

# Random Partitions of Samples\*

Klaus Th. Hess

Institut für Mathematische Stochastik

Technische Universität Dresden †

## Abstract

In the present paper we construct a decomposition of a sample into a finite number of subsamples in the case where the sample size is random and the decomposition depends on the values of the sampling variables. We investigate the basic properties of the subsamples and compute the first and second order moments of their sample sums, sample means, and sample variances.

## 1 Introduction

In the present paper we construct a decomposition of a sample into a finite number of subsamples in the case where the sample size is random and the decomposition depends on the values of the sampling variables. We investigate the basic properties of the subsamples and compute the first and second order moments of their sample sums, sample means, and sample variances.

Throughout the paper let

- $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space,
- $(M, \mathcal{M})$  be a measurable space where  $M$  is a linear space,
- $H$  be a finite set of indices,
- $\{M_h\}_{h \in H}$  be a finite partition of  $M$ ,
- $N : \Omega \rightarrow \mathbf{N}_0$  be a random variable with  $\mathbf{P}[N \in \mathbf{N}] > 0$ , and
- $\{Y_i\}_{i \in \mathbf{N}}$  be a sequence of random variables  $\Omega \rightarrow M$

such that

- the sequence  $\{Y_i\}_{i \in \mathbf{N}}$  is i.i.d.,
- the pair  $\{N, \{Y_i\}_{i \in \mathbf{N}}\}$  is independent, and
- $\eta_h := \mathbf{P}[Y \in M_h] > 0$  holds for all  $h \in H$  (where  $Y$  denotes a random variable having the same distribution as each  $Y_i$ ).

---

\*This is a corrected reprint of the version from February 24, 2000. The author would like to thank Markus Hübel (München) for some helpful comments.

†Present Address: Universität Rostock, Institut für Mathematik, D-18055 Rostock, Germany

Interpretation: The family  $Y_1, \dots, Y_N$  is a sample with random sample size  $N$ . We want to construct a decomposition into some random subsamples  $Y_1^{(h)}, \dots, Y_{N^{(h)}}^{(h)}$  with random sample sizes  $N^{(h)}$ , such that  $Y_i^{(h)} \in M_h$  for each  $h \in H$ . The aim of this paper is to prove some properties of these random subsamples and to calculate the first two moments of the sample sum, the sample mean, and the sample variance.

The present paper partly generalizes the results of Franke and Macht (1995) and of Hess, Macht, and Schmidt (1995), which are included in Schmidt (1996), in the sense that we consider a more general structure of the sample and a decomposition into more than two subsamples. Properties of thinned samples were also studied by Stenger (1986). He described the procedure of thinning and called it Poisson-sampling.

The paper is organized as follows: In Section 2 we give the results on decomposed samples, and in Section 3 we establish the first and second order moments of the sample sum, sample mean, and sample variance of the subsamples by reducing our situation to the classical case (see e. g. Cramér (1946)). In Section 4 we present some applications in insurance. Section 5 includes the proof of Theorem 2.1, which is the main result of this paper.

## 2 Decomposition of samples

For the moment fix  $h \in H$ . First let

$$N^{(h)} := \sum_{i=1}^N \chi_{\{Y_i \in M_h\}}$$

Then  $N^{(h)}$  is the random sample size of group  $h$ .

Second we define recursively a sequence of stopping times

$$\begin{aligned} \nu_0^{(h)} &:= 0 \\ \nu_i^{(h)} &:= \inf\{j \in \mathbf{N} \mid \nu_{i-1}^{(h)} < j, Y_j \in M_h\} \end{aligned}$$

and a sequence of random variables

$$Y_i^{(h)} := \sum_{j=1}^{\infty} \chi_{\{\nu_i^{(h)}=j\}} Y_j$$

for all  $i \in \mathbf{N}$ . Then  $\{Y_i^{(h)}\}_{i \in \mathbf{N}}$  is the sequence of sampling variables of group  $h$  and  $Y_i^{(h)} \in M_h$ .

The pair  $\left\{ \left\{ Y_i^{(h)} \right\}_{i \in \mathbf{N}}, N^{(h)} \right\}$  is called a random subsample of group  $h$ .

The next theorem shows the properties of random subsamples:

**2.1 Theorem.**

- (a) For each  $h \in H$  the sequence  $\{Y_i^{(h)}\}_{i \in \mathbf{N}}$  is i.i.d. with

$$\mathbf{P} [Y^{(h)} \in A] = \mathbf{P}(Y \in A | Y \in M_h)$$

for all  $A \in \mathcal{M}$ .

- (b) The family

$$\left\{ \left\{ Y_i^{(h)} \right\}_{i \in \mathbf{N}} \right\}_{h \in H}$$

is independent.

- (c) The pair

$$\left\{ \left\{ \left\{ Y_i^{(h)} \right\}_{i \in \mathbf{N}} \right\}_{h \in H}, \left\{ N^{(h)} \right\}_{h \in H} \right\}$$

is independent.

- (d) The conditional joint distribution of  $\{N^{(h)}\}_{h \in H}$  given  $N$  is the conditional multinomial distribution with parameters  $N$  and  $\{\eta_h\}_{h \in H}$ .

**Proof.** See Section 5. □

In general the family of random sample sizes of all groups is not independent:

**2.2 Corollary.** *The following are equivalent:*

- (i) The family  $\{N^{(h)}\}_{h \in H}$  is independent.  
(ii)  $N$  has a non-degenerate Poisson distribution.

