equalities holds

$$\begin{aligned}\mu_\pm(0) = \mathbb{E}_\mathbb{Q}[z_\pm(0)] = \mathbb{E}_\mathbb{Q}[\xi(0)] = \mathbb{E}_\mathbb{Q}\left[\mathbb{E}_\mathbb{Q}[\check{\varrho}(0)|\mathcal{G}_0]\right]\\ = \mathbb{E}_\mathbb{Q}[\check{\varrho}(0)] = \mathbb{E}_\mathbb{Q}[\varrho_0] = \eta_0\,;\end{aligned} \tag{7.57}$$

so, in conclusion, we have

$$\mu_+(t) = \mathrm{e}^{\left\{\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{8}-\frac{\varepsilon_3^2}{8}\right\}t}\,\mathrm{e}^{\check{\mathcal{K}}t}[\eta_0]\,. \tag{7.58}$$

On the other hand, we have

$$z_-(t) = z_+(t)^*, \qquad \mu_-(t) = \mu_+(t)^*. \tag{7.59}$$

So, it follows that

$$\begin{aligned}\mathbb{E}^T_{\eta_0}[\dot{W}_1(t)] &= \mathbb{E}_\mathbb{Q}[\mathrm{Tr}\,\{\mathcal{R}_1(t)[\varrho(t)]\}] = \mathbb{E}_\mathbb{Q}[\mathrm{Tr}\,\{\check{\mathcal{R}}_1(t)[\check{\varrho}(t)]\}] = \mathbb{E}_\mathbb{Q}[\mathrm{Tr}\,\{\check{\mathcal{R}}_1(t)[\xi(t)]\}]\\ &= \mathrm{Tr}\,\{\mathbb{E}_\mathbb{Q}\,[\check{\mathcal{R}}_1(t)[\xi(t)]]\} = \sqrt{\gamma}\alpha_1\,\mathrm{Tr}\,\{\sigma_-\mathbb{E}_\mathbb{Q}\,[z_+(t)]\} + \sqrt{\gamma}\overline{\alpha}_1\,\mathrm{Tr}\,\{\mathbb{E}_\mathbb{Q}\,[z_-(t)]\,\sigma_+\}\\ &= \sqrt{\gamma}\alpha_1\mathrm{e}^{\left\{\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{8}-\frac{\varepsilon_3^2}{8}\right\}t}\,\mathrm{Tr}\left\{\sigma_-\mathrm{e}^{\check{\mathcal{K}}t}[\eta_0]\right\} + \mathrm{c.c.}\,,\end{aligned}$$

where we write c.c. instead of the conjugate component. Then we have

$$\mathbb{E}^T_{\eta_0}[\dot{W}_1(t)] = 2\sqrt{\gamma}\,\mathrm{Re}\,\mathrm{Tr}\left\{\mathrm{e}^{\left\{\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{8}-\frac{\varepsilon_3^2}{8}\right\}t}\alpha_1\sigma_-\mathrm{e}^{\check{\mathcal{K}}t}[\eta_0]\right\}. \tag{7.60}$$

7.5.2 Second moments

To study the second moments of the output of the channel 1 of our quantum system we shall use Eq. (6.34), but using the propagator of Eq. (7.49). First of all, we introduce the propagator $\check{\Lambda}$ of the process $\check{\varrho}$: by Eq. (7.43), we have that the $\check{\Lambda}$ satisfy the stochastic differential equation

$$\begin{cases}\mathrm{d}\check{\Lambda}(t,s) = \check{\mathcal{L}}\circ\check{\Lambda}(t,s)\mathrm{d}t + \displaystyle\sum_{j=1}^{2}\check{\mathcal{R}}_j(t)\circ\check{\Lambda}(t,s)\mathrm{d}W_j(t) - \mathrm{i}\frac{\varepsilon_3}{2}[\sigma_z,(\cdot)]\circ\check{\Lambda}(t,s)\mathrm{d}B_3(t)\,,\\ \check{\Lambda}(s,s) = \mathrm{Id}_n\,.\end{cases} \tag{7.61}$$

On the other hand, if we name $\Xi(t,s)$ the propagator Eq. (7.49), it satisfies the following SDE

$$\begin{cases} \mathrm{d}\Xi(t,s) = \check{\mathcal{L}} \circ \Xi(t,s)\mathrm{d}t - \mathrm{i}\dfrac{\varepsilon_3}{2}[\sigma_z,(\cdot)] \circ \Xi(t,s)\mathrm{d}B_3(t)\,, \\ \\ \Xi(s,s) = \mathrm{Id}_n\,. \end{cases} \tag{7.62}$$

Let us stress that this is a SDE with constant coefficients, then the uniqueness of the solution is an immediate consequence of classical results for SDEs. By the uniqueness of the solution of this equation we can claim that

$$\Xi(t,s) = \mathbb{E}_{\mathbb{Q}}\left[\check{\Lambda}(t,s)|\mathcal{G}_t\right]\,, \quad \mathbb{Q}\text{-a.s.} \tag{7.63}$$

Indeed, conditoning Eq. (7.61) with respect to $\mathcal{G}_t$ and using Eq. (7.48), it is trivial to show that the two sides of the equation above satisfy the same SDE. The statement follows by uniqueness.

Then, by the measurability of $\check{\mathcal{R}}_1(t)$ with respect to $\mathcal{G}_t$ and the properties of the conditional expectations, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{R}_1(t)\circ\Lambda(t,s)\circ\mathcal{R}_1(s)[\varrho(s)]\right\}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\check{\mathcal{R}}_1(t)\circ\check{\Lambda}(t,s)\circ\check{\mathcal{R}}_1(s)[\check{\varrho}(s)]\right\}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\check{\mathcal{R}}_1(t)\circ\Xi(t,s)\circ\check{\mathcal{R}}_1(s)[\xi(s)]\right\}\right]\,. \end{aligned} \tag{7.64}$$

In conclusion we have to study the map-valued processes

$$\begin{aligned} &\exp\left\{\pm\left[\mathrm{i}\,(\nu-\omega)\,t - \frac{\mathrm{i}}{2}\left(\varepsilon_1 B_1(t) - \varepsilon_3 B_3(t)\right)\right]\right\}\Xi(t,s) \\ &\qquad = \exp\left\{\pm\left[\mathrm{i}\,(\nu-\omega)\,s - \frac{\mathrm{i}}{2}\left(\varepsilon_1 B_1(s) - \varepsilon_3 B_3(s)\right)\right]\right\}Z_\pm(t,s), \end{aligned}$$

where we have defined

$$Z_\pm(t,s) = \mathrm{e}^{\pm\left[\mathrm{i}(\nu-\omega)(t-s) - \frac{\mathrm{i}}{2}(\varepsilon_1(B_1(t)-B_1(s)) - \varepsilon_3(B_3(t)-B_3(s)))\right]}\Xi(t,s)\,. \tag{7.65}$$

Moreover, the following relation holds

$$Z_-(t,s) = \Big(Z_+(t,s)[\tau^*]\Big)^*\,.$$

Let us consider the process Z_+ (Z_- is its adjoint process). It is easy to show that this process fulfills the following SDE

$$\begin{aligned} \mathrm{d}Z_+(t,s) = &\left\{\left(\mathrm{i}(\nu-\omega) - \frac{\varepsilon_1^2}{8} - \frac{\varepsilon_3^2}{8}\right)Z_+(t,s) + \check{\mathcal{L}}\circ Z_+(t,s) + \frac{\varepsilon_3^2}{8}\left[\sigma_z,(\cdot)\right]\circ Z_+(t,s)\right\}\mathrm{d}t \\ &+ \mathrm{i}\frac{\varepsilon_3}{2}\left(Z_+(t,s) - \frac{1}{2}\left[\sigma_z,(\cdot)\right]\circ Z_+(t,s)\right)\mathrm{d}B_3(t) - \mathrm{i}\frac{\varepsilon_1}{2}Z_+(t,s)\mathrm{d}B_1(t)\,. \end{aligned} \tag{7.66}$$

Then, if we define

$$M_+(t,s) := \mathbb{E}_{\mathbb{Q}}[Z_+(t,s)]\,, \tag{7.67}$$

with the same calculations as before, we have

$$M_+(t,s) = \mathrm{Id}_n + \left[\mathrm{i}(\nu-\omega) - \frac{\varepsilon_1^2}{8} - \frac{\varepsilon_3^2}{8}\right]\int_s^t M_+(q,s)\mathrm{d}q + \int_s^t \check{\mathcal{K}} \circ M_+(q,s)\mathrm{d}q\,,$$

whose solution is

$$M_+(t,s) = \mathrm{e}^{\left\{\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{8}-\frac{\varepsilon_3^2}{8}\right\}(t-s)}\mathrm{e}^{\check{\mathcal{K}}(t-s)}\,, \tag{7.68}$$

where $\check{\mathcal{K}}$ has been defined in Eq. (7.56). Let us note that M_+ does not depend on t and s but only on their difference and then we make the identification

$$M_+(t,s) \equiv M_+(t-s)\,.$$

For the process $Z_-(t,s)$, we set

$$M_-(t-s) := \mathbb{E}_{\mathbb{Q}}[Z_-(t,s)] \tag{7.69}$$

and we obtain

$$M_-(t-s)[\tau] = \Big(M_+(t-s)[\tau^*]\Big)^*, \quad \forall \tau \in M_2(\mathbb{C})\,. \tag{7.70}$$

By direct calculation we have

$$\begin{aligned}\check{\mathcal{R}}_1(t) \circ \Xi(t,s) \circ \check{\mathcal{R}}_1[\xi(s)] = \gamma|\alpha_1|^2 \left\{Z_-(t,s)[\sigma_-\xi(s)]\sigma_+ + \sigma_- Z_+(t,s)[\xi(s)\sigma_+]\right\}\\ + \gamma\alpha_1^2\sigma_- Z_+(t,s)\left[\xi(s)\mathrm{e}^{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1 B_1(s)-\varepsilon_3 B_3(s))}\sigma_-\right]\\ + \gamma\overline{\alpha}_1^2 Z_-(t,s)\left[\mathrm{e}^{-2\mathrm{i}(\nu-\omega)s+\mathrm{i}(\varepsilon_1 B_1(s)-\varepsilon_3 B_3(s))}\xi(s)\sigma_+\right]\sigma_+\,.\end{aligned}$$

Let us observe that the processes $Z_\pm(t,s)$ are $\mathcal{G}_s$-independent, by construction of the model that we are analysing. Furthermore, $\xi(s)$ is $\mathcal{G}_s$-measurable.

Then, by the previous identity and the properties of conditional expectations, we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\check{\mathcal{R}}_1(t)\circ\Xi(t,s)\circ\check{\mathcal{R}}_1(s)[\xi(s)]\right]=\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\check{\mathcal{R}}_1(t)\circ\Xi(t,s)\circ\check{\mathcal{R}}_1(s)[\xi(s)]\big|\mathcal{G}_s\right]\right]\\
&=\gamma|\alpha_1|^2\left\{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_-(t,s)\big|\mathcal{G}_s\right][\sigma_-\xi(s)]\sigma_+\right]+\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_+(t,s)\big|\mathcal{G}_s\right][\xi(s)\sigma_+]\right]\right\}\\
&\quad+\gamma\alpha_1^2\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_+(t,s)\big|\mathcal{G}_s\right]\left[\mathrm{e}^{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\sigma_-\xi(s)\right]\right]\\
&\quad+\gamma\overline{\alpha}_1^2\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_-(t,s)\big|\mathcal{G}_s\right]\left[\mathrm{e}^{-2\mathrm{i}(\nu-\omega)s+\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\sigma_+\right]\right]\sigma_+\\
&=\gamma|\alpha_1|^2\left\{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_-(t,s)\right][\sigma_-\xi(s)]\sigma_+\right]+\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_+(t,s)\right][\xi(s)\sigma_+]\right]\right\}\\
&\quad+\gamma\alpha_1^2\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_+(t,s)\right]\left[\mathrm{e}^{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\sigma_-\xi(s)\right]\right]\\
&\quad+\gamma\overline{\alpha}_1^2\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[Z_-(t,s)\right]\left[\mathrm{e}^{-2\mathrm{i}(\nu-\omega)s+\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\sigma_+\right]\right]\sigma_+\\
&=\gamma|\alpha_1|^2\left\{\mathbb{E}_{\mathbb{Q}}\left[M_-(t-s)[\sigma_-\xi(s)]\sigma_+\right]+\sigma_-\mathbb{E}_{\mathbb{Q}}\left[M_+(t-s)[\xi(s)\sigma_+]\right]\right\}\\
&\quad+\gamma\alpha_1^2\sigma_-\mathbb{E}_{\mathbb{Q}}\left[M_+(t-s)\left[\mathrm{e}^{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\sigma_-\xi(s)\right]\right]\\
&\quad+\gamma\overline{\alpha}_1^2\mathbb{E}_{\mathbb{Q}}\left[M_-(t-s)\left[\mathrm{e}^{-2\mathrm{i}(\nu-\omega)s+\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\sigma_+\right]\right]\sigma_+\\
&=\gamma|\alpha_1|^2\left\{M_-(t-s)[\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\xi(s)\right]]\sigma_++\sigma_-M_+(t-s)[\mathbb{E}_{\mathbb{Q}}\left[\xi(s)\right]\sigma_+]\right\}\\
&\quad+\gamma\alpha_1^2\sigma_-M_+(t-s)\left[\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\right]\right]\\
&\quad+\gamma\overline{\alpha}_1^2M_-(t-s)\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\mathrm{i}(\nu-\omega)s+\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\right]\sigma_+\right]\sigma_+\\
&=\gamma|\alpha_1|^2\left\{M_-(t-s)\left[\sigma_-\mathrm{e}^{\check{\mathcal{L}}s}[\eta_0]\right]\sigma_++\sigma_-M_+(t-s)\left[\mathrm{e}^{\check{\mathcal{L}}s}[\eta_0]\sigma_+\right]\right\}\\
&\quad+\gamma\alpha_1^2\sigma_-M_+(t-s)\left[\sigma_-\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\right]\right]\\
&\qquad+\gamma\overline{\alpha}_1^2M_-(t-s)\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\mathrm{i}(\nu-\omega)s+\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))}\xi(s)\right]\sigma_+\right]\sigma_+\,.
\end{aligned}
$$

