Affine processes on positive semidefinite matrices

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Introduction

- Affine processes on S_d^+, the cone of positive semidefinite matrices, are stochastically continuous time-homogeneous Markov processes with state space S_d^+, whose Laplace transform have exponential-affine dependence on the initial state,

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  - Sufficient conditions for the existence of affine processes on S_d^+.
  - Probabilistic properties like infinite divisibility, excursion theory, etc.

Motivation: Affine stochastic covariance models

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- Risk neutral dynamics for the log-price process Y_t and the $\mathbb{R}_+$-valued variance process X_t:

$$dX_t = (b + \beta X_t)\, dt + \sigma \sqrt{X_t}\, dW_t + dJ_t, \qquad X_0 = x,$$
$$dY_t = \left(r - \frac{X_t}{2}\right) dt + \sqrt{X_t} dB_t, \qquad Y_0 = y.$$

  - B, W: correlated Brownian motions,
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- Efficient valuation of European options since the moment generating function is explicitly known (up to the solution of an ODE) and is of the following form

$$\mathbb{E}_{x,y}\left[e^{-zX_t+vY_t}\right] = e^{\Phi(t,z,v)+\Psi(t,z,v)x+vy}.$$

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- ...obtain a consistent pricing framework for multi-asset options such as basket options,
- ...use them for model-based decision-making in the area of portfolio optimization and hedging of correlation risk.

Multivariate affine stochastic covariance models

- Multivariate stochastic covariance models consist of a d-dimensional logarithmic price process with risk-neutral dynamics

$$dY_t = \left(r\mathbf{1} - \frac{1}{2} X_t^{\mathrm{diag}} \right) dt + \sqrt{X_t} dB_t, \quad Y_0 = y,$$

  and stochastic covariation process $\int_0^{\cdot} X = \langle Y, Y \rangle$.
  - B: d-dimensional Brownian motion.
  - r: constant interest rate
  - $\mathbf{1}$: the vector whose entries are all equal to one.
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  - $\mathbf{1}$: the vector whose entries are all equal to one.
  - X^{diag}: the vector containing the diagonal entries of X.
- In order to qualify for a covariation process, X must be specified as a process in S_d^+. An affine dynamics for X is interesting out of several reasons.

Multivariate affine stochastic covariance models

- The following affine dynamics for X have been proposed in the literature:

$$dX_t = (b + HX_t + X_t H^\top)dt + \sqrt{X_t}dW_t\Sigma + \Sigma^\top dW_t^\top \sqrt{X_t} + dJ_t,$$
$$X_0 = x \in S_d^+,$$

  - b: suitably chosen matrix in S_d^+,
  - H, Σ: invertible matrices,
  - W a standard $d \times d$-matrix of Brownian motions possibly correlated with B,
  - J a pure jump process whose compensator is an affine function of X.

Multivariate affine stochastic covariance models

- Analytic tractability: Under some (mild) technical conditions, the following affine transform formula holds:

$$\mathbb{E}_{x,y}\left[e^{-\operatorname{Tr}(zX_t)+v^\top Y_t}\right] = e^{\Phi(t,z,v)+\operatorname{Tr}(\Psi(t,z,v)x)+v^\top y}$$

  for appropriate arguments $z \in S_d \times \mathrm{i}S_d$ and $v \in \mathbb{C}^d$. The functions Φ and Ψ solve a system of generalized Riccati ODEs. $\Rightarrow$ Option pricing via Fourier methods

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- $v^\top y$ in the moment generating function corresponds to a homogeneity assumption and implies that the covariation process X is a Markov process in its own filtration, which motivates the analysis of affine processes on S_d^+.

Related Literature

- Affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$:
  - D. Duffie, D. Filipović and W. Schachermayer [11]: Characterization of affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$.
  - D. Filipović and E. Mayerhofer [12]: Characterization of affine Diffusion processes.
  - M. Keller-Ressel, W. Schachermayer and J. Teichmann [16]: Regularity of affine processes.
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- Affine processes on S_d^+ - Theory of Wishart processes:
  - M. Bru [3]: Existence and uniqueness of Wishart processes of type

$$dX_t = (\delta I_d)dt + \sqrt{X_t}dW_t + dW_t^\top \sqrt{X_t}, \quad X_0 \in S_d^+,$$

    for $\delta \geq d - 1$.
  - C. Donati-Martin, Y. Doumerc, H. Matsumoto and M. Yor [10]: Properties of Wishart processes.