**Proof.** The assertion follows from Theorem 2.1 (d); see e. g. Hess and Schmidt (1994). □

Furthermore we have the following properties of the random sample sizes:

**2.3 Corollary.**

- (a) For each  $h \in H$  the conditional distribution of  $N^{(h)}$  given  $N$  is the conditional binomial distribution with the parameters  $N$  and  $\eta_h$ .  
(b) If  $\mathbf{E}[N] < \infty$ , then  $\mathbf{E}[N^{(h)}] = \eta_h \mathbf{E}[N]$  holds for all  $h \in H$ .  
(c) If  $\mathbf{E}[N^2] < \infty$ , then  $\mathbf{cov}[N^{(h)}, N^{(j)}] = \eta_h \eta_j (\mathbf{var}[N] - \mathbf{E}[N]) + \delta_{hj} \eta_h \mathbf{E}[N]$  holds for all  $h, j \in H$ .  
(d) If  $N$  has the binomial distribution with parameters  $n \in \mathbf{N}$  and  $\vartheta \in (0, 1)$ , then  $\{N^{(h)}\}_{h \in H}$  has the multinomial distribution with parameters  $n$  and  $\{\vartheta \eta_h\}_{h \in H}$  and each  $N^{(h)}$  has the binomial distribution with parameters  $n$  and  $\vartheta \eta_h$ .  
(e) If  $N$  has the Poisson distribution with parameter  $\lambda \in (0, \infty)$ , then  $\{N^{(h)}\}_{h \in H}$  is independent and each  $N^{(h)}$  has the Poisson distribution with parameter  $\lambda \eta_h$ .  
(f) If  $N$  has the negativebinomial distribution with parameters  $\rho \in (0, \infty)$  and  $\vartheta \in (0, 1)$ , then  $\{N^{(h)}\}_{h \in H}$  has the negativemultinomial distribution with parameters  $\rho$  and  $\{\vartheta \eta_h\}_{h \in H}$  and each  $N^{(h)}$  has the negativebinomial distribution with parameters  $\rho$  and  $(\vartheta \eta_h)/(1 - \vartheta + \vartheta \eta_h)$ .

**Proof.** Straightforward. Partly see Hess and Schmidt (1994) and Schmidt and Wünsche (1998). □

### 3 Moments of sample moments

Let  $\{g_h\}_{h \in H}$  be a family of measurable functions  $g_h : M \rightarrow \mathbf{R}$ , and define sequences of real random variables by

$$X_i^{(h)} := g_h \circ Y_i^{(h)}$$

for all  $i \in \mathbf{N}$  and  $h \in H$ . Then Theorem 2.1 remains valid with  $X_i^{(h)}$  instead of  $Y_i^{(h)}$  and

$$\mathbf{P} \left[ X^{(h)} \in A \right] = \mathbf{P} \left( Y \in g_h^{-1}(A) \mid Y \in M_h \right)$$

for all  $A \in \mathcal{B}(\mathbf{R})$ .

Define

$$\mathcal{N} := \sigma(\{N^{(h)}\}_{h \in H})$$

Because of Theorem 2.1 (c), the assertions (a) and (b) of Theorem 2.1 also hold conditionally with respect to  $\mathcal{N}$ .

Now we consider some sample functions and using Theorem 2.1 we can calculate the first two moments of these sample functions.

First denote by

$$S^{(h)} := \sum_{i=1}^{N^{(h)}} X_i^{(h)}$$

the sample sum of group  $h$ .

**3.1 Theorem.** *Assume that  $\mathbf{E}[N^2] < \infty$  and  $\mathbf{E}[X^2] < \infty$ . Then the equations*

$$\begin{aligned} \mathbf{E}[S^{(h)}] &= \mathbf{E}[N^{(h)}] \mathbf{E}[X^{(h)}] \\ \mathbf{cov}[S^{(h)}, S^{(j)}] &= \mathbf{cov}[N^{(h)}, N^{(j)}] \mathbf{E}[X^{(h)}] \mathbf{E}[X^{(j)}] + \delta_{hj} \mathbf{E}[N^{(h)}] \mathbf{var}[X^{(h)}] \end{aligned}$$

hold for all  $h, j \in H$ .

**Proof.** For  $h = j \in H$  the assertions follow from Wald's equalities (see e. g. Schmidt (1996)); for  $h, j \in H$  with  $h \neq j$  we get

$$\begin{aligned} \mathbf{cov}[S^{(h)}, S^{(j)}] &= \mathbf{cov}[\mathbf{E}(S^{(h)} | \mathcal{N}), \mathbf{E}(S^{(j)} | \mathcal{N})] + \mathbf{E}[\mathbf{cov}(S^{(h)}, S^{(j)} | \mathcal{N})] \\ &= \mathbf{cov}[N^{(h)} \mathbf{E}(X^{(h)} | \mathcal{N}), N^{(j)} \mathbf{E}(X^{(j)} | \mathcal{N})] \\ &= \mathbf{cov}[N^{(h)} \mathbf{E}[X^{(h)}], N^{(j)} \mathbf{E}[X^{(j)}]] \\ &= \mathbf{cov}[N^{(h)}, N^{(j)}] \mathbf{E}[X^{(h)}] \mathbf{E}[X^{(j)}] \end{aligned}$$

and the assertion follows. □

Next denote by

$$\bar{S}^{(h)} := \begin{cases} \frac{1}{N^{(h)}} \sum_{i=1}^{N^{(h)}} X_i^{(h)} & \text{if } N^{(h)} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

the sample mean of group  $h$ . The concept of the sample mean is useful only if we have at least one observation in group  $h$ , which means that  $N^{(h)} > 0$ . Therefore we will consider only the conditional moments for the sample mean under the condition that we have at least one observation. For each non-empty  $J \subseteq H$  define

$$A_J := \bigcap_{h \in J} \{N^{(h)} > 0\} \cap \bigcap_{h \in H \setminus J} \{N^{(h)} = 0\}$$

Then we have  $\sigma(A_J) \subseteq \mathcal{N}$  and, by Theorem 2.1 (d),  $\mathbf{P}[A_J] > 0$ . Now we can calculate the first two conditional moments of the sample mean:

**3.2 Theorem.** *Let  $J \subseteq H$  with  $J \neq \emptyset$  and assume that  $\mathbf{E}[N^2] < \infty$  and  $\mathbf{E}[X^2] < \infty$ . Then the equations*

$$\begin{aligned} \mathbf{E}\left(\bar{S}^{(h)} \middle| A_J\right) &= \mathbf{E}\left[X^{(h)}\right] \\ \mathbf{cov}\left(\bar{S}^{(h)}, \bar{S}^{(j)} \middle| A_J\right) &= \delta_{hj} \mathbf{E}\left(\frac{1}{N^{(h)}} \middle| A_J\right) \mathbf{var}\left[X^{(h)}\right] \end{aligned}$$

hold for all  $h, j \in J$ .