To conclude we have to study the mean of the process

$$\exp\left\{\pm\left[2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))\right]\right\}\xi(s)\,.$$

The reasoning is the same as before and we obtain

$$
\begin{aligned}
\theta(s):=\mathbb{E}_{\mathbb{Q}}&\left[\exp\left\{2\mathrm{i}(\nu-\omega)s-\mathrm{i}(\varepsilon_1B_1(s)-\varepsilon_3B_3(s))\right\}\xi(s)\right]\\
&=\mathrm{e}^{\left\{2\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{2}-\frac{\varepsilon_3^2}{2}\right\}s}\mathrm{e}^{\check{\mathcal{S}}s}[\eta_0]
\end{aligned}
\tag{7.71}
$$

where we have defined the map $\check{\mathcal{S}}$ as

$$\check{\mathcal{S}}[\tau]:=\check{\mathcal{L}}[\tau]+\mathrm{i}\frac{\varepsilon_3^2}{4}[\sigma_z,\tau]\,,\quad\forall\tau\in M_n(\mathbb{C})\,,$$

So, we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\check{\mathcal{R}}_1(t)\circ\Xi(t,s)\circ\check{\mathcal{R}}_1(s)[\xi(s)]\right]\\
&\qquad=\gamma|\alpha_1|^2\sigma_-M_+(t-s)\left[\mathrm{e}^{\check{\mathcal{L}}s}[\eta_0]\sigma_+\right]\\
&\qquad\quad+\gamma\alpha_1^2\sigma_-M_+(t-s)\left[\sigma_-\mathrm{e}^{\check{\mathcal{S}}s}[\eta_0]\right]+\mathrm{h.c.}\,,
\end{aligned}
$$

where we write h.c. instead of the hermitian conjugate.

With a totally symmetric reasoning we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\check{\mathcal{R}}_1(s)\circ\Xi(s,t)\circ\check{\mathcal{R}}_1(t)[\xi(t)]\right] \\ = \gamma|\alpha_1|^2\sigma_- M_+(t-s)\left[\mathrm{e}^{\check{\mathcal{L}}t}[\eta_0]\sigma_+\right] \\ + \gamma\alpha_1^2\sigma_- M_+(t-s)\left[\sigma_-\mathrm{e}^{\check{\mathcal{S}}t}[\eta_0]\right] + \text{h.c.}\,.\end{aligned}$$

In conclusion we have obtained the explicit expression of the second moments of the output: it is enough to take the trace in the previous formulas and to insert them in Eq. (6.34), recalling Eq. (7.64), that is

$$\begin{aligned}\mathbb{E}^T_{\eta_0}[\dot{W}_1(t)\dot{W}_1(s)] = \delta(t-s) + 1_{(0,+\infty)}(t-s)\left\{g(t,s)+h(t,s)\right\} \\ + 1_{(0,+\infty)}(s-t)\left\{g(s,t)+h(s,t)\right\}\,,\end{aligned} \tag{7.72}$$

where we set

$$g(t,s) := 2\gamma|\alpha_1|^2\,\mathrm{Re}\,\mathrm{Tr}\left\{\sigma_- M_+(t-s)\left[\mathrm{e}^{\check{\mathcal{L}}s}[\eta_0]\sigma_+\right]\right\}\,; \tag{7.73a}$$

$$h(t,s) := 2\gamma\,\mathrm{Re}\,\mathrm{Tr}\left\{\alpha_1^2\sigma_- M_+(t-s)\left[\sigma_-\mathrm{e}^{\check{\mathcal{S}}s}[\eta_0]\right]\right\}\,. \tag{7.73b}$$

7.5.3 The Bloch representation of the means

We are interested in Eq. (7.72) for big times: to carry out this analysis we need to show that the semigroups appearing in this formula do not explode when $t\to\infty$. A useful way to study the long time behavior of Eq. (7.72) is to represent each one of the semigroups involved in the Bloch form.

First of all, we shall study the dynamic of the semigroup generated by $\check{\mathcal{L}}$: we mentioned that this is a Liouville operator and then the generated semigroup is a quantum dynamical semigroup, because $\check{\mathcal{L}}$ can be written in the Lindblad form, see [11]. For this reason we shall prove that there exists an equilibrium state $\eta^{\,\mathrm{eq}}\in S(\mathcal{H})$ for $\mathrm{e}^{\check{\mathcal{L}}t}$:

$$\check{\mathcal{L}}[\eta^{\,\mathrm{eq}}] = 0$$

and, $\forall\eta_0\in S(\mathcal{H})$,

$$\lim_{t\to\infty}\mathrm{e}^{\check{\mathcal{L}}t}[\eta_0] = \eta^{\,\mathrm{eq}}\,. \tag{7.74}$$

Then, we shall go to the analysis of the semigroups generated by $\check{\mathcal{S}}$ and $\check{\mathcal{K}}$, which are not quantum dynamical semigroups because their generators can not be written in the Lindblad form. Indeed, it is easy to see that they preserve the trace, but not the positivity. In this case we are not interested in the existence of an equilibrium of the generated semigroups because it is no a state of the quantum system and, so, we can not use the equilibrium as initial condition. By the way we shall prove that, for long times, the means μ_+ and θ exponentially vanish: if we look at Eq. (7.58) and (7.71), it is clear that to show this claim it is enough to prove that the generated semigroups does not explode for long times.

The mean ζ

Let us consider now Eq. (7.52) when the initial condition belongs to $S(\mathcal{H})$. The Bloch representation of this equation, when the initial condition is a statistical operator is

$$\zeta(t) = \frac{1}{2}\left(\mathbb{1} + \vec{p}(t)\cdot\vec{\sigma}\right) . \tag{7.75}$$

To determine the coefficient vector $\vec{p}(t)$, we recall that

$$p_i(t) = \mathrm{Tr}\{\sigma_i \zeta(t)\} .$$

Furthermore we have

$$\dot{\zeta}(t) = \check{\mathcal{L}}[\zeta(t)] = \frac{1}{2}\left(\check{\mathcal{L}}[\mathbb{1}] + \vec{p}(t)\cdot\check{\mathcal{L}}[\vec{\sigma}]\right) ,$$

where

$$\check{\mathcal{L}}[\vec{\sigma}] = \begin{pmatrix} \check{\mathcal{L}}[\sigma_1] \\ \check{\mathcal{L}}[\sigma_3] \\ \check{\mathcal{L}}[\sigma_3] \end{pmatrix} .$$

In conclusion we have that the Bloch representation of Eq. (7.52) is

$$\dot{p}_i(t) = \frac{1}{2}\,\mathrm{Tr}\left\{\sigma_i\left(\check{\mathcal{L}}[\mathbb{1}] + \vec{p}(t)\cdot\check{\mathcal{L}}[\vec{\sigma}]\right)\right\} , \quad i = 1,2,3 . \tag{7.76}$$

To give the explicit form of the previous system of ODEs we have to study the action of the generator $\check{\mathcal{L}}$ on the Pauli basis $(\mathbb{1}\,, \sigma_1\,, \sigma_2\,, \sigma_3)$, that is

$$\check{\mathcal{L}}[\mathbb{1}] = -\gamma\sigma_3 ,$$

$$\check{\mathcal{L}}[\sigma_1] = \Delta\omega\sigma_2 - \left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]\sigma_1 ,$$

$$\check{\mathcal{L}}[\sigma_2] = -\Delta\omega\sigma_1 - \left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]\sigma_2 + \Omega\sigma_3 ,$$

$$\check{\mathcal{L}}[\sigma_3] = -\gamma\left[2\bar{n} + 1\right]\sigma_3 - \Omega\sigma_2 .$$

By using

$$\mathrm{Tr}\{\sigma_i\sigma_j\} = 2\delta_{ij} , \quad i,j = 1,2,3 , \tag{7.77}$$

we get the following linear system of ODEs representing Eq. (7.51):

$$\begin{cases} \dot{p}_1(t) = -\left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]p_1(t) - \Delta\omega p_2(t) , \\ \dot{p}_2(t) = \Delta\omega p_1(t) - \left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]p_2(t) - \Omega p_3(t) , \\ \dot{p}_3(t) = \Omega p_2(t) - \left[2\gamma\bar{n} + \gamma\right]p_3(t) - \gamma , \end{cases} \tag{7.78}$$

that is

$$\dot{\vec{p}}(t) = -C\vec{p}(t) - \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.79}$$

where we set

$$C := \begin{pmatrix} \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} & \Delta\omega & 0 \\ -\Delta\omega & \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} & \Omega \\ 0 & -\Omega & 2\gamma\bar{n} + \gamma \end{pmatrix} . \tag{7.80}$$

The solution of the system (7.78) is

$$\vec{p}(t) = \mathrm{e}^{-Ct}\vec{p}(0) - \gamma \int_0^t \mathrm{e}^{-C(t-u)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{d}u \,. \tag{7.81}$$

To find the equilibrium solution of the autonomous and non-homogeneous system of ODEs (7.78) we proceed in the usual way by setting to zero the left side member. To simplify the calculation we introduce the following quantities

$$x := \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} \,, \tag{7.82}$$

$$y := \Delta\omega \,, \quad z := 2\gamma\bar{n} + \gamma \,. \tag{7.83}$$

Thanks to Assumption 7.1, we have

$$x > 0 \,, \quad z > 0 \,.$$

The system matrix C becomes

$$C = \begin{pmatrix} x & y & 0 \\ -y & x & \Omega \\ 0 & -\Omega & z \end{pmatrix}$$

and its determinant is

$$\det(C) = z(x^2 + y^2) + \Omega^2 x > 0 \,,$$

which is positive by Assumption 7.1.

The equilibrium solution satisfies the following system of equations

$$\begin{cases} x p_1^{\mathrm{eq}} + y p_2^{\mathrm{eq}} = 0 \\ y p_1^{\mathrm{eq}} - x p_2^{\mathrm{eq}} - \Omega p_3^{\mathrm{eq}} = 0 \\ \Omega p_2^{\mathrm{eq}} - z p_3^{\mathrm{eq}} = \gamma \,. \end{cases} \tag{7.84}$$

The previous system gives the necessary equilibrium conditions: it has a unique solution because of the invertibility of the matrix C and, so, if the semigroup generated by $\check{\mathcal{L}}$ has an equilibrium, it is unique. The solution of (7.84) is

$$\vec{p}^{\,\mathrm{eq}} = -\frac{\gamma}{C}\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \text{or} \quad \begin{aligned} p_1^{\mathrm{eq}} &= -\gamma\Omega\frac{y}{\det(C)} \\ p_2^{\mathrm{eq}} &= \gamma\Omega\frac{x}{\det(C)} \\ p_3^{\mathrm{eq}} &= -\gamma\frac{[x^2+y^2]}{\det(C)}\,. \end{aligned} \tag{7.85}$$

By direct computation we have $\|p^{\mathrm{eq}}\| < 1$ and then the uniquely determined matrix η^{eq}, with the Bloch representation

$$\eta^{\mathrm{eq}} = \frac{1}{2}(\mathbb{1} + \vec{p}^{\,\mathrm{eq}}\cdot\vec{\sigma})\,. \tag{7.86}$$

is a statistical operator but *it is not a pure state.*

The unique solution of the system (7.84), given in (7.85), is the candidate equilibrium. To show that this point is the equilibrium of $\mathrm{e}^{\check{\mathcal{L}}t}$ we have to prove that, for every initial state, η^{eq} is reached.