Related Literature

- Affine processes on S_d^+ – Applications from mathematical Finance:
  - O. Barndorff-Nielsen and R. Stelzer [2]: Matrix-valued Lévy driven Ornstein-Uhlenbeck processes of finite variation.
  - B. Buraschi et al. [5, 4]: Correlation risk and portfolio optimization.
  - J. Da Fonseca et al. [6, 7, 8, 9]: Multivariate stochastic covariance and option pricing.
  - C. Gourieroux and R. Sufana [13, 14]: Wishart quadratic term structure models.
  - M. Leippold and F. Trojani [17]: S_d^+-valued affine jump diffusions and financial applications (multivariate option pricing, interest rate models, etc.)

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- $(P_t)_{t \geq 0}$: associated semigroup acting on bounded measurable functions $f : S_d^+ \to \mathbb{R}$,

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- X might be not conservative. Standard extension of p_t to $S_d^+ \cup \{\Delta\}$ with cemetery Δ.

Definition

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The Markov process X is called affine if

1. it is stochastically continuous, that is, $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly on S_d^+ for every t and $x \in S_d^+$, and
2. its Laplace transform has exponential-affine dependence on the initial state:

$$P_t e^{-\langle u,x\rangle} = \int_{S_d^+} e^{-\langle u,\xi\rangle} p_t(x,d\xi) = e^{-\phi(t,u)-\langle\psi(t,u),x\rangle},$$

for all t and $u, x \in S_d^+$, for some functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$.

Regularity and Feller property

Definition

The affine process X is called regular if the derivatives

$$F(u) = \left.\frac{\partial \phi(t,u)}{\partial t}\right|_{t=0+}, \qquad R(u) = \left.\frac{\partial \psi(t,u)}{\partial t}\right|_{t=0+} \tag{1}$$

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Theorem

Let X be an affine process with state space S_d^+. Then, we have:

1. *X is regular.*
2. *X is a Feller process.*

Remarks on the proof

The proof relies on several properties of the functions ϕ and ψ:

- Semi-flow property: For all $t, s \in \mathbb{R}_+$

$$\phi(t+s,u) = \phi(t,u) + \phi(s,\psi(t,u)), \tag{2}$$

$$\psi(t+s,u) = \psi(s,\psi(t,u)). \tag{3}$$

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- Order relations: For all $u,\ v \in S_d^+$ with $v \preceq u$ and for all $t \geq 0$,

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This implies

Lemma

The function $\psi(t,u) \in S_d^{++}$ for all $u \in S_d^{++}$ and $t \geq 0$.

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- Regularity can be proved by methods on the regularity of (semi-)flows from Montgommery and Zippin.

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- Regularity can be proved by methods on the regularity of (semi-)flows from Montgommery and Zippin.
- The Feller property follows from regularity and the fact that $\{e^{-\langle u,x\rangle} \mid u \in S_d^{++}\}$ is dense in $C_0(S_d^+)$ (Stone-Weierstrass).

Consequences of regularity:

From regularity we can conclude local characteristics:

- Since the following limit exists,

$$\mathcal{A}e^{-\langle u,x\rangle} = e^{-\langle u,x\rangle} \lim_{t\to 0} \frac{1}{t} E\left(e^{-\langle u,X_t - x\rangle} - 1\right),$$

  we know that the limit is of Lévy-Khintchine form on the cone $S_d^+ - \mathbb{R}x$.
- Due to the affine property the previous equation is also an equation for the derivatives of ϕ and ψ.
- By the Markov property (semigroup property) these equations turn into ODEs for ϕ and ψ.

The function ϕ and ψ are solutions of ODEs:

- Hence we obtain

$$\frac{\partial \phi(t,u)}{\partial t} = F(\psi(t,u)), \quad \phi(0,u) = 0, \tag{4}$$

$$\frac{\partial \psi(t,u)}{\partial t} = R(\psi(t,u)), \quad \psi(0,u) = u \in S_d^+, \tag{5}$$

  where F and R are defined in (1).