**Proof.** We get for all  $h, j \in J$

$$\begin{aligned} \mathbf{E}\left(\bar{S}^{(h)} \middle| A_J\right) &= \mathbf{E}\left(\mathbf{E}\left(\bar{S}^{(h)} \middle| \mathcal{N}\right) \middle| A_J\right) \\ &= \mathbf{E}\left(\mathbf{E}\left(X^{(h)} \middle| \mathcal{N}\right) \middle| A_J\right) \\ &= \mathbf{E}\left[X^{(h)}\right] \end{aligned}$$

and

$$\begin{aligned} \mathbf{cov}\left(\bar{S}^{(h)}, \bar{S}^{(j)} \middle| A_J\right) &= \mathbf{cov}\left(\mathbf{E}\left(\bar{S}^{(h)} \middle| \mathcal{N}\right), \mathbf{E}\left(\bar{S}^{(j)} \middle| \mathcal{N}\right) \middle| A_J\right) + \mathbf{E}\left(\mathbf{cov}\left(\bar{S}^{(h)}, \bar{S}^{(j)} \middle| \mathcal{N}\right) \middle| A_J\right) \\ &= \mathbf{cov}\left(\mathbf{E}\left(X^{(h)} \middle| \mathcal{N}\right), \mathbf{E}\left(X^{(j)} \middle| \mathcal{N}\right) \middle| A_J\right) + \delta_{hj} \mathbf{E}\left(\frac{1}{N^{(h)}} \mathbf{var}\left(X^{(h)} \middle| \mathcal{N}\right) \middle| A_J\right) \\ &= \delta_{hj} \mathbf{E}\left(\frac{1}{N^{(h)}} \middle| A_J\right) \mathbf{var}\left[X^{(h)}\right] \end{aligned}$$

which proves the assertion. □

We conclude this section with analogous results for the sample variance, which is defined for each group  $h$  by

$$V^{(h)} := \begin{cases} \frac{1}{N^{(h)} - 1} \sum_{i=1}^{N^{(h)}} \left(X_i^{(h)} - \bar{S}^{(h)}\right)^2 & \text{if } N^{(h)} \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

For the sample variance we need at least two observations. Therefore we define for all non-empty  $J \subseteq H$

$$B_J := \bigcap_{h \in J} \{N^{(h)} > 1\} \cap \bigcap_{h \in H \setminus J} \{N^{(h)} \leq 1\}$$

Then we have again  $\sigma(B_J) \subseteq \mathcal{N}$  and, by Theorem 2.1 (d),  $\mathbf{P}[B_J] > 0$ , if we assume in addition, that  $\mathbf{P}[N \geq 2] > 0$  holds. Now we can calculate the first two conditional moments of the sample variance:

**3.3 Theorem.** *Let  $J \subseteq H$  with  $J \neq \emptyset$  and assume that  $\mathbf{P}[N \geq 2] > 0$ ,  $\mathbf{E}[N^2] < \infty$ , and  $\mathbf{E}[X^4] < \infty$ . Then the equations*

$$\begin{aligned} \mathbf{E}(V^{(h)} | B_J) &= \mathbf{var}[X^{(h)}] \\ \mathbf{cov}(V^{(h)}, V^{(j)} | B_J) &= \delta_{hj} \left( \mathbf{E} \left( \frac{3 - N^{(h)}}{N^{(h)}(N^{(h)} - 1)} \middle| B_J \right) (\mathbf{var}[X^{(h)}])^2 \right. \\ &\quad \left. + \mathbf{E} \left( \frac{1}{N^{(h)}} \middle| B_J \right) \mathbf{E} \left[ (X^{(h)} - \mathbf{E}[X^{(h)}])^4 \right] \right) \end{aligned}$$

hold for all  $h, j \in J$ .

**Proof.** We get for all  $h, j \in J$

$$\begin{aligned} \mathbf{E}(V^{(h)} | B_J) &= \mathbf{E}(\mathbf{E}(V^{(h)} | \mathcal{N}) | B_J) \\ &= \mathbf{E}(\mathbf{var}(X^{(h)} | \mathcal{N}) | B_J) \\ &= \mathbf{var}[X^{(h)}] \end{aligned}$$

and

$$\begin{aligned} \mathbf{cov}(V^{(h)}, V^{(j)} | B_J) &= \mathbf{cov}(\mathbf{E}(V^{(h)} | \mathcal{N}), \mathbf{E}(V^{(j)} | \mathcal{N}) | B_J) + \mathbf{E}(\mathbf{cov}(V^{(h)}, V^{(j)} | \mathcal{N}) | B_J) \\ &= \mathbf{cov}(\mathbf{var}(X^{(h)} | \mathcal{N}), \mathbf{var}(X^{(j)} | \mathcal{N}) | B_J) \\ &\quad + \delta_{hj} \left( \mathbf{E} \left( \frac{3 - N^{(h)}}{N^{(h)}(N^{(h)} - 1)} (\mathbf{var}(X^{(h)} | \mathcal{N}))^2 \middle| B_J \right) \right. \\ &\quad \left. + \mathbf{E} \left( \frac{1}{N^{(h)}} \mathbf{E} \left( (X^{(h)} - \mathbf{E}(X^{(h)} | \mathcal{N}))^4 \middle| \mathcal{N} \right) \middle| B_J \right) \right) \\ &= \delta_{hj} \left( \mathbf{E} \left( \frac{3 - N^{(h)}}{N^{(h)}(N^{(h)} - 1)} \middle| B_J \right) (\mathbf{var}[X^{(h)}])^2 \right. \\ &\quad \left. + \mathbf{E} \left( \frac{1}{N^{(h)}} \middle| B_J \right) \mathbf{E} \left[ (X^{(h)} - \mathbf{E}[X^{(h)}])^4 \right] \right) \end{aligned}$$

as was to be shown. □

## 4 Examples

The following example on reinsurance was considered by Hess, Macht, and Schmidt (1995). The sampling problem in health insurance occurs in Siegel (1995).