First of all, it is useful to observe that

$$C + C^T := \begin{pmatrix} \frac{\varepsilon_3^2}{4} + 4\gamma k_d + 2\gamma\bar{n} + \gamma & 0 & 0 \\ 0 & \frac{\varepsilon_3^2}{4} + 4\gamma k_d + 2\gamma\bar{n}\gamma & 0 \\ 0 & 0 & 2(2\gamma\bar{n}+\gamma) \end{pmatrix}.$$

Then, we set

$$B := \gamma\,(2\bar{n}+1)\,b\mathbb{1} - C\,, \quad b := \min\left\{1, \frac{1}{2} + \frac{2k_d + \varepsilon_3^2/2}{2\bar{n}+1}\right\} \geq \frac{1}{2}\,.$$

We prove now the following statement

Proposition 7.1. e^{Bt} *is a semigroup of contractions.*

Proof. We want to prove that

$$\left\|\mathrm{e}^{Bt}\vec{x}\right\| \leq \|\vec{x}\|\,, \quad \forall\vec{x}\in\mathbb{R}^3\,.$$

Then we prove that the euclidean norm of $\mathrm{e}^{Bt}\vec{x}$ is time decreasing.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathrm{e}^{Bt}\vec{x}\right\| = \langle \mathrm{e}^{Bt}\vec{x}|(B+B^T)\mathrm{e}^{Bt}\vec{x}\rangle\,.$$

But it is easy to see that $(B+B^T)$ is a diagonal matrix whose diagonal elements are non-positive, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathrm{e}^{Bt}\vec{x}\right\| \leq 0\,, \quad \forall\vec{x}\in\mathbb{R}^3\,.$$

$\square$

Now we have to show that the state which has the Bloch representation (7.86) is the equilibrium for Eq. (7.52). The following proposition holds

Proposition 7.2. *Under Assumptin 7.1, $\vec{p}^{\,\mathrm{eq}}$ is an equilibrium for the system of ODEs* (7.78) *or, equivalently, $\eta^{\,\mathrm{eq}}$ is an equilibrium for Eq.* (7.52)*, this is*

$$\lim_{t\to\infty}\zeta(t)=\lim_{t\to\infty}\mathrm{e}^{\check{\mathcal{L}}t}[\eta_0]=\eta^{\,\mathrm{eq}}\,. \tag{7.87}$$

Furthermore the convergence to the equilibrium is exponential.

Proof. First of all, from Eq. (7.81) and thanks to Assumption 7.1, we have that the solution of the system (7.79) can be written as

$$\vec{p}(t)=\mathrm{e}^{-Ct}\vec{p}(0)-\gamma\frac{1-\mathrm{e}^{-Ct}}{C}\begin{pmatrix}0\\0\\1\end{pmatrix}\,,$$

or equivalently, taking into account (7.85),

$$\vec{p}(t)=\mathrm{e}^{-Ct}\left(\vec{p}(0)-\vec{p}^{\,\mathrm{eq}}\right)+\vec{p}^{\,\mathrm{eq}}\,. \tag{7.88}$$

In the next passage we shall use the notion of nor one of an operator, stated in Chapter 1 and that $b\geq\frac{1}{2}$. By using the Bloch representation of $\zeta(t)$, the definition of B and the properties of its generated semigroup, given in Proposition 7.1, we have

$$\begin{aligned}\left\|\zeta(t)-\eta^{\,\mathrm{eq}}\right\|_1&=\left\|\vec{p}(t)-\vec{p}^{\,\mathrm{eq}}\right\|=\left\|\mathrm{e}^{-Ct}\left(\vec{p}(0)-\vec{p}^{\,\mathrm{eq}}\right)\right\|\\&=\left\|\mathrm{e}^{Bt-\gamma(2\bar{n}+1)bt\mathbb{1}}\left(\vec{p}(0)-\vec{p}^{\,\mathrm{eq}}\right)\right\|=\mathrm{e}^{-\gamma(2\bar{n}+1)bt}\left\|\mathrm{e}^{Bt}\left(\vec{p}(0)-\vec{p}^{\,\mathrm{eq}}\right)\right\|\\&\leq\mathrm{e}^{-\frac{1}{2}\gamma(2\bar{n}+1)t}\left\|\vec{p}(0)-\vec{p}^{\,\mathrm{eq}}\right\|\,,\end{aligned}$$

and, so,

$$\lim_{t\to\infty}\left\|\zeta(t)-\eta^{\,\mathrm{eq}}\right\|_1=0\,. \tag{7.89}$$

By the previous inequalities the convergence velocity is exponential. □

In conclusion, we showed that the semigroup generated by $\check{\mathcal{L}}$ has an equilibrium state in the sense of (7.74).

The mean θ

Let us recall that

$$\theta(t)=\mathrm{e}^{\left\{2\mathrm{i}(\nu-\omega)-\frac{\varepsilon_1^2}{2}-\frac{\varepsilon_1^3}{2}\right\}t}\,\mathrm{e}^{\check{\mathcal{S}}t}[\eta_0]\,.$$

The first term of this expression exponentially vanishes: then, to prove that θ for long time vanishes too it is enough to prove that the semigroup generated by $\check{\mathcal{S}}$ does not explode.

We define

$$\theta_0(t):=\mathrm{e}^{\check{\mathcal{S}}t}[\eta_0]. \tag{7.90}$$

As in the previous case, we are interested in the action of $\mathrm{e}^{\check{\mathcal{S}}t}$ when the initial condition is a statistical operator. We already said that this semigroup is not a quantum dynamical semigroup because it does not preserve the positivity, but it preserves the trace. In conclusion the Bloch representation of θ_0 is

$$\theta_0(t) = \frac{1}{2}\left(\mathbb{1} + \vec{d}(t)\cdot\vec{\sigma}\right). \tag{7.91}$$

The determination of the system of ODEs satisfied by $\vec{d}(t)$ goes through the same arguments that we saw in the previous paragraph. Indeed, we have

$$d_i(t) = \mathrm{Tr}\left\{\sigma_i\theta_0(t)\right\}, \quad i = 1,2,3,$$

so

$$\dot{d}_i(t) = \frac{1}{2}\,\mathrm{Tr}\left\{\sigma_i\left(\check{\mathcal{S}}[\mathbb{1}] + \vec{d}(t)\cdot\check{\mathcal{S}}[\vec{\sigma}]\right)\right\}.$$

The action of $\check{\mathcal{S}}$ on the Pauli basis is

$$\check{\mathcal{S}}[\mathbb{1}] = -\gamma\sigma_3,$$

$$\check{\mathcal{S}}[\sigma_1] = -\left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]\sigma_1 + \left[\Delta\omega - \frac{\varepsilon_3^2}{2}\right]\sigma_2,$$

$$\check{\mathcal{S}}[\sigma_2] = -\left[\Delta\omega - \frac{\varepsilon_3^2}{2}\right]\sigma_1 - \left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]\sigma_2 + \Omega\sigma_3;$$

$$\check{\mathcal{S}}[\sigma_3] = -\left[2\gamma\bar{n} + \gamma\right]\sigma_3 - \Omega\sigma_2.$$

Thanks to Eqs. (7.77), we end up with the following system

$$\begin{cases} \dot{d}_1(t) = -\left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]d_1(t) - \left[\Delta\omega - \frac{\varepsilon_3^2}{2}\right]d_2(t) \\ \dot{d}_2(t) = \left[\Delta\omega - \frac{\varepsilon_3^2}{2}\right]d_1(t) - \left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]d_2(t) - \Omega d_3(t) \\ \dot{d}_3(t) = \Omega d_2(t) - 2\gamma(\bar{n}+1)d_3(t) - \gamma. \end{cases} \tag{7.92}$$

If we introduce the matrix A as

$$A := \begin{pmatrix} \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} & \left[\Delta\omega - \frac{\varepsilon_3^2}{2}\right] & 0 \\ -\left[\Delta\omega - \frac{\varepsilon_3^2}{2}\right] & \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} & \Omega \\ 0 & -\Omega & 2\gamma(\bar{n}+1) \end{pmatrix} \tag{7.93}$$

the previous system becomes

$$\dot{\vec{d}}(t) = -A\vec{d}(t) - \gamma\begin{pmatrix}0\\0\\1\end{pmatrix}$$

whose solution is

$$\vec{d}(t) = \mathrm{e}^{-At}\vec{d}(0) - \gamma \int_0^t \mathrm{e}^{-A(t-s)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{d}s \,.$$

The invertibility of the matrix A can be obtained in the same way that we used in the previous paragraph to obtain the invertibility of the matrix C (it is enough to redefine y in (7.82)). Then, the solution of (7.92) can be written as

$$\vec{d}(t) = \mathrm{e}^{-At} \left(\vec{d}(0) + \frac{\gamma}{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) - \frac{\gamma}{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,.$$

The first addend of the latter expression does not explode because $-A$ is the infinitesimal generator of a contractive semigroup. Indeed, we have that the matrix $A + A^T$ is a diagonal matrix whose eigenvalues are strictly positive and, so,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathrm{e}^{-At}\vec{x}\|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \mathrm{e}^{-At}\vec{x} \middle| \mathrm{e}^{-At}\vec{x} \right\rangle = - \left\langle \mathrm{e}^{-At}\vec{x} \middle| (A + A^T)\mathrm{e}^{-At}\vec{x} \right\rangle < 0 \,, \quad \forall \vec{x} \in \mathbb{R}^3 \,.$$

In conclusion we have that *exponentially*

$$\theta(t) = \mathrm{e}^{\left\{ 2\mathrm{i}(\nu - \omega) - \frac{\varepsilon_1^2}{2} - \frac{\varepsilon_1^3}{2} \right\} t} \theta_0(t) \xrightarrow[t \to \infty]{} 0 \,. \tag{7.94}$$

The mean μ_+

The situation is completely similar to the previous one but now we have to study the Bloch representation starting from a general matrix $\tau \in M_2(\mathbb{C})$ and not necessarily from a statistical operator. Indeed, if we look at Eqs. (7.72) and (7.73), we note that the argument of the semigroup generated by $\check{\mathcal{K}}$ is not a statistical operator because of the presence of the multiplying factors $\sigma_\pm$.

We start from the relation

$$\mu_+(t) = \mathrm{e}^{\left\{ \mathrm{i}(\nu - \omega) - \frac{\varepsilon_1^2}{8} - \frac{\varepsilon_3^2}{8} \right\} t} \mathrm{e}^{\check{\mathcal{K}}t}[\tau] \,, \quad \forall \tau \in M_2(\mathbb{C}) \,,$$

and we introduce μ_0 as

$$\mu_0(t) := \mathrm{e}^{\check{\mathcal{K}}t}[\tau] \,.$$

The Bloch representation of μ_0 is

$$\mu_0 = \frac{1}{2} \left(c_0 \mathbb{1} + \vec{q}(t) \cdot \vec{\sigma} \right) \,, \quad c_0 = \mathrm{Tr}\{\tau\} \,.$$

Let us stress that c_0 does not depend on time because, as we already said, $\mathrm{e}^{\check{\mathcal{K}}t}$ is trace preserving. The action of $\check{\mathcal{K}}$ on the Pauli basis is

$$\check{\mathcal{K}}[\mathbb{1}] = -\gamma\sigma_3\,;$$

$$\check{\mathcal{K}}[\sigma_1] = -\left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]\sigma_1 + \left[\Delta\omega - \frac{\varepsilon_3^4}{2}\right]\sigma_2\,,$$

$$\check{\mathcal{K}}[\sigma_2] = -\left[\Delta\omega - \frac{\varepsilon_3^2}{4}\right]\sigma_1 - \left[\frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2}\right]\sigma_2 + \Omega\sigma_3\,,$$

$$\check{\mathcal{K}}[\sigma_3] = -\left[2\gamma\bar{n} + \gamma\right]\sigma_3 - \Omega\sigma_2\,.$$

and, so, we have

$$\dot{\vec{q}}(t) = -D\vec{q}(t) - c_0\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}\,, \tag{7.95}$$

where we have defined

$$D := \begin{pmatrix} \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} & \left[\Delta\omega - \frac{\varepsilon_3^2}{4}\right] & 0 \\ -\left[\Delta\omega - \frac{\varepsilon_3^2}{4}\right] & \frac{\varepsilon_3^2}{8} + 2\gamma k_d + \gamma\bar{n} + \frac{\gamma}{2} & \Omega \\ 0 & -\Omega & 2\gamma(\bar{n}+1) \end{pmatrix}\,, \tag{7.96}$$

which is an invertible matrix because its determinant is strictly positive. The solution of the previous system is

$$\vec{q}(t) = \mathrm{e}^{-Dt}\left(\vec{q}(0) + \frac{c_0\gamma}{D}\begin{pmatrix}0\\0\\1\end{pmatrix}\right) - \frac{c_0\gamma}{D}\begin{pmatrix}0\\0\\1\end{pmatrix}\,. \tag{7.97}$$

As in the case for θ it is possible to prove that $-D$ is the infinitesimal generator of a contractive semigroup and then we can conclude that, *exponentially*,

$$\mu_+(t) = \mathrm{e}^{\left\{\mathrm{i}(\nu-\omega) - \frac{\varepsilon_1^2}{8} - \frac{\varepsilon_3^2}{8}\right\}t}\mu_0(t) \xrightarrow[t\,\to\,\infty]{} 0\,. \tag{7.98}$$

Remark 7.3 (The physical mean of the output). We note that the limit (7.98) shows that the mean of the output of the measurement vanishes for long times. Indeed, if we take the limit in Eq. (7.60), by using (7.98), we obtain

$$\mathbb{E}^T_{\eta_0}[\dot{W}_1(t)] \xrightarrow[t\,\to\,\infty]{} 0\,. \tag{7.99}$$

7.6 The output current and the electrical power

We introduced the electrical current I_1 and its electrical power P_1 in Eqs. (7.14) and (7.16) respectively. In this section we want to study the long time behaviour of the mean of the electrical power as a function of the local oscillator frequency ν.