- Due to the particular form of F and R we shall call them generalized Riccati equations.

The functions F and R – the question of existence of affine processes

Theorem

Moreover, $\phi(t,u)$ *and* $\psi(t,u)$ *solve the differential equations* (4) *and* (5), *where* F *and* R *have the following form*

$$F(u) = \langle b, u \rangle + c - \int_{S_d^+ \setminus \{0\}} (e^{-\langle u, \xi \rangle} - 1) m(d\xi),$$

$$R(u) = -2u\alpha u + B^\top(u) + \gamma - \int_{S_d^+ \setminus \{0\}} \left(e^{-\langle u, \xi \rangle} - 1 + \langle \chi(\xi), u \rangle \right) \frac{\mu(d\xi)}{\|\xi\|^2 \wedge 1}.$$

Conversely, let $(\alpha, b, \beta^{ij}, c, \gamma, m, \mu)$ *be an admissible parameter set. Then there exists a unique affine process on* S_d^+ *with infinitesimal generator* (6).

Infinitesimal generator

Theorem

If X is an affine process on S_d^+, then its infinitesimal generator is of affine form:

$$\begin{aligned}\mathcal{A}f(x) = {} & \frac{1}{2}\sum_{i,j,k,l} A_{ijkl}(x)\frac{\partial^2 f(x)}{\partial x_{ij}\partial x_{kl}} + \langle b + B(x), \nabla f(x)\rangle - (c + \langle \gamma, x\rangle)f(x) \\ & + \int_{S_d^+\setminus\{0\}} (f(x+\xi) - f(x))\, m(d\xi) \\ & + \int_{S_d^+\setminus\{0\}} (f(x+\xi) - f(x) - \langle \chi(\xi), \nabla f(x)\rangle)\, M(x, d\xi),\end{aligned} \tag{6}$$

for some truncation function χ and admissible parameters

$$\left(\alpha, b, B(x) = \sum_{i,j} \beta^{ij} x_{ij}, c, \gamma, m(d\xi), M(x, d\xi) = \frac{\langle x, \mu\rangle}{\|\xi\|^2 \wedge 1}\right),$$

where $A_{ijkl}(x) = x_{ik}\alpha_{jl} + x_{il}\alpha_{jk} + x_{jk}\alpha_{il} + x_{jl}\alpha_{ik}$.

Relation to semimartingales

Corollary

Let X be a conservative affine process on S_d^+. Then X is a semimartingale. Furthermore, there exists, possibly on an enlargement of the probability space, a $d \times d$-matrix of standard Brownian motions W such that X admits the following representation

$$\begin{aligned} X_t = x &+ \int_0^t \left(b + \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) + B(X_s) \right) ds, \\ &+ \int_0^t \left(\sqrt{X_s} dW_s \Sigma + \Sigma^\top dW_s \sqrt{X_s} \right) \\ &+ \int_0^t \int_{S_d^+ \setminus \{0\}} \chi(\xi) \left(\mu^X(dt, d\xi) - (m(d\xi) + M(X_t, d\xi))\, dt \right) \\ &+ \int_0^t \int_{S_d^+ \setminus \{0\}} (\xi - \chi(\xi)) \mu^X(dt, d\xi), \end{aligned}$$

where Σ is a $d \times d$ matrix satisfying $\Sigma^\top \Sigma = \alpha$ and μ^X denotes the random measure associated with the jumps of X.

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- linear jump coefficient: $d \times d$-matrix $\mu = (\mu_{ij})$ of finite signed measures on $S_d^+ \setminus \{0\}$ with
  - $\mu(E) \in S_d^+$ for all $E \in \mathcal{B}(S_d^+ \setminus \{0\})$
  - $M(x, d\xi) := \frac{\langle x, \mu(d\xi)\rangle}{\|\xi\|^2 \wedge 1}$ satisfies $\int_{S_d^+ \setminus \{0\}} \langle \chi(\xi), u \rangle M(x, d\xi) < \infty$ for all $x, u \in S_d^+$ with $\langle x, u \rangle = 0$,

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- constant drift term: $b \succeq (d-1)\alpha$,
- constant jump term: Borel measure m on $S_d^+ \setminus \{0\}$ with $\int_{S_d^+ \setminus \{0\}} (\|\xi\| \wedge 1) \, m(d\xi) < \infty$,

