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On the Decomposition of Mixed Poisson Processes*

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Abstract

In the present paper we decompose a mixed Poisson process into a finite number of claim number processes, which are constructed according to the values of an i.i.d. sequence independent of the mixed Poisson process. We show that the decomposed claim number process is a multivariate mixed Poisson process. We also show that the coordinates of this multivariate mixed Poisson process are independent if and only if the given claim number process is a (homogeneous) Poisson process.

1 Introduction

Mixed Poisson processes are frequently used as a model in motor car insurance. In the presence of different kinds of accidents, like accidents involving damage of persons or not, it is necessary to replace the univariate claim number process by a multivariate one.

In the present paper we consider a decomposition of a mixed Poisson process with respect to an auxiliary sequence of random variables which is assumed to be i.i.d. and independent of the mixed Poisson process. For example, in motor car insurance, the auxiliary random variables may be interpreted as labels indicating the kind of the accident. Another possible actuarial application is loss reserving, where the auxiliary random variables may be interpreted as labels indicating the delay in reporting a claim.

We shall show that the decomposition of a mixed Poisson process produces a multivariate mixed Poisson process, and we shall also show that the coordinates of the multivariate mixed Poisson process are independent if and only if the original mixed Poisson process is a (homogeneous) Poisson process.

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The paper generalizes results of Schmidt (1996; Section 6.3) on Poisson processes.

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

2 Mixed Poisson Processes

A stochastic process $\{N_t\}_{t \in \mathbf{R}_+}$ is called a (*univariate*) *claim number process* if
- $N_0(\omega) = 0$,
- $N_t(\omega) \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$ for all $t \in (0, \infty)$,
- $N_t(\omega) = \inf_{s \in (t,\infty)} N_s(\omega)$ for all $t \in (0, \infty)$,
- $\sup_{s \in [0,t)} N_s(\omega) \leq N_t(\omega) \leq \sup_{s \in [0,t)} N_s(\omega) + 1$ for all $t \in (0, \infty)$, and
- $\sup_{t \in \mathbf{R}_+} N_t(\omega) = \infty$

holds for all $\omega \in \Omega$.

A claim number process $\{N_t\}_{t \in \mathbf{R}_+}$ is called a *mixed Poisson process* if there exists a random variable Λ with $\mathbf{P}[\Lambda \in (0, \infty)] = 1$ such that

$$\mathbf{P}\left[\bigcap_{j=1}^{m} \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] = \int_{\mathbf{R}} \prod_{j=1}^{m} \mathrm{e}^{-\lambda(t_j - t_{j-1})} \frac{(\lambda(t_j - t_{j-1}))^{n_j}}{n_j!} \, \mathrm{d}\mathbf{P}_\Lambda(\lambda)$$

holds for all $m \in \mathbf{N}$, $t_0, t_1, \ldots, t_m \in \mathbf{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$, and $n_1, \ldots, n_m \in \mathbf{N}_0$.

A mixed Poisson process is called a *Poisson process* if Λ is constant.

Every mixed Poisson process fulfills the multinomial criterion; see Schmidt (1996; Lemma 4.2.2):

2.1 Proposition (Multiomial Criterion). *Let $\{N_t\}_{t \in \mathbf{R}_+}$ be a mixed Poisson process. Then*

$$\mathbf{P}\left[\bigcap_{j=1}^{m} \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] = \frac{n!}{\prod_{j=1}^{m} n_j!} \cdot \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j} \cdot \mathbf{P}[N_{t_m} = n]$$

holds for all $m \in \mathbf{N}$, $t_0, t_1, \ldots, t_m \in \mathbf{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$, $n_j \in \mathbf{N}_0$ with $j \in \{1, \ldots, m\}$, and $n := \sum_{j=1}^{m} n_j$.