If we look at Eqs. (7.14) and (7.16), we realise that they depend on the detector response function F and on the output signal $\mathrm{d}W_1$ of the channel 1, but not on ν. Nevertheless, we are interested in the mean under the physical probabilities and, as we can see from Eqs. (7.29), (7.35) and (7.36), they do depend on ν. To explicitly see this dependence, we shall write $\mathbb{E}_{\eta_0}^{t,\nu}$ for the quantum mean. Furthermore, thanks to the consistence property, we have

$$\mathbb{E}_{\eta_0}^{T,\nu}[P_1(t)] = \mathbb{E}_{\eta_0}^{t,\nu}[P_1(t)]\,, \quad \forall T \geq t \geq 0\,.$$

In the heterodyne detection scheme the local oscillator and the stimulating light come out from two different lasers and it is impossible to maintain a stable relative phase between the two. In our model both the stimulating laser of frequency ω and the local oscillator of frequency ν are colored lasers and, so, smoothing effects are present. Then, the mean power at large times for the heterodyne detection is

$$P_{\text{het}}(\nu) = \lim_{t\to\infty} \mathbb{E}_{\eta_0}^{t,\nu}[P_1(t)] = k_2 \lim_{t\to\infty} \mathbb{E}_{\eta_0}^{t,\nu}[I_1(t)^2]. \tag{7.100}$$

As a function of ν, $P_{\text{het}}(\nu)$ is the mean observed *power spectrum.* In our model it is possible to state an analytic expression of $P_{\text{het}}(\nu)$, but not a suitable decomposition in elastic and inelastic part, as in [3].

Another difference with respect to [3] is that the limit in (7.100) is an ordinary limit and not a limit in the sense of distributions. This is due to the terms coming out from Assumption 7.2.

7.6.1 The power spectrum

Proposition 7.3. *The mean observed power spectrum* (7.100) *can be written as*

$$P_{\text{het}}(\nu) = \frac{k_1^2 k_2}{4\pi} + k_1^2 k_2 |\alpha_1|^2\, \Sigma(\nu-\omega)\,, \tag{7.101}$$

where

$$\begin{aligned}\Sigma(\nu-\omega) = \frac{\gamma}{\pi}\, &\frac{\frac{\tilde{\varkappa}}{2}(p_1^{\text{eq}} w_1 + p_2^{\text{eq}} w_2) + (p_1^{\text{eq}} w_2 - p_2^{\text{eq}} w_1)(\nu-\omega)}{\frac{\tilde{\varkappa}^2}{4} + (\nu-\omega)^2} \\ &+ \frac{\gamma}{2\pi}\, \mathrm{Re}\left(\begin{pmatrix} 1, & -\mathrm{i}, & 0 \end{pmatrix} \cdot \frac{1}{D + \frac{\tilde{\varkappa}}{2} - \mathrm{i}(\nu-\omega)}\, \vec{v} \right) .\end{aligned} \tag{7.102}$$

The vector $\vec{p}^{\,\text{eq}}$ and the matrix D have been given in Eqs. (7.85) *and* (7.96) *respectively. Recalling Eq.* (7.97)*, we have defined*

$$\vec{v} := \vec{q}(0) + \frac{c_0\gamma}{D} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{w} := -\frac{\gamma}{D} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.103}$$

Moreover we have

$$\tilde{\varkappa} := \varkappa + \frac{\varepsilon_1^2}{2} + \frac{\varepsilon_3^2}{2}\,. \tag{7.104}$$

Proof. By the definition of the the electrical power, given in Eq. (7.16), and by Eqs. (6.34) and (6.35) we have

$$\mathbb{E}^{t,\nu}_{\eta_0}[P_1(t)] = k_2\mathbb{E}^{t,\nu}_{\eta_0}[I_1(t)] = \frac{k_1^2 k_2 \varkappa}{4\pi}\,\mathbb{E}^{t,\nu}_{\eta_0}\left[\int_0^t \mathrm{d}s \int_0^t \mathrm{d}r\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\dot{W}_1(s)\dot{W}_1(r)\right]$$

$$= \frac{k_1^2 k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right) + \frac{k_1^2 k_2 \varkappa}{4\pi}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\Big\{\mathcal{R}_1(s)\circ\Lambda(s,r)\circ\mathcal{R}_1(r)[\varrho(r)]\Big\}\Big]$$

$$+ \frac{k_1^2 k_2 \varkappa}{4\pi}\int_0^t \mathrm{d}s \int_s^t \mathrm{d}r\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\Big\{\mathcal{R}_1(r)\circ\Lambda(r,s)\circ\mathcal{R}_1(s)[\varrho(s)]\Big\}\Big]$$

$$= \frac{k_1^2 k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right) + \frac{k_1^2 k_2 \varkappa}{4\pi}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\Big\{\mathcal{R}_1(s)\circ\Lambda(s,r)\circ\mathcal{R}_1(r)[\varrho(r)]\Big\}\Big]$$

$$+ \frac{k_1^2 k_2 \varkappa}{4\pi}\int_0^t \mathrm{d}r \int_0^r \mathrm{d}s\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\Big\{\mathcal{R}_1(r)\circ\Lambda(r,s)\circ\mathcal{R}_1(s)[\varrho(s)]\Big\}\Big]$$

$$= \frac{k_1^2 k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right) + \frac{k_1^2 k_2 \varkappa}{2\pi}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\Big\{\mathcal{R}_1(s)\circ\Lambda(s,r)\circ\mathcal{R}_1(r)[\varrho(r)]\Big\}\Big]$$

$$= \frac{k_1^2 k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right) + \frac{k_1^2 k_2 \varkappa}{2\pi}\int_0^t \mathrm{d}r \int_r^t \mathrm{d}r\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\Big\{\mathcal{R}_1(s)\circ\Lambda(s,r)\circ\mathcal{R}_1(r)[\varrho(r)]\Big\}\Big].$$

Using the definition of mean power spectrum stated in Eq. (7.100) we have

$$\frac{4\pi}{k_1^2 k_2}P_{\text{het}}(\nu) - 1 = 2\varkappa \lim_{t\leftarrow\infty}\int_0^t \mathrm{d}r \int_r^t \mathrm{d}s\; \mathrm{e}^{-\varkappa\left[t-\frac{(s+r)}{2}\right]}\mathbb{E}\left[\operatorname{Tr}\left\{\mathcal{R}_1(s)\circ\Lambda(s,r)\circ\mathcal{R}_1(r)[\varrho(r)]\right\}\right].$$

Now we use the transformations $r_1 = s - r$, $r_2 = r$ and $s = r_1$, $r = t - r_2$ in the previous integral and then, by expressions (7.72) and (7.73), we obtain

$$\frac{4\pi}{k_1^2 k_2}P_{\text{het}}(\nu) - 1 = 2\varkappa \lim_{t\to\infty}\int_0^t \mathrm{d}r_2 \int_0^{t-r_2} \mathrm{d}r_1\; \mathrm{e}^{-\varkappa\left[t-\frac{2r_2+r_1}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\left\{\mathcal{R}_1(r_1+r_2)\circ\Lambda(r_1+r_2,r_2)\circ\mathcal{R}_1(r_2)[\varrho(r_2)]\right\}\Big]$$

$$= 2\varkappa \lim_{t\to\infty}\int_0^t \mathrm{d}r \int_0^r \mathrm{d}s\; \mathrm{e}^{-\varkappa\left[r-\frac{s}{2}\right]}\mathbb{E}\Big[\operatorname{Tr}\left\{\mathcal{R}_1(s+t-r)\circ\Lambda(s+t-r,t-r)\circ\mathcal{R}_1(t-r)[\varrho(t-r)]\right\}\Big]$$

$$= 4\varkappa \lim_{t\to\infty}\int_0^t \mathrm{d}r \int_0^r \mathrm{d}s\; \mathrm{e}^{-\varkappa\left[r-\frac{s}{2}\right]}\operatorname{Re}\operatorname{Tr}\left\{\gamma|\alpha_1|^2\sigma_- M_+(s)\left[\mathrm{e}^{\check{\mathcal{L}}(t-r)}[\eta_0]\sigma_+\right] + \gamma\alpha_1^2 M_+(s)[\sigma_-\mathrm{e}^{\check{\mathcal{S}}(t-r)}[\eta_0]]\right\}.$$

In the last formula, we see that M_+ does not depend on t. Moreover, in the previous section, we studied the properties of the semigroups generated by $\check{\mathcal{L}}$ and $\check{\mathcal{S}}$ for big times: they converge exponentially to the equilibrium and to zero respectively. Therefore, we get

$$\begin{aligned}
\frac{4\pi}{k_1^2 k_2} P_{\text{het}}(\nu) - 1 &= 4\varkappa\gamma|\alpha_1|^2 \operatorname{Re} \int_0^\infty \mathrm{d}r \int_0^r \mathrm{d}s \, \mathrm{e}^{-\varkappa[r-\frac{s}{2}]} \operatorname{Tr}\{\sigma_- M_+(s)[\eta^{\,\mathrm{eq}}\sigma_+]\} \\
&= 4\gamma|\alpha_1|^2 \operatorname{Re} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-\varkappa\frac{s}{2}} \operatorname{Tr}\{\sigma_- M_+(s)[\eta^{\,\mathrm{eq}}\sigma_+]\} \\
&= 4\gamma|\alpha_1|^2 \operatorname{Re} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{\mathrm{i}(\nu-\omega)t - \frac{1}{2}\left(\varkappa + \frac{\varepsilon_1^2+\varepsilon_3^2}{2}\right)t} \operatorname{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{K}}t}[\eta^{\,\mathrm{eq}}\sigma_+]\right\} \\
&\equiv 4\gamma|\alpha_1|^2 \operatorname{Re} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{\mathrm{i}(\nu-\omega)t - \frac{\tilde{\varkappa}}{2}t} \operatorname{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{K}}t}[\eta^{\,\mathrm{eq}}\sigma_+]\right\} . \qquad (7.105)
\end{aligned}$$

To conclude, we have to represent the integrand in the formula above, in the Bloch form. We recall that the Bloch representation of $\eta^{\,\mathrm{eq}}$ is

$$\eta^{\,\mathrm{eq}} = \frac{1}{2}\left(\mathbb{1} + p^{\,\mathrm{eq}} \cdot \vec{\sigma}\right) ,$$

and that $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$. Then, we have

$$\eta^{\mathrm{eq}}\sigma_+ = \frac{1}{2}\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + p_1^{\mathrm{eq}} P_- + \mathrm{i}p_2^{\mathrm{eq}} P_- + \frac{p_3^{\mathrm{eq}}}{2}(\sigma_x + \mathrm{i}\sigma_y)\right) .$$

Moreover, the Bloch representation of $\mathrm{e}^{\check{\mathcal{K}}t}[\tau]$, $\forall \tau \in M_2(\mathbb{C})$, is

$$\mathrm{e}^{\check{\mathcal{K}}t}[\tau] = \frac{1}{2}\left(c_0\mathbb{1} + \vec{q}(t)\cdot\vec{\sigma}\right) , \quad c_0 = \operatorname{Tr}\{\tau\} ,$$

and the vector $\vec{q}(t)$, using the notation introduced in the statement of this proposition, is

$$\vec{q}(t) = \mathrm{e}^{-Dt}\vec{v} + c_0\vec{w} . \qquad (7.106)$$

To completely determine the form of $\vec{q}(t)$ we have to calculate $\vec{v}$ and $\vec{w}$ but, if we look at Eq. (7.103), we see that it is enough to calculate $\vec{q}(0)$. By using that $q_i(0) = \operatorname{Tr}\{\sigma_i \eta^{\,\mathrm{eq}}\sigma_+\}$, we get

$$\vec{q}(0) = \begin{pmatrix} \frac{1+p_3^{\mathrm{eq}}}{2} \\ \mathrm{i}\frac{1+p_3^{\mathrm{eq}}}{2} \\ -\frac{p_1^{\mathrm{eq}} + \mathrm{i}p_2^{\mathrm{eq}}}{2} \end{pmatrix} , \qquad c_0 = \frac{p_1^{\mathrm{eq}} + \mathrm{i}p_2^{\mathrm{eq}}}{2} . \qquad (7.107)$$