**Excess-of-Loss Reinsurance:** Let  $(M, \mathcal{M}) := (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ . We consider the collective model of risk theory given by the random variable  $N$  which represents the number of claims (occurring in one year) and the sequence  $\{Y_i\}_{i \in \mathbf{N}}$  where  $Y_i$  represents the claim amount of claim  $i \in \mathbf{N}$ . It is assumed that the sequence  $\{Y_i\}_{i \in \mathbf{N}}$  is i.i.d. and independent of  $N$ . In excess of loss reinsurance, the reinsurer covers for each individual claim that part of the claim amount which exceeds a given priority  $c > 0$ . The aggregate claim amount of the reinsurer is then given by

$$S := \sum_{i=1}^N (Y_i - c)^+$$

In general, the probability  $\mathbf{P}[(Y - c)^+ = 0]$  is large. It is therefore convenient to consider the thinned sequence of all claims exceeding the priority  $c$ . We define  $H := \{0, 1\}$  and  $M_1 := (c, \infty)$ . Then

$$N^{(1)} = \sum_{i=1}^N \chi_{\{Y_i > c\}}$$

is the number of all claims exceeding the priority  $c$ . The aggregate claim amount of the reinsurer is the sample sum

$$S = \sum_{i=1}^{N^{(1)}} (Y_i^{(1)} - c)$$

where  $\{Y_i^{(1)}\}_{i \in \mathbf{N}}$  is i.i.d. with  $\mathbf{P}[Y^{(1)} > c] = 1$  and independent of  $N^{(1)}$  by Theorem 2.1.

**Health Insurance:** We consider a portfolio of  $n$  risks, that means  $\mathbf{P}[N = n] = 1$ . The annual cost per head depends on the age of the insured and on the observation year. Let  $\mathcal{X} \subseteq \mathbf{N}_0^2$  be the finite set consisting of the possible pairs of ages and observation years and define  $(M, \mathcal{M}) = (\mathcal{X} \times \mathbf{R}, 2^{\mathcal{X}} \otimes \mathcal{B}(\mathbf{R}))$ . Then the sequence of random variables  $\{Y_i\}_{i \in \mathbf{N}}$  is assumed to be i.i.d. with  $Y_i = (X_i, T_i, K_i)$  where  $X_i$  is the age of the insured,  $T_i$  is the observation period, and  $K_i$  is the annual cost. In order to estimate the average cost per head for each age and each observation year, the sample  $\{Y_i\}_{i \in \{1, \dots, n\}}$  has to be decomposed according to the values of the sample  $\{(X_i, T_i)\}_{i \in \{1, \dots, n\}}$ . Therefore we define  $H := \mathcal{X}$  and  $M_h := \{h\} \times \mathbf{R}$ . Then for each  $(x, t) \in \mathcal{X}$

$$N^{(x,t)} = \sum_{i=1}^n \chi_{\{(X_i, T_i) = (x,t)\}}$$

is the random number of insured of age  $x$  and observed in period  $t$ . The sampling variables of group  $(x, t)$  are  $Y_i^{(x,t)} = (X_i^{(x,t)}, T_i^{(x,t)}, K_i^{(x,t)}) = (x, t, K_i^{(x,t)})$ . We are

interested in an estimator of the average cost per head  $\mathbf{E} \left[ K_i^{(x,t)} \right]$  in each group. By Theorem 3.2 the sample mean of the thinned sequence

$$\widehat{K}^{x,t} := \frac{1}{N(x,t)} \sum_{i=1}^{N(x,t)} K_i^{(x,t)}$$

is an unbiased estimator of  $\mathbf{E} \left[ K_i^{(x,t)} \right]$ , if we have at least one observation in group  $(x,t) \in \mathcal{X}$ . For further calculations like regression the theorem gives the variance-covariance structure of the estimators.

## 5 Proof of Theorem 2.1

We first prove that the sequence of the sampling variables of all groups is i.i.d. (assertion (a)).

**5.1 Theorem.** *For each  $h \in H$  the sequence  $\{Y_i^{(h)}\}_{i \in \mathbf{N}}$  is i.i.d. with*

$$\mathbf{P} \left[ Y^{(h)} \in A \right] = \mathbf{P}(Y \in A | Y \in M_h)$$

for all  $A \in \mathcal{M}$ .

**Proof.** Let  $h \in H$ . For all  $k \in \mathbf{N}$ , let  $\mathcal{E}(k)$  denote the collection of all strictly increasing sequences  $\{m_i\}_{i \in \{1, \dots, k\}} \subseteq \mathbf{N}$ . For  $E = \{m_i\}_{i \in \{1, \dots, k\}} \in \mathcal{E}(k)$ , define  $J(E) := \{1, \dots, m_k\} \setminus E$ . Then the identities

$$\bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} = \bigcap_{m \in E} \{Y_m \in M_h\} \cap \bigcap_{m \in J(E)} \{Y_m \notin M_h\}$$

and hence

$$\mathbf{P} \left[ \bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} \right] = \eta_h^k (1 - \eta_h)^{m_k - k}$$

hold for all  $k \in \mathbf{N}$  and for all  $E = \{m_i\}_{i \in \{1, \dots, k\}} \in \mathcal{E}(k)$ .