3 Multivariate Mixed Poisson Processes

A multivariate stochastic process $\{(N_t^{(1)} \ldots N_t^{(d)})'\}_{t \in \mathbf{R}_+}$ is called a *multivariate claim number process* with dimension $d \in \mathbf{N}$ if
- every coordinate process $\{N_t^{(h)}\}_{t \in \mathbf{R}_+}$ with $h \in \{1, \ldots, d\}$ and
- the sum of the coordinates $\{N_t^{(1)} + \ldots + N_t^{(d)}\}_{t \in \mathbf{R}_+}$

is a claim number process.

A multivariate claim number process $\{(N_t^{(1)} \dots N_t^{(d)})'\}_{t \in \mathbf{R}_+}$ is called a *multivariate mixed Poisson process* if there exists a random vector $(\Lambda_1 \dots \Lambda_d)'$ with $\mathbf{P}\left[(\Lambda_1 \dots \Lambda_d)' \in (0,\infty)^d\right] = 1$ such that

$$\begin{aligned}
&\mathbf{P}\left[\bigcap_{h=1}^{d} \bigcap_{j=1}^{m} \{N_{t_j}^{(h)} - N_{t_{j-1}}^{(h)} = n_j^{(h)}\}\right] \\
&= \int_{\mathbf{R}^d} \prod_{h=1}^{d} \prod_{j=1}^{m} \mathrm{e}^{-\lambda_h (t_j - t_{j-1})} \frac{(\lambda_h (t_j - t_{j-1}))^{n_j^{(h)}}}{n_j^{(h)}!} \, d\mathbf{P}_{(\Lambda_1 \dots \Lambda_d)'}(\lambda_1, \dots, \lambda_d)
\end{aligned}$$

holds for all $m \in \mathbf{N}$, $t_0, t_1, \dots, t_d \in \mathbf{R}_+$ with $0 = t_0 < t_1 < \dots < t_d$ and $n_j^{(h)} \in \mathbf{N}_0$ with $h \in \{1, \dots, d\}$ and $j \in \{1, \dots, m\}$.

A multivariate mixed Poisson process is called a *multivariate Poisson process*, if $(\Lambda_1 \dots \Lambda_d)'$ is constant.

4 The Decomposition Theorem

For the remainder of this paper, let $d \in \mathbf{N}$ with $d \geq 2$, let $\{N_t\}_{t \in \mathbf{R}_+}$ be a claim number process, and let $\{X_i\}_{i \in \mathbf{N}}$ be sequence of random variables such that
- $\{X_i\}_{i \in \mathbf{N}}$ is independent of $\{N_t\}_{t \in \mathbf{R}_+}$,
- $\{X_i\}_{i \in \mathbf{N}}$ is i.i.d., and
- $\mathbf{P}[X_1 \in \{1, \dots, d\}] = 1$ and $\mathbf{P}[X_1 = h] > 0$ holds for all $h \in \{1, \dots, d\}$.

We define $\eta_h := \mathbf{P}[X_1 = h]$.

This model has the following interpretation: Let $\{T_n\}_{n \in \mathbf{N}}$ be the claim arrival sequence related to $\{N_t\}_{t \in \mathbf{R}_+}$, which means that the nth claim occurs at the random time T_n. The random variable X_n indicates to which of the d categories of claims the nth claim belongs. Our aim is to construct claim number processes for the different categories of claims.

For each $h \in \{1, \dots, d\}$ and $t \in \mathbf{R}_+$, we define

$$N_t^{(h)} := \sum_{i=1}^{N_t} \chi_{\{X_i = h\}}$$

The multivariate process $\{(N_t^{(1)} \dots N_t^{(d)})'\}$ is called the *decomposed claim number process.*

4.1 Lemma. *The decomposed claim number process is a multivariate claim number process.*

Proof. For each $h \in \{1, \dots, d\}$, the process $\{N_t^{(h)}\}_{t \in \mathbf{R}_+}$ is a claim number process, since its jump times are a subset of the jump times of the given claim number process $\{N_t\}_{t \in \mathbf{R}_+}$.