In conclusion we have

$$\operatorname{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{K}}t}[\eta^{\,\mathrm{eq}}\sigma_+]\right\} = \frac{1}{2}(q_1(t) - \mathrm{i}q_2(t))$$

where $q_1(t)$ and $q_2(t)$ are completely determined by Eqs. (7.106) and (7.107). Inserting the latter formula in Eq. (7.105), we obtain

$$
\begin{aligned}
\frac{4\pi}{k_1^2 k_2} P_{\mathrm{het}}(\nu) - 1 &= 4\gamma|\alpha_1|^2 \,\mathrm{Re} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\nu-\omega)t - \frac{\tilde{\varkappa}}{2}t}\, \mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{K}}t}[\eta^{\mathrm{eq}}\sigma_+]\right\} \\
&= 2\gamma|\alpha_1|^2 \,\mathrm{Re} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\nu-\omega)t - \frac{\tilde{\varkappa}}{2}t}(q_1(t) - \mathrm{i}q_2(t)) \\
&= 2\gamma|\alpha_1|^2 \,\mathrm{Re} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\nu-\omega)t - \frac{\tilde{\varkappa}}{2}t} \left\{c_0(w_1 - \mathrm{i}w_2) + \left[\left(\mathrm{e}^{-Dt}\vec{v}\right)_1 - \mathrm{i}\left(\mathrm{e}^{-Dt}\vec{v}\right)_2\right]\right\} \\
&= 2\gamma|\alpha_1|^2 \,\mathrm{Re} \left\{ \frac{c_0(w_1 - \mathrm{i}w_2)}{\frac{\tilde{\varkappa}}{2} - \mathrm{i}(\nu-\omega)} + \begin{pmatrix} 1, & -\mathrm{i}, & 0 \end{pmatrix} \cdot \frac{1}{D + \frac{\tilde{\varkappa}}{2} - \mathrm{i}(\nu-\omega)}\, \vec{v} \right\} \\
&= \gamma|\alpha_1|^2 \frac{\frac{\tilde{\varkappa}}{2}(p_1^{\mathrm{eq}}w_1 + p_2^{\mathrm{eq}}w_2) + (p_1^{\mathrm{eq}}w_2 - p_2^{\mathrm{eq}}w_1)(\nu-\omega)}{\frac{\tilde{\varkappa}^2}{4} + (\nu-\omega)^2} \\
&\quad + 2\gamma|\alpha_1|^2 \,\mathrm{Re} \left\{ \begin{pmatrix} 1, & -\mathrm{i}, & 0 \end{pmatrix} \cdot \frac{1}{D + \frac{\tilde{\varkappa}}{2} - \mathrm{i}(\nu-\omega)}\, \vec{v} \right\},
\end{aligned}
$$

which is the thesis. $\square$

Remark 7.4. Let us stress that, the vector $\vec{w}$ is the equilibrium of the semigroup generated by $\check{\mathcal{K}}$, if the trace of the initial condition τ is equal to one. Then, we can explicitly determine $\vec{w}$ in a similar way as we did for $\vec{p}^{\mathrm{eq}}$:

$$
\vec{w} = \frac{\gamma}{\det(D)} \begin{pmatrix} -\left(y - \frac{\varepsilon_3^2}{4}\right)\Omega \\ x\Omega \\ -x^2 - \left(y - \frac{\varepsilon_3^2}{4}\right)^2 \end{pmatrix},
$$

where the quantities x, y, z have been introduced in (7.82). By Eqs. (7.85) we can conclude that the quantity $p_1^{\mathrm{eq}}w_1 + p_2^{\mathrm{eq}}w_2$ appearing in $\Sigma(\nu-\omega)$ is positive. Indeed, we have

$$
\begin{aligned}
p_1^{\mathrm{eq}}w_1 + p_2^{\mathrm{eq}}w_2 &= \frac{\gamma^2}{\det(C)\det(D)} \left[y\left(y - \frac{\varepsilon_3^2}{4}\right) + x^2 \right] \Omega^2 \\
&= \frac{\gamma^2}{\det(C)\det(D)} \left[\Delta\omega^2 - \frac{\varepsilon_3^2}{4}\Delta\omega + x^2 \right] \Omega^2 .
\end{aligned}
$$

We have already observed that $\det(C)$ and $\det(D)$ are strictly positive. Moreover, if we look at the definition of x, we have $x^2 = \frac{\varepsilon_3^4}{16} + c^2 + \frac{\varepsilon_3^2}{2}c$, where $c > 0$ is easily determined. In conclusion we have

$$
p_1^{\mathrm{eq}}w_1 + p_2^{\mathrm{eq}}w_2 = \frac{\gamma^2}{\det(C)\det(D)} \left[\frac{3}{4}\Delta\omega^2 + \left(\frac{\varepsilon_3^2}{4} - \frac{1}{2}\Delta\omega\right)^2 + c^2 + \frac{\varepsilon_3^2}{2}c \right] \Omega^2 > 0 .
$$

Let us stress that the other terms appearing in the expression of $P_{\text{het}}(\nu)$ can be negative but their sum is, by definition, greater then zero. In conclusion, as we mentioned, we can not decompose the observed mean power spectrum in the *elastic* or *coherent* part and in the *inelastic* or *incoherent* part, as in [3].

Remark 7.5. If $\varepsilon_1 = \varepsilon_2 = 0$, one has $\vec{w} = \vec{p}^{\text{eq}}$ and the interference term with $(\nu - \omega)$ in the numerator disappears from Eq. (7.102). In this case the spectrum becomes symmetric in $\nu - \omega$. So, the presence of random phases gives also the asymmetry of the spectrum.

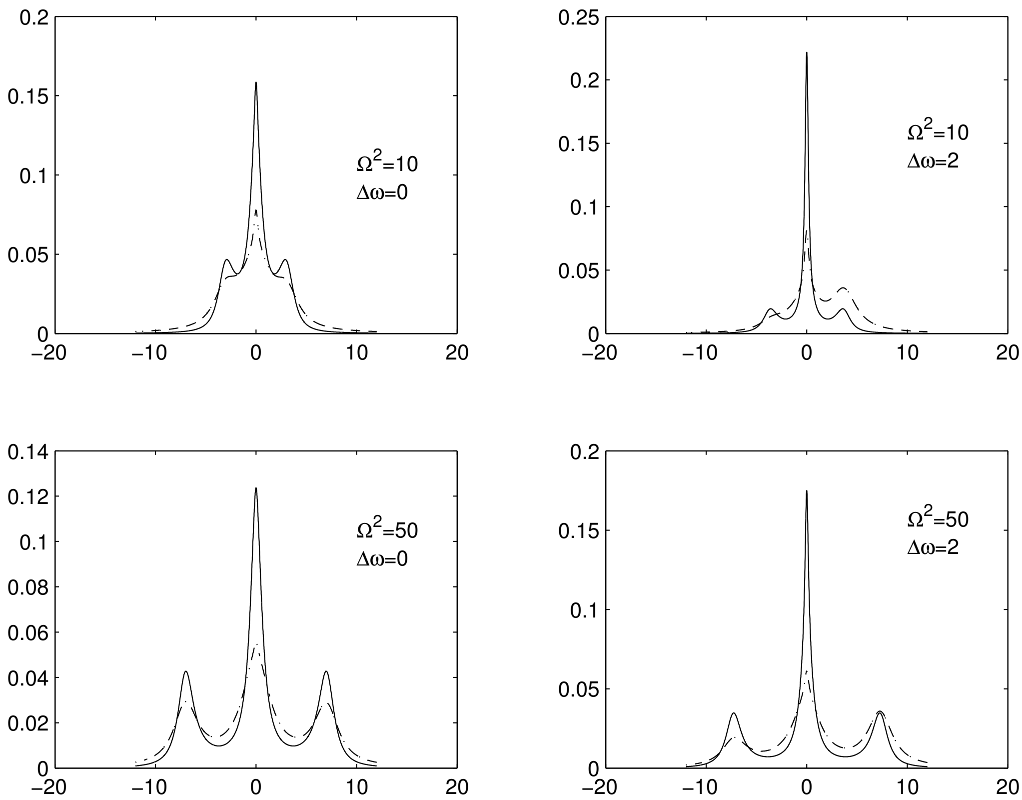

Figure 7.3: The mean observed power spectrum $P_{\text{het}}(\nu)$ for $\gamma = 1$, $\varkappa = 0.4$, $\overline{n} = 0$ $k_{\text{d}} = 0$. The continuous line represents the situation with $\varepsilon_1^2 = \varepsilon_3^2 = 0$ while the dotted line represents $\varepsilon_1^2 = \varepsilon_3^2 = 0.2$.

7.6.2 Graphical examples of the mean observed power spectrum

In this paragraph we want to give some graphical examples of the power spectrum. In [3] graphical examples have been obtained by plotting with *Matlab* the graphic of $\Sigma(\nu - \omega)$ defined in Eq. (7.102). As we mentioned we assume that the stimulating laser and the local oscillator have a stochastic phase, thus we are assuming that $\varepsilon_1^2 + \varepsilon_3^2 > 0$, while in [3] the case $\varepsilon_1 = \varepsilon_3 = 0$ has been studied. Now we shall chose the parameters involved in Eq. (7.102) as in [3], but under the Assumption 7.2. We shall see that the presence of the stochastic phases reduce the visibility of the mean observed power spectrum.

We shall set the following values for the parameters: $\bar{n} = 0$, $k_d = 0$, $\gamma = 1$, $\varkappa = 0.4$, $\Delta\omega = 0$ or $\Delta\omega = 2$, $\Omega^2 = 10$ or $\Omega^2 = 50$, and we shall compare the case $\varepsilon_1^2 = \varepsilon_3^2 = 0.2$ with $\varepsilon_1^2 = \varepsilon_3^2 = 0$. Then, we shall increase the value of ε_1 and ε_3 to better observe the effect of a big stochastic phase on the mean observed power spectrum: when the stochastic contribution to the phase of the lasers grows, $P_{\mathrm{het}(\nu)}$ decays in a slower way and it is less visible with respect to the case of a smaller stochastic contribution.

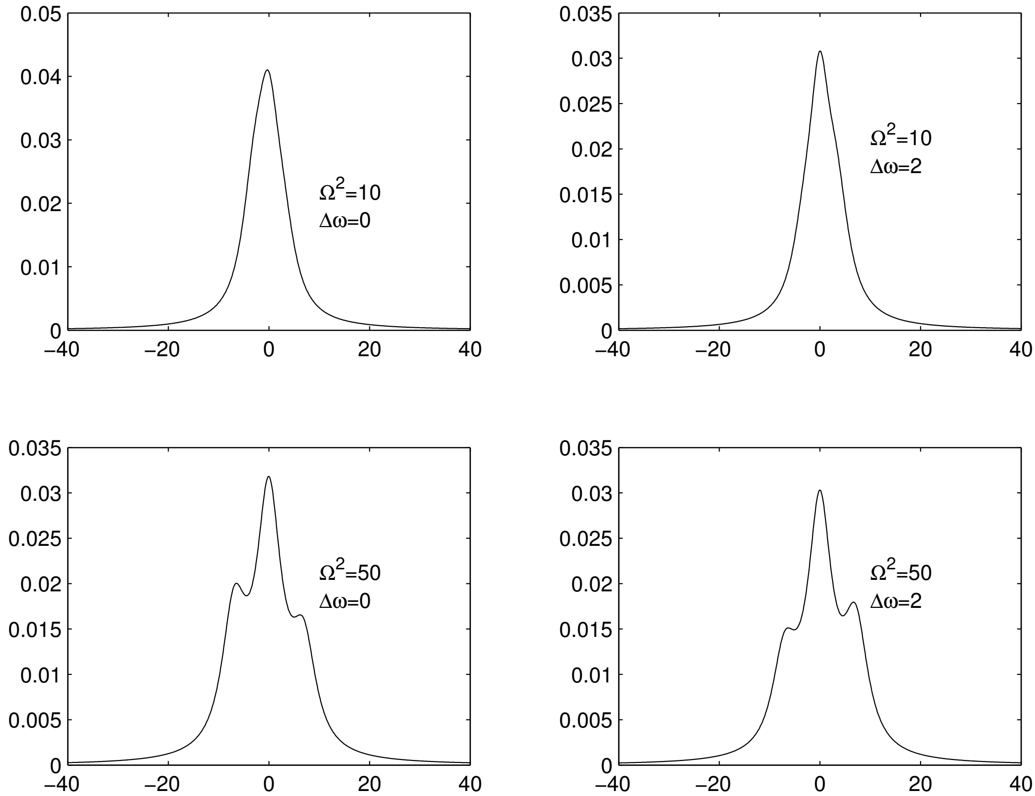

Figure 7.4: The mean observed power spectrum $P_{\mathrm{het}}(\nu)$ for $\gamma = 1$, $\varkappa = 0.4$, $\overline{n} = 0$ $k_{\mathrm{d}} = 0$ and $\varepsilon_1^2 = \varepsilon_3^2 = 4$.

7.7 Homodyne detection

As we mentioned, in the homodyne detection scheme, the local oscillator is in resonance with the carrier frequency of the field reaching the measuring apparatus. Moreover, we need not only $\omega = \nu$, but also that the stimulating laser and the local oscillator are produced by the same source. In conclusion, we can say that all the considerations that we made in the case of heterodyne detection hold, but now we have that the source of randomness in the local oscillator is the same of that one in the stimulating laser. In other words we have to put the following further assumptions in the model.

Assumption 7.3. In the homodyne detection scheme we have

1. $\nu = \omega$;
2. $B_1 \equiv B_2 \equiv B_3 =: B$.