Admissible parameters

- linear diffusion coefficient: $\alpha \in S_d^+$,
- linear jump coefficient: $d \times d$-matrix $\mu = (\mu_{ij})$ of finite signed measures on $S_d^+ \setminus \{0\}$ with
  - $\mu(E) \in S_d^+$ for all $E \in \mathcal{B}(S_d^+ \setminus \{0\})$
  - $M(x, d\xi) := \frac{\langle x, \mu(d\xi)\rangle}{\|\xi\|^2 \wedge 1}$ satisfies $\int_{S_d^+ \setminus \{0\}} \langle \chi(\xi), u\rangle M(x, d\xi) < \infty$ for all $x, u \in S_d^+$ with $\langle x, u\rangle = 0$,
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- Jumps described by m are of finite variation, for the linear jump part we have finite variation for the directions orthogonal to the boundary while parallel to the boundary general jump behavior is allowed.

Remark on the admissible parameters

Parameters of prototype equation are admissible:

$$dX_t = (b + HX_t + X_t H^\top)dt + \sqrt{X_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{X_t} + dJ_t,$$

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- $c = 0$, $\gamma = 0$,
- J a compound Poisson process with intensity $l + \langle\lambda, x\rangle$ and jump distribution ν supported on S_d^+: $m(d\xi) = l\nu(d\xi)$, $M(x, d\xi) = \langle\lambda, x\rangle\nu(d\xi)$.

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$$B(x) = \begin{pmatrix} x_{22} & x_{12} \\ x_{12} & x_{11} \end{pmatrix}.$$
  - Full generality of jumps (quadratic variation jumps parallel to the boundary).

Ideas and methods applied in the proof

Necessary conditions:

- Since the limit exists

$$\mathcal{A}e^{-\langle u,x\rangle} = e^{-\langle u,x\rangle} \lim_{t\to 0} \frac{1}{t} E\left(e^{-\langle u,X_t - x\rangle} - 1\right),$$

  we know that the limit is of Lévy-Khintchine form on the cone $S_d^+ - \mathbb{R}x$.
- The structure of infinitely divisible random variables on cones implies the form of F and R and it determines the form of the infinitesimal generator since $\mathcal{A}e^{-\langle u,x\rangle} = (-F(u) - \langle R(u), x\rangle)e^{-\langle u,x\rangle}$.
- In order to derive the condition on b we consider $\det(X_t)$. Invariance conditions for $\mathbb{R}_+$-valued process imply then $b \succeq (d-1)\alpha$.

Ideas and methods applied in the proof

Sufficient conditions

- Problems: Unbounded coefficients, $\sqrt{x}$ is not Lipschitz continuous at the boundary, infinitely active jumps.
- Regularization of the coefficients (replace $\sqrt{x}$ by $\sqrt{x+\epsilon I_d}-\sqrt{\epsilon I_d}$).
- Through stochastic invariance, we establish existence of an S_d^+-valued solution of the martingale problem for the generator of the regularized process $\Rightarrow$ Existence of an S_d^+-valued solution of the martingale problem for $\mathcal{A}$.
- Uniqueness and existence of global solutions ϕ and ψ of the generalized Riccati equations (4) and (5) yield uniqueness of the solution of the martingale problem $\Rightarrow$ Existence of an affine (Markov) process on S_d^+.

Infinite divisibility

- On S_d^+, the laws of

$$dX_t = \delta I_d dt + \sqrt{X_t} dW_t + dW_t^\top \sqrt{X_t}$$

are those of non central Wishart distributions and are not infinitely divisible.

As a consequence of the condition on the constant drift, we obtain

Corollary

X is affine and its one-dimensional marginal distributions are infinitely divisible if and only if $\alpha = 0$ or $d = 1$.

Conclusion and Outlook

- Conclusion
  - Characterization of S_d^+-valued affine processes.
  - Exhaustive model specification.
  - Characterization of infinitely divisible affine processes.
- Outlook
  - Calibration and parameter estimation of multivariate stochastic covariance models.
  - Affine processes on symmetric cones.
  - Calculation of stochastic quantities beyond marginals in the affine setting.
  - Infinite dimensional affine processes.

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