For two distinct sequences  $\{m_i\}_{i \in \{1, \dots, k\}} \in \mathcal{E}(k)$  and  $\{m'_i\}_{i \in \{1, \dots, k\}} \in \mathcal{E}(k)$  we have

$$\bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} \cap \bigcap_{i=1}^k \{\nu_i^{(h)} = m'_i\} = \emptyset$$

Furthermore we get

$$\begin{aligned} \sum_{E \in \mathcal{E}(k)} \mathbf{P} \left[ \bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} \right] &= \sum_{E \in \mathcal{E}(k)} \eta_h^k (1 - \eta_h)^{m_k - k} \\ &= \sum_{m_k=k}^{\infty} \binom{m_k - 1}{m_k - k} \eta_h^k (1 - \eta_h)^{m_k - k} \\ &= \sum_{l=0}^{\infty} \binom{k + l - 1}{l} \eta_h^k (1 - \eta_h)^l \\ &= 1 \end{aligned}$$

For all  $A_1, \dots, A_k \in \mathcal{M}$  and  $E = \{m_i\}_{i \in \{1, \dots, k\}} \in \mathcal{E}(k)$  we have

$$\begin{aligned}
& \mathbf{P} \left[ \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i\} \cap \bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} \right] \\
&= \mathbf{P} \left[ \bigcap_{i=1}^k \{Y_{m_i} \in A_i\} \cap \bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} \right] \\
&= \mathbf{P} \left[ \bigcap_{i=1}^k \{Y_{m_i} \in A_i\} \cap \bigcap_{i=1}^k \{Y_{m_i} \in M_h\} \cap \bigcap_{l \in J(E)} \{Y_l \notin M_h\} \right] \\
&= \mathbf{P} \left[ \bigcap_{i=1}^k (\{Y_{m_i} \in A_i\} \cap \{Y_{m_i} \in M_h\}) \cap \bigcap_{l \in J(E)} \{Y_l \notin M_h\} \right] \\
&= \prod_{i=1}^k \mathbf{P} [\{Y_{m_i} \in A_i\} \cap \{Y_{m_i} \in M_h\}] \cdot \prod_{l \in J(E)} \mathbf{P} [Y_l \notin M_h] \\
&= \prod_{i=1}^k \mathbf{P}(Y_{m_i} \in A_i | Y_{m_i} \in M_h) \cdot \prod_{i=1}^k \mathbf{P}[Y_{m_i} \in M_h] \cdot \prod_{l \in J(E)} \mathbf{P}[Y_l \notin M_h] \\
&= \prod_{i=1}^k \mathbf{P}(Y_{m_i} \in A_i | Y_{m_i} \in M_h) \cdot \mathbf{P} \left[ \bigcap_{i=1}^k \{Y_{m_i} \in M_h\} \cap \bigcap_{l \in J(E)} \{Y_l \notin M_h\} \right] \\
&= \prod_{i=1}^k \mathbf{P}(Y \in A_i | Y \in M_h) \cdot \mathbf{P} \left[ \bigcap_{i=1}^k \{\nu_i^{(h)} = m_i\} \right]
\end{aligned}$$

Summation over all sequences in  $\mathcal{E}(k)$  yields

$$\mathbf{P} \left[ \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i\} \right] = \prod_{i=1}^k \mathbf{P}(Y \in A_i | Y \in M_h)$$

Using this identity we get

$$\mathbf{P} [Y_i^{(h)} \in A_i] = \mathbf{P}(Y \in A_i | Y \in M_h)$$

for all  $i \in \{1, \dots, k\}$  and therefore

$$\mathbf{P} \left[ \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i\} \right] = \prod_{i=1}^k \mathbf{P} [Y_i^{(h)} \in A_i]$$

which completes the proof.  $\square$

For the proof of the assertion (b) we need a family of sequences which generalizes the set  $\mathcal{E}(k)$  from the last proof.

For  $s \in \mathbf{N}$  and  $k_1, \dots, k_s \in \mathbf{N}_0$ , such that  $\max\{k_1, \dots, k_s\} \geq 1$ , and for each  $r \in \{1, \dots, s\}$  denote by  $\mathcal{D}^{(r)}(k_1, \dots, k_s)$  the collection of all  $s$ -tuples of strictly increasing sequences  $\{m_i^{(j)}\}_{i \in \{1, \dots, l_j\}} \subseteq \mathbf{N}$  satisfying  $l_r = k_r$  and  $l_j \geq k_j$  as well as  $m_{l_j} < m_{l_r} = l_1 + \dots + l_s$  for all  $j \in \{1, \dots, s\} \setminus \{r\}$ , such that some of these sequences

may be empty and the (disjoint) union of these sequences is  $\{1, \dots, l_1 + \dots + l_s\}$ . Further we define

$$\mathcal{D}(k_1, \dots, k_s) := \sum_{r=1}^s \mathcal{D}^{(r)}(k_1, \dots, k_s)$$

Note that  $\mathcal{D}^{(r)}(k_1, \dots, k_s) = \emptyset$  if  $k_r = 0$ . Furthermore there exists a bijection between  $\mathcal{D}(k, 0)$  and  $\mathcal{E}(k)$  as used in the proof of Theorem 5.1. For the proof of assertion (b) we need the following lemma.

**5.2 Lemma.** *For all  $s \in \mathbb{N}$  and  $k_1, \dots, k_s \in \mathbb{N}_0$  with  $\max\{k_1, \dots, k_s\} \geq 1$  the equation*

$$\sum_{\mathcal{D}(k_1, \dots, k_s)} \prod_{i=1}^s \vartheta_i^{l_i} = 1$$

holds for each  $\vartheta_1, \dots, \vartheta_s \in (0, 1]$  with  $\vartheta_1 + \dots + \vartheta_s = 1$ .

**Proof.** We will prove this lemma by induction over both  $s$  and  $k_1 + \dots + k_s$ . If  $s = 1$  and  $k_1 = 1$  the assertion follows immediately.

