Stochastic solutions and Feynman-Kac formulae for generalized time-fractional evolution equations

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Keywords: time-fractional evolution equations, fractional calculus, randomly slowed down Markov processes, randomely scaled linear fractional Lévy motion, generalized grey Brownian motion, inverse subordinators, Saigo-Maeda generalized fractional operators, Appell functions, three parameter Mittag-Leffler function, Feynman-Kac formulae, anomalous diffusion

MSC2020 codes: 35R11, 35S10, 45D05, 45K05, 60G18, 60G22,60G51, 60G52, 60H30

Abstract. This is a joint work with Christian Bender, Saarland University. We consider a general class of integro-differential evolution equations which includes the governing equation of the generalized grey Brownian motion and the time- and space-fractional heat equation:

$$u(t,x) = u_0(x) + \int_0^t k(t,s)Lu(s,x)ds, \qquad t>0, \quad x\in\mathbb{R}^d, \tag{1}$$

where $k(t,s)$, $0<s<t<\infty$, is a general memory kernel and L is a generator of a strongly continuous semigroup on some Banach space X corresponding to some Markov process $(\xi_t)_{t\geq0}$. Such equations arise, in particular, in models of anomalous diffusion.

We show that

$$u(t,x) := \mathbb{E}^x\left[u_0\left(\xi_{A(t)}\right)\right]$$

solves equation (1) with any process $(A(t))_{t\geq0}$ having prescribed one-dimensional marginal distributions. More precisely, we derive a series representation in terms of the time kernel k for the Laplace transform of random variables $A(t)$, $t\geq0$. In the special case when k is of convolution type and is related to a Lévy subordinator, one may choose $(A(t))_{t\geq0}$ to be a corresponding inverse subordinator. In the special case of homogeneous kernels (this case includes the kernel $k(t,s)=(t-s)^{\beta-1}/\Gamma(\beta)$ of time-fractional evolution equations and, more generally, kernels corresponding to Saigo-Maeda fractional diffintegration operators), one may separate randomness and time dependence in $(A(t))_{t\geq0}$. This leads to stochastic representations of solutions of equation (1) in terms of randomely slowed down Markov processes and, in the case when $(\xi_t)_{t\geq0}$ is stable, in terms of randomely scaled linear fractional stable processes (the latter are self-similar processes with stationary increments).

Our general construction allows to obtain new connections $k(t,s) \leftrightsquigarrow A(t)$ and hence to obtain new Feynman-Kac formulae for evolution equations of type (1). In particular, the connection between Saigo-Maeda fractional diffintegration operators and positive random variables with Laplace transform given by Prabhakar's three parameter generalization of the Mittag-Leffler function is established. These results yield a stochastic representation for solution of equation (1) with a Saigo-Maeda kernel in terms of a randomly slowed down Markov process $(\xi_{At^\beta})_{t\geq0}$, where A is an independent random variable with Laplace transform given by the three-parameter Mittag-Leffler function, and β corresponds to the degree of homogeneity of the kernel.

Similar results hold also in the case of equation (1) with more general operator L (it is enough to assume that L generates a strongly continuous semigroup on some Banach space) and lead to Feynman-Kac formulae for such equations (e.g., if $L := L_0 + V$ where L_0 generates a Markov process and V is a suitable potential).

References:

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