7.7.1 The final model

The final model for the homodyne detection for a two level atom is obtained by using Assumption 7.3 in Eqs. (7.29), that is

$$R_1(t) = \exp\left\{\mathrm{i}\omega t - \mathrm{i}\frac{\varepsilon}{2}B(t)\right\}\sqrt{\gamma}\alpha_1\sigma_- \,, \tag{7.108a}$$

$$R_2(t) = \exp\left\{\mathrm{i}\omega t - \mathrm{i}\frac{\varepsilon}{2}B(t)\right\}\sqrt{\gamma}\alpha_2\sigma_- \,, \tag{7.108b}$$

$$R_3(t) = \sqrt{\gamma}\alpha_3\sigma_- + \lambda\exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon}{2}B(t)\right\}\mathbb{1}\,, \tag{7.108c}$$

$$\sum_{i=1}^{3}|\alpha_i|^2 = 1\,, \quad \alpha_1\,,\ \alpha_2\,,\ \alpha_3 \in \mathbb{C}\,, \tag{7.108d}$$

$$R_4 = 0\,, \tag{7.108e}$$

$$R_5(t) \equiv R_5 = \sqrt{\gamma k_{\mathrm{d}}}\sigma_z\,, \tag{7.108f}$$

$$R_6(t) \equiv R_6 = \sqrt{\gamma\bar{n}}\sigma_-\,, \quad R_7(t) \equiv R_7 = \sqrt{\gamma\bar{n}}\sigma_+\,, \tag{7.108g}$$

$$H_0 = \frac{\omega_0}{2}\sigma_z\,, \tag{7.108h}$$

$$B = W_4\,. \tag{7.108i}$$

The generator of the reduced dynamic

As we did in the case of heterodyne detection, we obtain the generator of the reduced dynamics in the homodyne measurement by using Eqs. (7.108) in Eq. (7.9) and proceeding by direct computation. We give directly the generator with respect to the Rabi frequency Ω, introduced in Eq. (7.30). Then, we end up with the following formula:

$$\begin{aligned}\mathcal{L}(t)[\tau] = -\frac{\mathrm{i}}{2}\,\omega_0[\sigma_z,\tau] + \gamma k_{\mathrm{d}}\left(\sigma_z\tau\sigma_z - \tau\right) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right)\\ + \gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\{P_-,\tau\}\right) - \mathrm{i}\frac{\Omega}{2}\exp\left\{\mathrm{i}\omega t - \mathrm{i}\frac{\varepsilon}{2}B(t)\right\}[\sigma_-,\tau]\\ - \mathrm{i}\frac{\Omega}{2}\exp\left\{-\mathrm{i}\omega t + \mathrm{i}\frac{\varepsilon}{2}B(t)\right\}[\sigma_+,\tau]\,.\end{aligned} \tag{7.109}$$

Let us stress that the Liouvillian is random because it depends on B and not on other stochastic terms.

Summary of the involved parameters and quantities in the homodyne model. As we did for the heterodyne case, we gather in this paragraph the parameters and the most important quantities involved in our model.

- Side channel 1: the channel of the detected light.
- Side channel 2: the channel for the feedback proposals.
- Forward chanel: the channel of the lost light and of the stimulating laser.
- Component of the Wiener process used to introduce a random phase in the two local oscillators and in the stimulating laser: $W_4 \equiv B$.
- Carrier frequency of the two local oscillators and of the stimulating laser: $\omega > 0$.
- Amplitude of the stimulating laser: λ.
- Intensity parameter of the random phase of the local oscillators and of the stimulating laser: ε.
- Natural linewidth of the atom: $\gamma > 0$.
- Thermal bath parameter: $\bar{n} \geq 0$.
- Dephasing parameter: $k_{\mathrm{d}} \geq 0$.
- Resonance frequency of the atom: $\omega_0 > 0$.
- Detuning: $\Delta\omega = \omega - \omega_0$.
- Proportions of the light in the side channels and in the forward channel respectively: α_1, α_2 and α_2 such that $\sum_{k=1}^{3} |\alpha_k|^2 = 1$.
- Module and argument of α_1: r and ϑ_1 respectively.
- Detection random operator: $R_1(t) = \gamma r \mathrm{e}^{\mathrm{i}(\omega+\vartheta_1)-\mathrm{i}\frac{\varepsilon}{2}B(t)}\sigma_-$.
- Rabi frequency: $\Omega = 2|\lambda||\alpha_3|$. We assume $\arg(\mathrm{i}\alpha_3\overline{\lambda}) = 0$.
- Output current of the channel 1: $I_1(t) = \int_0^t F(t-s)\mathrm{d}W_1(s)$.
- Detector response function: $F(t) = k_1\sqrt{\frac{\varkappa}{4\pi}}\exp\left\{-\frac{\varkappa}{2}t\right\}\,, \quad \varkappa > 0\,, \quad k_1 \neq 0$.

The reduced linear stochastic master equation

From the notations stated in the introduction to this chapter we have

$$d = 7\,; \quad m = 4\,; \quad \bar{m} = 2\,. \tag{7.110}$$

As in the case of heterodyne detection, we chose to not observe the component W_3 of the Wiener process W. Then, we have that the filtration $\{\mathcal{E}_t^0\}_{t\geq 0}$ is

$$\mathcal{E}_t^0 = \sigma\Big\{W_1(r),\ W_2(r),\ W_4(r)\,; \quad r \in [0,t]\Big\} \vee \mathcal{N}\,, \quad \forall t \geq 0\,. \tag{7.111}$$

The reduced linear stochastic master equation is

$$\varrho(t) = \eta_0 + \int_0^t \mathcal{L}(s)[\varrho(s)]\mathrm{d}s + \sum_{j=1}^{2} \int_0^t \mathcal{R}_j(s)[\varrho(s)]\mathrm{d}W_j(s), \tag{7.112}$$

where the coefficients now are those stated in Eqs. (7.108) and (7.109).

Rotating frame

To obtain the moments of the output process, we study the conditional state ϱ in a rotating frame. We introduce the process $\check{\varrho}$ by means of a unitary transformation of the process ϱ. We define

$$\check{\varrho}(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon}{2} B(t)\right)\sigma_z} \varrho(t) \mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\omega t - \frac{\varepsilon}{2} B(t)\right)\sigma_z} . \tag{7.113}$$

By the same calculations carried out in the heterodyne case, it follows that the equation fulfilled by the process $\check{\varrho}$ is

$$\mathrm{d}\check{\varrho}(t) = \check{\mathcal{L}}[\check{\varrho}(t)]\mathrm{d}t + \sum_{j=1}^{2} \check{\mathcal{R}}_j[\check{\varrho}(t)]\mathrm{d}W_j(t) - \frac{\mathrm{i}\varepsilon}{4}\left[\sigma_z, \check{\varrho}(t)\right] \mathrm{d}B(t)\,, \tag{7.114}$$

where

$$\begin{aligned}\check{\mathcal{L}}[\tau] = -\frac{\mathrm{i}}{2}\Delta\omega[\sigma_z,\tau] + \left(\gamma k_{\mathrm{d}} + \frac{\varepsilon^2}{16}\right)(\sigma_z\tau\sigma_z - \tau) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right) \\ + \gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\{P_-,\tau\}\right) - \mathrm{i}\frac{\Omega}{2}\left[\sigma_z,\tau\right],\end{aligned} \tag{7.115}$$

and

$$\check{R}_1 := \sqrt{\gamma}\alpha_1\sigma_-\,, \qquad \check{R}_2 := \sqrt{\gamma}\alpha_2\sigma_-\,. \tag{7.116}$$

Then, we define the operators $\check{\mathcal{R}}_j$ as

$$\check{\mathcal{R}}_j[\tau] := \check{R}_j\tau + \tau\check{R}_j^*\,, \quad j = 1,2\,, \quad \forall\tau \in M_n(\mathbb{C})\,. \tag{7.117}$$

Let us stress that in the homodyne detection case all the operators involved in Eq. (7.114) are deterministic and time constant. This is because, with respect to the heterodyne model, we obtained some simplifications due to Assumption 7.3.

7.7.2 Moments

In the homodyne detection context the computations of the moments are more easy than in the heterodyne case. We give below the mean and the second moment. We shall end up with formulas which are formally the same to those obtained in [3, Chap. 9].

The mean

First of all we note that, because of the non randomness of the coefficients appearing in Eq. (7.114), we have that the mean of the process $\check{\varrho}$ fulfills a closed equation. Indeed, if we introduce the process $\check{\eta}$ as

$$\check{\eta}(t) = \mathbb{E}_{\mathbb{Q}}[\check{\varrho}(t)]\,, \tag{7.118}$$

from Eq. (7.114), we have

$$\frac{\check{\eta}(t)}{\mathrm{d}t} = \check{\mathcal{L}}[\check{\eta}(t)]\,, \quad \eta(t) = \eta_0 \in S(\mathcal{H}), \tag{7.119}$$

whose solution is

$$\check{\eta}(t) := \mathrm{e}^{\check{\mathcal{L}}t}[\eta_0]\,. \tag{7.120}$$

Then, the mean of the output is

$$\mathbb{E}^T_{\eta_0}\left[\dot{W}_1(t)\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\check{\mathcal{R}}_1[\check{\varrho}(t)]\right\}\right] = \mathrm{Tr}\left\{\check{\mathcal{R}}_1[\check{\eta}(t)]\right\}\,. \tag{7.121}$$

Second moments

For the second moments we note that the propagator of Eq. (7.119) is

$$\mathcal{T}(t,s) = \mathrm{e}^{\check{\mathcal{L}}(t-s)}. \tag{7.122}$$

Then, by Eq. (6.34), we have

$$\begin{aligned}\mathbb{E}^T_{\eta_0}\left[\dot{W}_1(t)\dot{W}_1(s)\right] &= \delta(t-s) + 1_{(0,+\infty)}(t-s)\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}(t-s)} \circ \check{\mathcal{R}}_1[\check{\eta}(s)]\right\} \\ &\quad + 1_{(0,+\infty)}(s-t)\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}(s-t)} \circ \check{\mathcal{R}}_1[\check{\eta}(t)]\right\}\,. \end{aligned} \tag{7.123}$$

In this case the homodyne spectrum is exactly as in [3], but with a bigger dephasing term. We sketch here below the most important results concerning the spectrum and we remand to [3, Chap. 9] for a detailed discussion.

The homodyne spectrum

First of all we observe that now $\check{\mathcal{L}}$ is the generator of a quantum dynamical semigroup because it can be written in the form stated by Lindblad in [11]. Equation (7.120) can be written in Bloch form as

$$\check{\eta}(t) = \frac{1}{2}\left(c_0 \mathbb{1} + \vec{p}(t)\cdot\vec{\sigma}\right),$$

where now we are considering a generic matrix as initial condition and c_0 is the trace of $\check{\eta}(t)$: it is fixed and does not depend on time. This property comes from the structure of the Liouvillian $\check{\mathcal{L}}$. From the previous discussion, we know that the semigroup $\mathrm{e}^{\check{\mathcal{L}}t}$ has a unique equilibrium: if we write the initial state η_0 as

$$\eta_0 = \eta^{\mathrm{eq}} + \Delta\eta_0\,, \qquad \Delta\eta_0 := \eta_0 - \eta^{\mathrm{eq}}\,, \tag{7.124}$$

the contribution to the spectrum of $\Delta\eta_0$ disappears for long times, as in the heterodyne case, because $e^{\check{\mathcal{L}}t}$ is linear and it exponentially sends all states to the equilibrium state η^{eq}. This way to write the initial condition is very useful to carry out the computation of the spectrum of the homodyne current because the equilibrium state η^{eq} has the following property

$$e^{\check{\mathcal{L}}t}[\eta^{\text{eq}}] \equiv \eta^{\text{eq}}\,.$$

In what follows, we shall write

$$\alpha_1 = |\alpha_1| e^{i\vartheta_1}\,, \quad \vartheta_1 := \arg(\alpha_1)\,.$$

We give now the results about the spectrum: as we already said they are completely equivalent to the case studied in [3], but now the Liouville operator of the system has a greater dephasing therm. The spectral density of the output current is

$$S_{I_1}(\mu,\vartheta_1) := \lim_{T\to\infty}\frac{1}{T}\,\mathbb{E}^T_{\eta_0}\left[\left|\int_0^T e^{i\mu t} I_1(t)\mathrm{d}t\right|^2\right]. \tag{7.125}$$

It is possible to show that

$$S_{I_1}(\mu,\vartheta_1) = \frac{k_1^2\varkappa}{\pi(\varkappa^2+4\mu^2)} S_{\text{hom}}(\mu,\vartheta_1),$$

where

$$S_{\text{hom}}(\mu,\vartheta_1) := S^{\text{el}}_{\text{hom}}(\mu,\vartheta_1) + S^{\text{inel}}_{\text{hom}}(\mu,\vartheta_1), \tag{7.126}$$

$$\begin{aligned} S^{\text{el}}_{\text{hom}}(\mu,\vartheta_1) &= 8\pi\gamma\,|\alpha_1|^2\left(\operatorname{Re}\left(e^{i\vartheta_1}\eta^{\text{eq}}_{12}\right)\right)^2\delta(\mu) \\ &= 2\pi\gamma\,|\alpha_1|^2\left(p_1^{\text{eq}}\cos\vartheta_1 + p_2^{\text{eq}}\sin\vartheta_1\right)^2\delta(\mu), \end{aligned} \tag{7.127}$$

$$S^{\text{inel}}_{\text{hom}}(\mu,\vartheta_1) = |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_1|^2 S^{\text{red}}_{\text{hom}}(\mu,\vartheta_1), \tag{7.128}$$

$$S^{\text{red}}_{\text{hom}}(\mu,\vartheta_1) = 1 + \operatorname{Re}\left\{\begin{pmatrix}1+e^{2i\vartheta_1}\,, & i\left(1-e^{2i\vartheta_1}\right)\,, & 0\end{pmatrix}\cdot\left(\frac{2\gamma C}{C^2+\mu^2}\,\vec{d}(0)\right)\right\}, \tag{7.129}$$

where

$$\vec{d}(0) := \frac{1}{2}\,\vec{a}^{\,(1)} + \frac{i}{2}\,\vec{a}^{\,(2)}\,,$$

$$\vec{a}^{\,(1)} := \begin{pmatrix} 1+p_3^{\text{eq}} - (p_1^{\text{eq}})^2 \\ -p_1^{\text{eq}}p_2^{\text{eq}} \\ -p_1^{\text{eq}}(1+p_3^{\text{eq}}) \end{pmatrix}, \qquad \vec{a}^{\,(2)} := \begin{pmatrix} p_1^{\text{eq}}p_2^{\text{eq}} \\ -(1+p_3^{\text{eq}}) + (p_2^{\text{eq}})^2 \\ p_2^{\text{eq}}(1+p_3^{\text{eq}}) \end{pmatrix}.$$

In the homodyne case a decomposition of the spectrum holds. Indeed, the components $S^{\text{el}}_{\text{hom}}(\mu,\vartheta_1)$ and $S^{\text{inel}}_{\text{hom}}(\mu,\vartheta_1)$ of the homodyne spectrum $S_{\text{hom}}(\mu,\vartheta_1)$, are respectively the *elastic* or *coherent* part and the *inelastic* or *incoherent* part, while the therm $S^{\text{red}}_{\text{hom}}(\mu,\vartheta_1)$ is its *reduced component.*

7.7.3 Randomness in the optical paths

To take into account random differences in the optical paths one can put a random phase in α_1, say a normal variable, independent of all the Wiener processes, centered on ϑ_1 and with a variance β^2 (which grows with the difference of the optical paths). To obtain the moments one takes the previous formulas and takes the mean over this new randomness, which appears only in $\check{\mathcal{R}}_1$.