Now we consider the case, that at least one of the  $k_j$ 's is equal to zero. Without loss of generality we assume that  $k_s = 0$ . In this case we have  $s > 1$  and hence  $\vartheta_i < 1$  for all  $i \in \{1, \dots, s\}$ . Thus we obtain

$$\begin{aligned} & \sum_{\mathcal{D}(k_1, \dots, k_s)} \prod_{i=1}^s \vartheta_i^{l_i} \\ &= \sum_{j=1}^s \sum_{\substack{(r_1, \dots, r_{s-1}, r_s) \geq (k_1, \dots, k_{s-1}, 0) \\ r_j = k_j}} \#\{D \in \mathcal{D}^{(j)}(k_1, \dots, k_s) \mid \forall i \in \{1, \dots, s\} : l_i = r_i\} \cdot \prod_{i=1}^s \vartheta_i^{l_i} \\ &= \sum_{j=1}^{s-1} \sum_{\substack{(r_1, \dots, r_{s-1}, r_s) \geq (k_1, \dots, k_{s-1}, 0) \\ r_j = k_j}} \#\{D \in \mathcal{D}^{(j)}(k_1, \dots, k_s) \mid \forall i \in \{1, \dots, s\} : l_i = r_i\} \cdot \prod_{i=1}^s \vartheta_i^{l_i} \\ &= \sum_{j=1}^{s-1} \sum_{\substack{(r_1, \dots, r_{s-1}) \geq (k_1, \dots, k_{s-1}) \\ r_j = k_j}} \sum_{r_s=0}^{\infty} \binom{r_1 + \dots + r_{s-1}}{r_1 \dots r_{j-1} \ r_j - 1 \ r_{j+1} \dots r_s} \cdot \prod_{i=1}^s \vartheta_i^{r_i} \\ &= \sum_{j=1}^{s-1} \sum_{\substack{(r_1, \dots, r_{s-1}) \geq (k_1, \dots, k_{s-1}) \\ r_j = k_j}} \frac{\Gamma(r_1 + \dots + r_{s-1})}{\Gamma(r_j) \cdot r_1! \dots r_{j-1}! r_{j+1}! \dots r_{s-1}!} \\ & \quad \cdot \sum_{r_s=0}^{\infty} \frac{\Gamma(r_1 + \dots + r_{s-1} + r_s)}{\Gamma(r_1 + \dots + r_{s-1}) \cdot r_s!} \vartheta_s^{r_s} (1 - \vartheta_s)^{r_1 + \dots + r_{s-1}} \cdot \prod_{i=1}^{s-1} \left( \frac{\vartheta_i}{1 - \vartheta_s} \right)^{r_i} \\ &= \sum_{j=1}^{s-1} \sum_{\substack{(r_1, \dots, r_{s-1}) \geq (k_1, \dots, k_{s-1}) \\ r_j = k_j}} \binom{r_1 + \dots + r_{s-1} - 1}{r_1 \dots r_{j-1} \ r_j - 1 \ r_{j+1} \dots r_{s-1}} \\ & \quad \cdot \prod_{i=1}^{s-1} \left( \frac{\vartheta_i}{1 - \vartheta_s} \right)^{r_i} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{s-1} \sum_{\substack{(r_1, \dots, r_{s-1}) \geq (k_1, \dots, k_{s-1}) \\ r_j = k_j}} \#\{D \in \mathcal{D}^{(j)}(k_1, \dots, k_{s-1}) \mid \forall i \in \{1, \dots, s-1\} : l_i = r_i\} \\
&\quad \cdot \prod_{i=1}^{s-1} \left( \frac{\vartheta_i}{1 - \vartheta_s} \right)^{r_i} \\
&= \sum_{\mathcal{D}(k_1, \dots, k_{s-1})} \prod_{i=1}^{s-1} \left( \frac{\vartheta_i}{1 - \vartheta_s} \right)^{l_i}
\end{aligned}$$

If all  $k_j > 0$ , then we split  $\mathcal{D}(k_1, \dots, k_s)$  into  $s$ -parts:

For each  $r \in \{1, \dots, s\}$  denote by  $\mathcal{D}_{(r)}(k_1, \dots, k_s)$  the collection of all  $s$ -tuples  $(\{m_i^{(1)}\}_{i=\{1, \dots, l_1\}}, \dots, \{m_i^{(s)}\}_{i=\{1, \dots, l_s\}}) \in \mathcal{D}(k_1, \dots, k_s)$  satisfying  $m_1^{(r)} = 1$ . Then we have

$$\mathcal{D}(k_1, \dots, k_s) = \sum_{r=1}^s \mathcal{D}_{(r)}(k_1, \dots, k_s)$$

Furthermore, there are obvious bijections between  $\mathcal{D}_{(r)}(k_1, \dots, k_{r-1}, k_r, k_{r+1}, \dots, k_s)$  and  $\mathcal{D}(k_1, \dots, k_{r-1}, k_r - 1, k_{r+1}, \dots, k_s)$ . Therefore the assertion follows by induction.  $\square$

Now we are able to prove that the sequences of sample variable of different groups are independent (assertion (b)). Furthermore, we shall prove that the family of all these sequences is independent of the families of stopping-times and the sample size. We will need this result for the proof of assertion (c).

**5.3 Theorem.** *The family of the thinned sequences*

$$\left\{ \left\{ Y_i^{(h)} \right\}_{i \in \mathbf{N}} \right\}_{h \in H}$$

and the pair

$$\left\{ \left\{ \left\{ Y_i^{(h)} \right\}_{i \in \mathbf{N}} \right\}_{h \in H}, \left\{ \left\{ \left\{ \nu_i^{(h)} \right\}_{i \in \mathbf{N}} \right\}_{h \in H}, N \right\} \right\}$$

are independent.