Since $\mathbf{P}[X_1 \in \{1, \ldots, d\}] = 1$, we have

$$N_t = N_t^{(1)} + \ldots + N_t^{(d)}$$

for each $t \in \mathbf{R}_+$. Hence, the sum of the coordinates is a claim number process, namely the given claim number process $\{N_t\}_{t \in \mathbf{R}_+}$. □

4.2 Lemma. *For each $n^{(1)}, \ldots, n^{(d)} \in \mathbf{N}$ and $n := \sum_{h=1}^d n^{(h)}$,*

$$\mathbf{P}\left[\bigcap_{h=1}^d \left\{\sum_{i=1}^n \chi_{\{X_i = h\}} = n^{(h)}\right\}\right] = \frac{n!}{\prod_{h=1}^d n^{(h)}!} \prod_{h=1}^d \eta_h^{n^{(h)}}$$

holds.

Proof. Let $n^{(1)}, \ldots, n^{(d)} \in \mathbf{N}$ and $n := \sum_{h=1}^d n^{(h)}$ be fixed. Let $\mathcal{I}$ denote the collection of all partitions $I = \{I_1, \ldots, I_d\}$ of $\{1, \ldots, n\}$ such that I_h has the cardinality $n^{(h)}$ for all $h \in \{1, \ldots, d\}$. There are

$$\frac{n!}{\prod_{h=1}^d n^{(h)}!}$$

such partitions. Since the sequence $\{X_i\}_{i \in \mathbf{N}}$ is i.i.d., we get

$$\begin{aligned}
\mathbf{P}\left[\bigcap_{h=1}^d \left\{\sum_{i=1}^n \chi_{\{X_i = h\}} = n^{(h)}\right\}\right] &= \mathbf{P}\left[\sum_{I \in \mathcal{I}} \bigcap_{h=1}^d \bigcap_{i \in I_h} \{X_i = h\}\right] \\
&= \sum_{I \in \mathcal{I}} \prod_{h=1}^d \prod_{i \in I_h} \mathbf{P}[X_i = h] \\
&= \sum_{I \in \mathcal{I}} \prod_{h=1}^d \eta_h^{n^{(h)}} \\
&= \frac{n!}{\prod_{h=1}^d n^{(h)}!} \prod_{h=1}^d \eta_h^{n^{(h)}}
\end{aligned}$$

as was to be proven. □

4.3 Theorem. *For each $t \in \mathbf{R}_+$,*

$$\mathbf{P}_{\left(N_t^{(1)} \ldots N_t^{(d)}\right)' | N_t} = \mathbf{Mult}(N_t; \eta_1, \ldots, \eta_d)$$

holds.

Proof. Let $t \in R_+$, let $n^{(1)}, \ldots, n^{(d)} \in \mathbf{N}_0$, and define $n := \sum_{h=1}^d n^{(h)}$. Using Lemma 4.2, we obtain

$$
\begin{aligned}
\mathbf{P}\left[\bigcap_{h=1}^{d}\{N_t^{(h)}=n^{(h)}\}\cap\{N_t=n\}\right] &= \mathbf{P}\left[\bigcap_{h=1}^{d}\left\{\sum_{i=1}^{N_t}\chi_{\{X_i=h\}}=n^{(h)}\right\}\cap\{N_t=n\}\right] \\
&= \mathbf{P}\left[\bigcap_{h=1}^{d}\left\{\sum_{i=1}^{n}\chi_{\{X_i=h\}}=n^{(h)}\right\}\cap\{N_t=n\}\right] \\
&= \mathbf{P}\left[\bigcap_{h=1}^{d}\left\{\sum_{i=1}^{n}\chi_{\{X_i=h\}}=n^{(h)}\right\}\right]\cdot\mathbf{P}\left[N_t=n\right] \\
&= \frac{n!}{\prod_{h=1}^{d} n^{(h)}!}\prod_{h=1}^{d}\eta_h^{n^{(h)}}\cdot\mathbf{P}\left[N_t=n\right]
\end{aligned}
$$

The assertion now follows. □

For the discussion whether the decomposed claim number process has independent coordinates, we need the following proposition; for a proof see Hess and Schmidt (2002; Lemma A.1).