We recall that the reference filtration is $\{\mathcal{F}_t\}_{t\geq 0}$. Then, we introduce the $\mathcal{F}_0$-measurable normal random variable X as

$$X \sim \mathcal{N}(\vartheta_1, \beta^2)\,, \quad \text{independent of } W\,.$$

Take $r \geq 0$, $r^2 < 1$, and let α_1 be such that

$$\alpha_1 = r\mathrm{e}^{\mathrm{i}X}.$$

The homodyne spectrum with the new source of randomness

To obtain the spectrum with this new randomness in the phase, it is enough to take the mean over X in Eq. (7.126). Then, we have

$$\hat{S}_{\text{hom}}(\mu, \vartheta_1, \beta) := \hat{S}^{\text{el}}_{\text{hom}}(\mu, \vartheta_1, \beta) + \hat{S}^{\text{inel}}_{\text{hom}}(\mu, \vartheta_1, \beta)\,, \tag{7.130}$$

where

$$\hat{S}^{\text{el}}_{\text{hom}}(\mu, \vartheta_1, \beta) := \mathbb{E}_{\mathbb{Q}}\left[S^{\text{el}}_{\text{hom}}(\mu, X)\right] \tag{7.131}$$

$$\hat{S}^{\text{inel}}_{\text{hom}}(\mu, \vartheta_1, \beta) := \mathbb{E}_{\mathbb{Q}}\left[S^{\text{inel}}_{\text{hom}}(\mu, X)\right]\,. \tag{7.132}$$

From Eqs. (7.128) and (7.129) we can immediately obtain $\hat{S}^{\text{inel}}_{\text{hom}}(\mu, \vartheta_1, \beta)$ because of its linearity in the therms depending on the random phase: by defining the reduced spectrum as

$$\hat{S}^{\text{red}}_{\text{hom}}(\mu, \vartheta_1, \beta) := \mathbb{E}_{\mathbb{Q}}\left[S^{\text{red}}_{\text{hom}}(\mu, X)\right] \tag{7.133}$$

we have

$$\hat{S}^{\text{red}}_{\text{hom}}(\mu, \vartheta_1, \beta) = 1 + \operatorname{Re}\left\{ \left(1 + \mathrm{e}^{2\mathrm{i}\vartheta_1 - \beta^2}\,, \quad \mathrm{i}\left(1 - \mathrm{e}^{2\mathrm{i}\vartheta_1 - \beta^2}\right)\,, \quad 0\right) \cdot \left(\frac{2\gamma C}{C^2 + \mu^2}\,\vec{d}(0)\right)\right\}. \tag{7.134}$$

For $\hat{S}^{\text{el}}_{\text{hom}}(\mu, \vartheta_1, \beta)$, by using Eq. (7.127), we have

$$\begin{aligned}\hat{S}^{\text{el}}_{\text{hom}}(\mu, \vartheta_1, \beta) &= 8\pi\gamma r^2 \mathbb{E}_{\mathbb{Q}}\left[\left(\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}X}\eta^{\text{eq}}_{12}\right)\right)^2\right]\delta(\mu) \\ &= 2\pi\gamma r^2 \mathbb{E}_{\mathbb{Q}}\left[\left(\eta^{\text{eq}}_{12}\mathrm{e}^{\mathrm{i}X} + \overline{\eta}^{\text{eq}}_{12}\mathrm{e}^{-\mathrm{i}X}\right)^2\right]\delta(\mu)\,.\end{aligned}$$

In conclusion, we obtain

$$\hat{S}^{\text{el}}_{\text{hom}}(\mu, \vartheta_1, \beta) = 2\pi r^2 \gamma \delta(\mu) \Bigg[\mathrm{e}^{-\beta^2} \left(p_1^{\text{eq}} \cos\vartheta_1 + p_2^{\text{eq}} \sin\vartheta_1 \right)^2 + \frac{1 - \mathrm{e}^{-\beta^2}}{2} \left((p_1^{\text{eq}})^2 + (p_2^{\text{eq}})^2 \right) \Bigg] \tag{7.135}$$

7.7.4 Squeezing: graphical examples

As can be easily seen by direct computation, in both the previous cases of homodyne detection, that is the case with a deterministic phase in α_1 and the case with a random phase, when the Rabi frequency Ω is equal to zero, the dependence on the phase disappears from the spectrum. By the way, when $\Omega > 0$, the components $S^{\text{el}}_{\text{hom}}(\mu, \vartheta_1)$ and $S^{\text{inel}}_{\text{hom}}(\mu, \vartheta_1)$ in Eq. (7.126) and the components $S^{\text{el}}_{\text{hom}}(\mu, \vartheta_1, \beta)$ and $S^{\text{inel}}_{\text{hom}}(\mu, \vartheta_1, \beta)$ in Eq. (7.130) turn out to be dependent on the phase. The dependence on the phase is the first peculiar difference between the homodyne and the heterodyne detection. In the homodyne contest, the white noise contribution, with our choice of normalisation, is 1: the phase dependence rises a typical quantum phenomenon which is known as *squeezing*. With squeezing we mean that the spectrum goes below the level of the shot noise 1 and, so, we can say that some negative correlation between signal and noise has been introduced. In both the cases that we have studied above, the therm responsible of this phenomenon is the reduced compoent $S^{\text{red}}_{\text{hom}}$ of the spectrum.

Here below we present some graphical examples: we plot together the graphics of $S^{\text{red}}_{\text{hom}}$ in both the cases of deterministic and random phase for different values of ϑ_1. From these example we see that the presence of the therm $\mathrm{e}^{-\beta^2}$, due to the variance of the random phase, reduce the squeezing visibility, either in the case of non random phase in the stimulating laser and in the local oscillator, i.e. $\varepsilon = 0$, or in the case $\varepsilon \neq 0$.

As we mentioned, the result obtained for the non random phase case are completely similar to those obtained in [3]: the unique difference is that now we can take into account the presence of a stochastic phase in the local oscillator and in the stimulating laser by taking $\varepsilon \neq 0$. Then, to see the difference that the random phase could introduce in the model presented in the book, first we set $\varepsilon \equiv 0$ and we plot the reduced component of the spectrum in the deterministic phase case (that is $\beta = 0$) and in the random phase case (thus, $\beta \neq 0$), for different values of ϑ_1, (Fig. 7.5). Then, we shall present the difference that the random phase introduces in our model: we shall plot the case $\varepsilon \neq 0$ in both the situations $\beta = 0$ and $\beta \neq 0$, (Fig. 7.6).

7.8 Dissipation with memory

By using the proposal of [5] we can modify the dephasing contribution and to introduce some memory in it, by using the Ornstein-Uhlenbeck process.

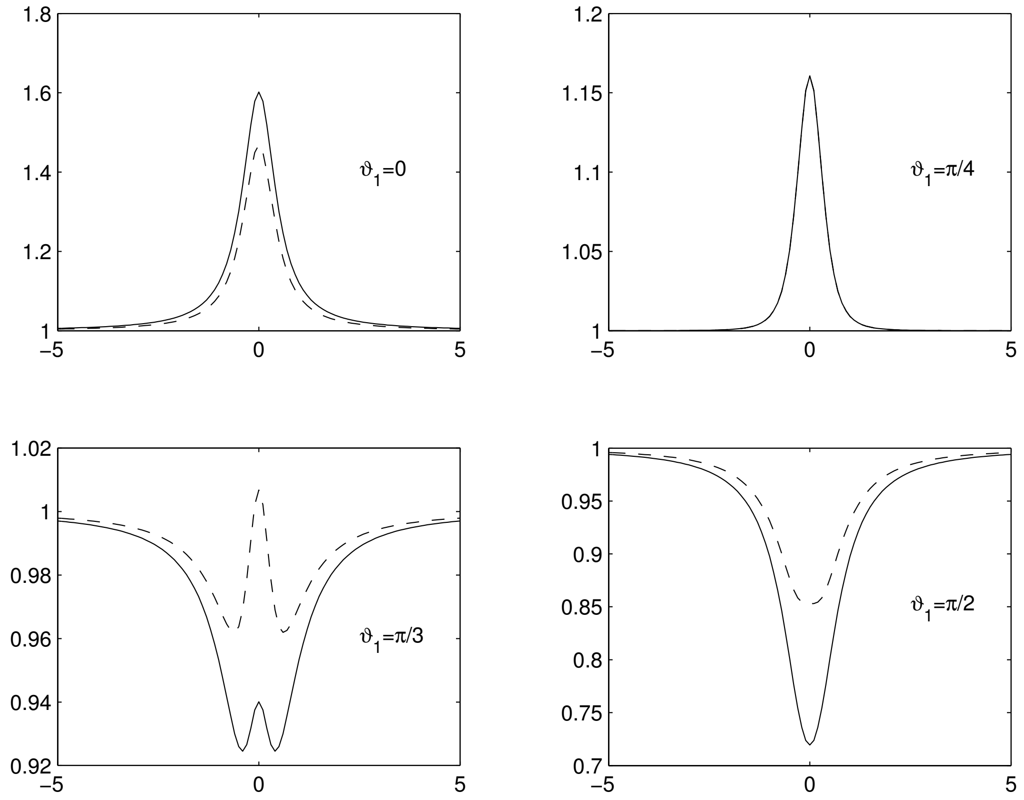

Figure 7.5: $S^{\mathrm{red}}_{\mathrm{mom}}$ for $\gamma = 1$, $\overline{n} = 0$, $k_{\mathrm{d}} = 0$, $\Omega = 0.2976$ and $\varepsilon \equiv 0$. The continuous line represents the situation with $\beta = 0$ while the dotted line represents $\beta = 0.6$.

By modifying to some extent the same proposal, we can obtain also some thermal-like dissipation term with memory.