**Proof.** For all families  $\{k_h\}_{h \in H} \subseteq \mathbf{N}_0$  such that  $\max\{k_h \mid h \in H\} \geq 1$  and for all  $\{\{m_i^{(h)}\}_{i=\{1, \dots, l_h\}}\}_{h \in H} \in \mathcal{D}(\{k_h\}_{h \in H})$  the identities

$$\bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} = \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{Y_{m_i^{(h)}} \in M_h\}$$

and hence

$$\mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \right] = \prod_{h \in H} \eta_h^{l_h}$$

hold. Using Lemma 5.2 we get

$$\sum_{\mathcal{D}(\{k_h\}_{h \in H})} \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \right] = 1$$

Let  $\{\{m_i^{(h)}\}_{i=\{1, \dots, l_h\}}\}_{h \in H} \in \mathcal{D}(\{k_h\}_{h \in H})$  and  $\{\{\tilde{m}_i^{(h)}\}_{i=\{1, \dots, l_h\}}\}_{h \in H} \in \mathcal{D}(\{k_h\}_{h \in H})$  be two distinct families. Then we have

$$\bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = \tilde{m}_i^{(h)}\} = \emptyset$$

By using Theorem 5.1 we get for all  $k \in \mathbf{N}$  and  $A_i^{(h)} \in \mathcal{M}$  (for  $i \in \{1, \dots, k\}$  and  $h \in H$ ), for all families  $\{\{m_i^{(h)}\}_{i=\{1, \dots, l_h\}}\}_{h \in H} \in \mathcal{D}(\{k_h\}_{h \in H})$ , and for all  $n \in \mathbf{N}_0$

$$\begin{aligned} & \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \{Y_j^{(h)} \in A_j^{(h)}\} \cap \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \\ &= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \{Y_{m_i^{(h)}} \in A_j^{(h)}\} \cap \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \\ &= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \{Y_{m_i^{(h)}} \in A_j^{(h)}\} \cap \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{Y_{m_i^{(h)}} \in M_h\} \cap \{N = n\} \right] \\ &= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \left( \{Y_{m_i^{(h)}} \in A_j^{(h)}\} \cap \{Y_{m_i^{(h)}} \in M_h\} \right) \right. \\ &\quad \left. \cap \bigcap_{h \in H} \bigcap_{i=k+1}^{l_h} \{Y_{m_i^{(h)}} \in M_h\} \cap \{N = n\} \right] \\ &= \prod_{h \in H} \prod_{j=1}^k \mathbf{P} \left[ \{Y_{m_i^{(h)}} \in A_j^{(h)}\} \cap \{Y_{m_i^{(h)}} \in M_h\} \right] \\ &\quad \cdot \prod_{h \in H} \prod_{i=k+1}^{l_h} \mathbf{P} \left[ Y_{m_i^{(h)}} \in M_h \right] \cdot \mathbf{P} [N = n] \\ &= \prod_{h \in H} \prod_{j=1}^k \mathbf{P} \left( Y_{m_i^{(h)}} \in A_j^{(h)} \mid Y_{m_i^{(h)}} \in M_h \right) \\ &\quad \cdot \prod_{h \in H} \prod_{i=1}^{l_h} \mathbf{P} \left[ Y_{m_i^{(h)}} \in M_h \right] \cdot \mathbf{P} [N = n] \\ &= \prod_{h \in H} \prod_{j=1}^k \mathbf{P} \left[ Y_j^{(h)} \in A_j^{(h)} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{Y_{m_i^{(h)}} \in M_h\} \cap \{N = n\} \right] \\ &= \prod_{h \in H} \prod_{j=1}^k \mathbf{P} \left[ Y_j^{(h)} \in A_j^{(h)} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \\ &= \prod_{h \in H} \mathbf{P} \left[ \bigcap_{j=1}^k \{Y_j^{(h)} \in A_j^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \end{aligned}$$

Summation over all  $\{\{m_i^{(h)}\}_{i=\{1,\dots,l_h\}}\}_{h \in H} \in \mathcal{D}(\{k\}_{h \in H})$  and all  $n \in \mathbf{N}$  yields

$$\mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \{Y_j^{(h)} \in A_j^{(h)}\} \right] = \prod_{h \in H} \mathbf{P} \left[ \bigcap_{j=1}^k \{Y_j^{(h)} \in A_j^{(h)}\} \right]$$

Hence it is clear, that the sequences of sampling variables of different groups are independent. By using the last equality we also get

$$\begin{aligned} & \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \{Y_j^{(h)} \in A_j^{(h)}\} \cap \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \\ &= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{j=1}^k \{Y_j^{(h)} \in A_j^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^{l_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \end{aligned}$$

It is easily seen that

$$\begin{aligned} & \mathbf{P} \left[ \bigcap_{h \in H_1} \bigcap_{j \in K_h} \{Y_j^{(h)} \in A_j^{(h)}\} \cap \bigcap_{h \in H_2} \bigcap_{i \in L_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \\ &= \mathbf{P} \left[ \bigcap_{h \in H_1} \bigcap_{j \in K_h} \{Y_j^{(h)} \in A_j^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H_2} \bigcap_{i \in L_h} \{\nu_i^{(h)} = m_i^{(h)}\} \cap \{N = n\} \right] \end{aligned}$$

holds for all  $H_1, H_2 \subseteq H$ , finite  $K_h, L_h \subseteq \mathbf{N}$  and  $A_j^{(h)} \in \mathcal{M}$  (for  $h \in H_1$  and  $j \in K_h$ ),  $m_i^{(h)} \in \mathbf{N}_0$  (for  $h \in H_2$  and  $i \in L_h$ ) and  $n \in \mathbf{N}_0$ , which completes the proof.  $\square$

Next we prove that the family of the sequences of sampling variables of each group is independent of the family of random sample sizes (assertion (c)).

**5.4 Theorem.** *The pair*

$$\left\{ \left\{ \{Y_i^{(h)}\}_{i \in \mathbf{N}} \right\}_{h \in H}, \{N^{(h)}\}_{h \in H} \right\}$$

*is independent.*

**Proof.** We have for each  $h \in H$  and  $n_h \in \mathbf{N}_0$  the identity

$$\{N^{(h)} = n_h\} = \{\nu_{n_h}^{(h)} \leq N < \nu_{n_h+1}^{(h)}\}$$

For all  $k \in \mathbf{N}$ ,  $A_i^{(h)} \in \mathcal{M}$  (for  $i \in \{1, \dots, k\}$  and  $h \in H$ ), and  $\{n_h\}_{h \in H} \subseteq \mathbf{N}_0$  define  $n := \sum_{h \in H} n_h$ . By using Theorem 5.3 we get

$$\begin{aligned} & \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \cap \bigcap_{h \in H} \{N^{(h)} = n_h\} \right] \\ &= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \cap \bigcap_{h \in H} \{N^{(h)} = n_h\} \cap \{N = n\} \right] \end{aligned}$$

$$\begin{aligned}
&= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \cap \bigcap_{h \in H} \{\nu_{n_h}^{(h)} \leq N < \nu_{n_h+1}^{(h)}\} \cap \{N = n\} \right] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \cap \bigcap_{h \in H} \{\nu_{n_h}^{(h)} \leq n < \nu_{n_h+1}^{(h)}\} \cap \{N = n\} \right] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \{\nu_{n_h}^{(h)} \leq n < \nu_{n_h+1}^{(h)}\} \cap \{N = n\} \right] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \{\nu_{n_h}^{(h)} \leq N < \nu_{n_h+1}^{(h)}\} \cap \{N = n\} \right] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \{N^{(h)} = n_h\} \cap \{N = n\} \right] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i=1}^k \{Y_i^{(h)} \in A_i^{(h)}\} \right] \cdot \mathbf{P} \left[ \bigcap_{h \in H} \{N^{(h)} = n_h\} \right]
\end{aligned}$$

and the assertion follows.  $\square$

We finish the proof of Theorem 2.1 by showing that the joint distribution of the random sample sizes of all groups is a conditional multinomial distribution given the sample size (assertion (d)).

**5.5 Theorem.** *The conditional joint distribution of  $\{N^{(h)}\}_{h \in H}$  given  $N$  is the conditional multinomial distribution with the parameters  $N$  and  $\{\eta_h\}_{h \in H}$ .*

**Proof.** For all  $\{n_h\}_{h \in H} \subseteq \mathbf{N}_0$  define  $n := \sum_{h \in H} n_h$ . If  $\mathbf{P}[N = n] > 0$ , then we get

$$\begin{aligned}
&\mathbf{P} \left( \bigcap_{h \in H} \{N^{(h)} = n_h\} \middle| N = n \right) \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \{N^{(h)} = n_h\} \cap \{N = n\} \right] / \mathbf{P}[N = n] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \left\{ \sum_{i=1}^N \chi_{\{Y_i \in M_h\}} = n_h \right\} \cap \{N = n\} \right] / \mathbf{P}[N = n] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \left\{ \sum_{i=1}^n \chi_{\{Y_i \in M_h\}} = n_h \right\} \cap \{N = n\} \right] / \mathbf{P}[N = n] \\
&= \mathbf{P} \left[ \bigcap_{h \in H} \left\{ \sum_{i=1}^n \chi_{\{Y_i \in M_h\}} = n_h \right\} \right] \\
&= \sum_{\substack{\{I_h\}_{h \in H} \subseteq \{1, \dots, n\} \\ \sum_{h \in H} I_h = \{1, \dots, n\} \\ \#I_h = n_h, h \in H}} \mathbf{P} \left[ \bigcap_{h \in H} \bigcap_{i \in I_h} \{Y_i \in M_h\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\{I_h\}_{h \in H} \subseteq \{1, \dots, n\} \\ \sum_{h \in H} I_h = \{1, \dots, n\} \\ \#I_h = n_h, h \in H}} \prod_{h \in H} \prod_{i \in I_h} \mathbf{P}[Y_i \in M_h] \\
&= \sum_{\substack{\{I_h\}_{h \in H} \subseteq \{1, \dots, n\} \\ \sum_{h \in H} I_h = \{1, \dots, n\} \\ \#I_h = n_h, h \in H}} \prod_{h \in H} \eta_h^{n_h} \\
&= \frac{n!}{\prod_{h \in H} n_h!} \prod_{h \in H} \eta_h^{n_h}
\end{aligned}$$

The assertion now follows. □

## Acknowledgment

The author wishes to thank Klaus D. Schmidt for various discussions and for his helpful suggestions.

## References

- Cramér, H. (1946):** *Mathematical methods of statistics*. Princeton N. J.: Princeton University Press.
- Franke, T.; Macht, W. (1995):** *Decomposition of Risk Processes*. Dresdner Schriften zur Versicherungsmathematik 2/1995.
- Hess, K. Th.; Schmidt, K. D. (1994):** *A Remark on Modelling IBNR Claim Numbers with Random Delay Pattern*. Dresdner Schriften zur Versicherungsmathematik 4/1994.
- Hess, K. Th.; Macht, W.; Schmidt, K. D. (1995):** *Thinning of Risk Processes*. Dresdner Schriften zur Versicherungsmathematik 1/1995.
- Schmidt, K. D. (1996):** *Lectures on Risk Theory*. Stuttgart: B. G. Teubner.
- Siegel, G. (1995):** *Gewichtete Ausgleichsverfahren für Kopfschadenreihen und Leistungsschätzungen in der PKV*. Blätter DGVM **22**, 419–441.
- Stenger, H. (1986):** *Stichproben*. Heidelberg–Wien: Physica–Verlag.

Klaus Th. Hess  
 Institut für Mathematische Stochastik  
 Technische Universität Dresden  
 D–01062 Dresden  
 Germany

April 2, 2009