Let $B(t)$ be a standard complex Wiener process, $B(t) = \frac{1}{\sqrt{2}} W_a(t) - \frac{\mathrm{i}}{\sqrt{2}} W_b(t)$, and W_0 a third real independent Wiener process. Take $\kappa \in \mathbb{C}$, $\operatorname{Re}\kappa > 0$, $g \geq 0$, $n \geq 0$, and define the complex Ornstein-Uhlenbeck process

$$X(t) := -\kappa\sqrt{gn} \int_0^t \mathrm{e}^{-\kappa(t-s)} \mathrm{d}B(s).$$

As a first proposal of a model of "coloured" thermal dissipation, we could introduce the following contribution to the linear master equation for $\sigma(t)$:

$$-\,\mathrm{i}\left[\overline{X(t)}\sigma_- + X(t)\sigma_+,\, \sigma(t)\right] \mathrm{d}t + g\left(\sigma_-\sigma(t)\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-,\, \sigma(t)\right\}\right) \mathrm{d}t \\ + \sqrt{g}\left(\sigma_-\sigma(t) + \sigma(t)\sigma_+\right) \mathrm{d}W_0(t).$$

In the limit $\operatorname{Re}\kappa \to +\infty$, one *should* get the usual Markov approximation in the master equation, which is

$$g\,(n+1)\left(\sigma_-\eta(t)\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-,\, \eta(t)\right\}\right) + gn\left(\sigma_+\eta(t)\sigma_- - \frac{1}{2}\left\{\sigma_-\sigma_+,\, \eta(t)\right\}\right).$$

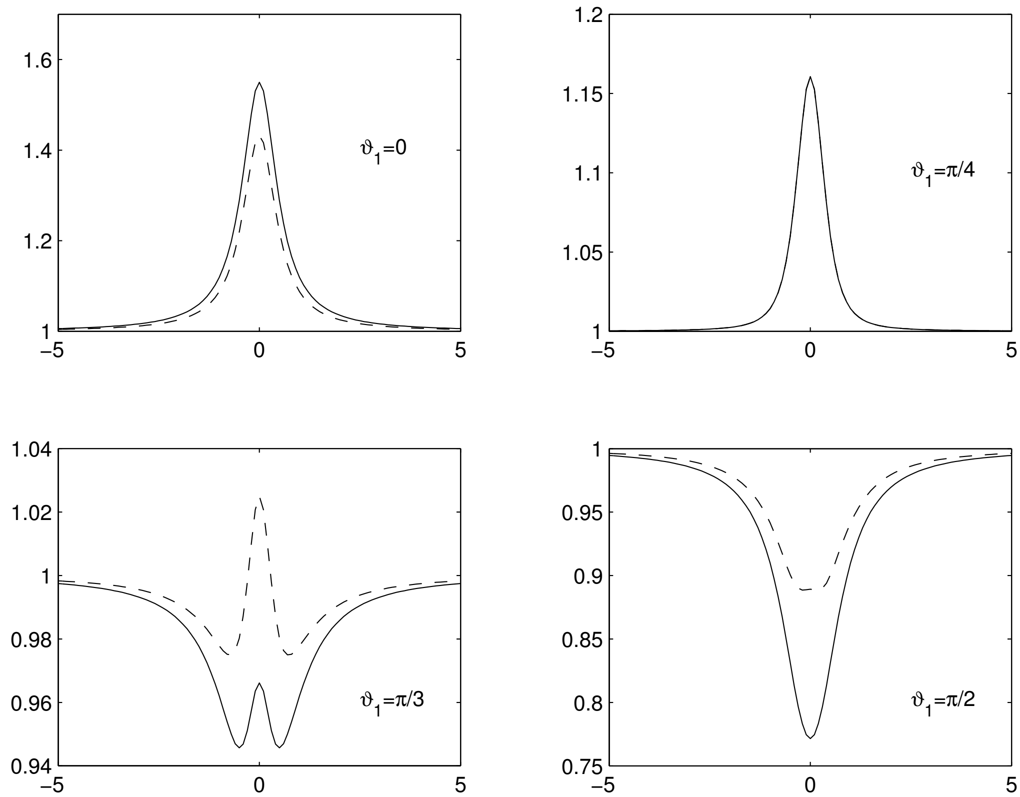

Figure 7.6: $S^{\text{red}}_{\text{mom}}$ for $\gamma = 1$, $\overline{n} = 0$, $k_{\text{d}} = 0$, $\Omega = 0.2976$ and $\varepsilon^2 = \sqrt{0.2}$. The continuous line represents the situation with $\beta = 0$ while the dotted line represents $\beta = 0.6$.

A second proposal could be based in using the increments of the Ornstein-Uhlenbeck process

$$\mathrm{d}Y(t) = -\kappa Y(t)\mathrm{d}t + \sqrt{gn}\,\mathrm{d}B(t), \qquad Y(t) := -\frac{X(t)}{\kappa}\,.$$

We can add to the linear stochastic master equation for $\sigma(t)$ the term

$$\begin{aligned} &-\mathrm{i}\left[\sigma_-\mathrm{d}\overline{Y(t)} + \sigma_+\mathrm{d}Y(t),\, \sigma(t)\right] + \sqrt{g}\left(\sigma_-\sigma(t) + \sigma(t)\sigma_+\right)\mathrm{d}W_0(t) \\ &\qquad + g(n+1)\left(\sigma_-\sigma(t)\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-,\, \sigma(t)\right\}\right)\mathrm{d}t \\ &\qquad\qquad + gn\left(\sigma_+\sigma(t)\sigma_- - \frac{1}{2}\left\{\sigma_-\sigma_+,\, \sigma(t)\right\}\right)\mathrm{d}t. \end{aligned}$$

In the limit $\kappa \to 0$, one obtains the usual thermal contribution without memory.

7.9 Feedback

We said at the beginning of this chapter that we shall use the output of channel 2 to introduce feedback in our model. In [3] the case of instantaneous homodyne feedback has been studied: there instantaneous signal of channel 2, which is formally proportional to $\dot{W}_2(t)$, is considered.

In this section we shall give some proposal, that could be developed in a future work, to introduce feedback in our model not considering only the instantaneous signal of channel 2, but the output of channel 2 itself.

The quantum system which we refer to, is always a two level atom stimulated by a laser. The feedback acts on the stimulating laser by means of an elecromodulator. The ideal configuration is given in Fig. 7.7.

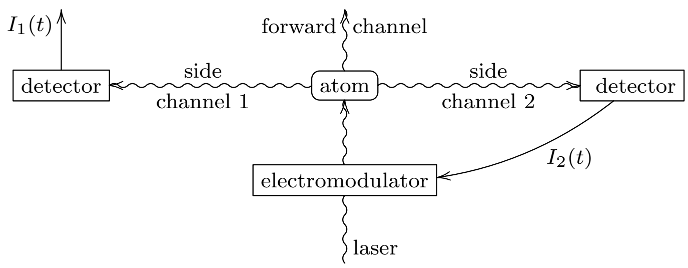

Figure 7.7: Channel 0: forward channel with laser; channel 1: side channel with feedback; channel 2: side channel without feedback.

Let us stress that we do not specify if the detectors are of homodyne or heterodyne kind: we want to give, for the same scheme, a proposal for both the situations.

The output current of channel 2

The output current of channel 2 is

$$I_2(t) := \int_0^t F_2(t-s)\mathrm{d}W_2(s)\,, \tag{7.136}$$

where F_2 has the same role of F in Eq. (7.14): it is the detector response function of channel 2 and it is given by

$$F_2(t) := k\sqrt{\frac{\varkappa_2}{4\pi}}\exp\left\{-\frac{\varkappa_2}{2}\,t\right\}\,, \quad \varkappa_2 > 0\,, \quad k \neq 0\,. \tag{7.137}$$

As before, the constant k and $\varkappa_2$ depend on the measuring apparatus; k has the dimensions of a current and $\frac{1}{\varkappa_2}$ the dimensions of a time. The constant $\varkappa_2$ controls the time resolution: for $\varkappa_2 \to +\infty$ the current $I_2(t)$ becomes formally proportional to the singular process $\dot{W}_2(t)$, and the past time are not involved: in other words, a big value of $\varkappa_2$ gives a good time resolution.

Feedback proposals

The feedback proposals that we are going to do are of Hamiltonian type, that is the contribution of the feedback to our system enters into play because it modify the Hamiltonian $H(t)$ of the system. More precisely, it enters into play not in the free Hamiltonian H_0 but in its time dependent component $H_f(t)$.

For the feedback proposal, we consider the case without memory, this is we set to zero the control parameters of the component of the Winer process W that we used to introduce randomness in $H_f(t)$.

In the homodyne case we take

$$H_f(t) = \frac{\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t}\sigma_- + \mathrm{e}^{-\mathrm{i}\omega t}\sigma_+\right) .$$

Then, we insert the feedback as follows

$$\begin{aligned} \Omega \mathrm{e}^{-\mathrm{i}\omega t} &\longrightarrow \mathrm{e}^{-\mathrm{i}\omega t}\left(\Omega + c\,\mathrm{e}^{\mathrm{i}\varphi}u(t)\right) , \\ \Omega \mathrm{e}^{\mathrm{i}\omega t} &\longrightarrow \mathrm{e}^{\mathrm{i}\omega t}\left(\Omega + c\,\mathrm{e}^{-\mathrm{i}\varphi}u(t)\right) , \end{aligned} \tag{7.138}$$

where $c \geq 0$ and φ is a phase. The function u is a real control, and then this is a function of the output current of channel 2 I_2: we have to choose it in a suitable form. First of all we observe that u has to be chosen as a smooth function. Indeed, we have to guarantee that the conditions introduced by Assumption 2.3, for the existence and uniqueness for the stochastic differential equation involved, are fulfilled. Of course, they are very strong conditions but it is what we need to introduce randomness just in the phase of the lasers and not in their intensity. On the other hand, we have to take into account the feedback delay: when we insert the current $I_2(t)$ as feedback it is reasonable that the feedback loop introduces a delay in the signal.

An interesting case that could be developed, and that satisfies to the previous conditions, is

$$u(t) = \beta \tanh\left(\frac{1}{\beta} I_2(t-\delta)\right) , \quad \delta > 0\,, \quad \beta > 0\,. \tag{7.139}$$

In conclusion we have that the new Hamiltonian of the system is

$$\tilde{H}(t) = H_0 + \tilde{H}_f(t)\,,$$

$$\begin{aligned} \tilde{H}_f(t) := H_f(t) &+ \frac{1}{2}\,c\,\beta\,\mathrm{e}^{\mathrm{i}(\varphi-\omega)} \tanh\left(\frac{1}{\beta} I_2(t-\delta)\right)\sigma_+ \\ &+ \frac{1}{2}\,c\,\beta\,\mathrm{e}^{-\mathrm{i}(\varphi-\omega)} \tanh\left(\frac{1}{\beta} I_2(t-\delta)\right)\sigma_- \,. \end{aligned} \tag{7.140}$$

The Liouville operator now is

$$\tilde{\mathcal{L}}(t)[\tau] = -\frac{\mathrm{i}}{2}\,\omega_0[\sigma_z,\tau] + \gamma k_\mathrm{d}\,(\sigma_z\tau\sigma_z - \tau) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right)$$
$$+\,\gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\{P_-,\tau\}\right) - \mathrm{i}\frac{\Omega}{2}\,\mathrm{e}^{\mathrm{i}\omega t}[\sigma_-,\tau] - \mathrm{i}\frac{\Omega}{2}\,\mathrm{e}^{-\mathrm{i}\omega t}[\sigma_+,\tau]$$
$$-\,\frac{\mathrm{i}}{2}\,c\,\beta\,\mathrm{e}^{\mathrm{i}(\varphi-\omega)}\tanh\left(\frac{1}{\beta}I_2(t-\delta)\right)[\sigma_+,\tau]$$
$$-\,\frac{1}{2}\,c\,\beta\,\mathrm{e}^{-\mathrm{i}(\varphi-\omega)}\tanh\left(\frac{1}{\beta}I_2(t-\delta)\right)[\sigma,-\tau]\,. \tag{7.141}$$

In the limit $\delta \downarrow 0$, $\beta \uparrow \infty$, $\varkappa_2 \uparrow \infty$ one should obtain the case in [3].

A proposal in the heterodyne case could be to take the control function u as

$$u(t) = \beta\tanh\left(\frac{1}{\beta}I_2(t-\delta)^2\right)\,, \quad \delta > 0\,, \quad \beta > 0\,, \tag{7.142}$$

that is, depending on the power of the current I_2. Then, one should study the electrical power of the current I_1, as we did in the no feedback case.

Of course one could think to insert further complications in the heterodyne model by allowing randomness in the phase of the stimulating laser. In this way one has

$$H_f(t) = \frac{\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t - \frac{\varepsilon_3}{2}B_3(t)}\sigma_- + \mathrm{e}^{-\mathrm{i}\omega t + \frac{\varepsilon_3}{2}B_3(t)}\sigma_+\right)\,.$$

With the previous proposal for the control the Hamiltonian H becomes

$$\tilde{H}(t) = H_0 + \tilde{H}_f(t)\,,$$

$$\tilde{H}_f(t) := H_f(t) + \frac{1}{2}\,c\,\beta\,\mathrm{e}^{\mathrm{i}(\varphi-\omega)}\tanh\left(\frac{1}{\beta}I_2(t-\delta)^2\right)\sigma_+ \tag{7.143}$$
$$+\frac{1}{2}\,c\,\beta\,\mathrm{e}^{-\mathrm{i}(\varphi-\omega)}\tanh\left(\frac{1}{\beta}I_2(t-\delta)^2\right)\sigma_-\,.$$

and so the liouvillian is

$$\tilde{\mathcal{L}}(t)[\tau] = -\frac{\mathrm{i}}{2}\,\omega_0[\sigma_z,\tau] + \gamma k_\mathrm{d}\,(\sigma_z\tau\sigma_z - \tau) + \gamma(\bar{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right)$$
$$+\,\gamma\bar{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\{P_-,\tau\}\right) - \mathrm{i}\frac{\Omega}{2}\,\mathrm{e}^{\mathrm{i}\omega t - \frac{\varepsilon_3}{2}B_3(t)}[\sigma_-,\tau] - \mathrm{i}\frac{\Omega}{2}\,\mathrm{e}^{-\mathrm{i}\omega t + \frac{\varepsilon_3}{2}B_3(t)}[\sigma_+,\tau]$$
$$-\,\frac{\mathrm{i}}{2}\,c\,\beta\,\mathrm{e}^{\mathrm{i}(\varphi-\omega)}\tanh\left(\frac{1}{\beta}I_2(t-\delta)^2\right)[\sigma_+,\tau]$$
$$-\,\frac{\mathrm{i}}{2}\,c\,\beta\,\mathrm{e}^{-\mathrm{i}(\varphi-\omega)}\tanh\left(\frac{1}{\beta}I_2(t-\delta)^2\right)[\sigma_-,\tau]\,. \tag{7.144}$$

We stress that these are all phenomenological proposals that have to be studied...

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