4.4 Proposition. *Let $Z_1,\ldots,Z_d,Z$ be random variables $\Omega\to\mathbf{N}_0$ with $\mathbf{E}[Z]>0$, $Z=\sum_{h=1}^{d}Z_h$, and $\mathbf{P}_{(Z_1\ldots Z_d)'|Z}=\mathbf{Mult}(Z;\vartheta_1,\ldots,\vartheta_d)$. Let $\lambda:=\mathbf{E}[Z]$. Then the following are equivalent:*
(i) *The random vector $(Z_1\ldots Z_d)'$ has independent coordinates.*
(ii) $\mathbf{P}_Z=\mathbf{Poi}(\lambda)$.
In this case, $\mathbf{P}_{Z_h}=\mathbf{Poi}(\lambda\vartheta_h)$ holds for all $h\in\{1,\ldots,d\}$.

Applying this proposition to Theorem 4.3, we get:

4.5 Corollary. *For each $t\in\mathbf{R}_+$, the random vector*

$$\left(N_t^{(1)}\ \ldots\ N_t^{(d)}\right)'$$

has independent coordinates if and only if N_t has a Poisson distribution.

The main result of this paper is following decomposition theorem for mixed Poisson processes.

4.6 Theorem. *Let $\{N_t\}_{t\in\mathbf{R}_+}$ be a mixed Poisson process with parameter Λ. Then*

$$\left\{\left(N_t^{(1)}\ \ldots\ N_t^{(d)}\right)'\right\}_{t\in\mathbf{R}}$$

is a multivariate mixed Poisson process with parameter $\Lambda(\eta_1\ \ldots\ \eta_k)'$.
Moreover, the coordinate processes of the decomposed claim number process are independent if and only if $\{N_t\}_{t\in\mathbf{R}_+}$ is a Poisson process.

Proof. Let $m\in\mathbf{N}$, $t_0,t_1,\ldots,t_d\in\mathbf{R}_+$ with $0=t_0<t_1<\ldots<t_d$ and $n_j^{(h)}\in\mathbf{N}_0$ with $h\in\{1,\ldots,d\}$ and $j\in\{1,\ldots,m\}$. We define $n_j:=\sum_{h=1}^{d}n_j^{(h)}$ for

all $j \in \{1, \dots, m\}$, $n^{(h)} := \sum_{j=1}^m n_j^{(h)}$ for all $h \in \{1, \dots, d\}$, and $n := \sum_{h=1}^d n_j$. Then, using the properties of $\{X_i\}_{i \in \mathbf{N}}$, Lemma 4.2, and the Multinomial Criterion, we get

$$
\begin{aligned}
&\mathbf{P}\left[\bigcap_{h=1}^d \bigcap_{j=1}^m \{N_{t_j}^{(h)} - N_{t_{j-1}}^{(h)} = n_j^{(h)}\}\right] \\
&= \mathbf{P}\left[\bigcap_{h=1}^d \bigcap_{j=1}^m \{N_{t_j}^{(h)} - N_{t_{j-1}}^{(h)} = n_j^{(h)}\} \cap \bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] \\
&= \mathbf{P}\left[\bigcap_{h=1}^d \bigcap_{j=1}^m \left\{ \sum_{i=N_{t_{j-1}}+1}^{N_{t_j}^{(h)}} \chi_{\{X_i = h\}} = n_j^{(h)} \right\} \cap \bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] \\
&= \mathbf{P}\left[\bigcap_{h=1}^d \bigcap_{j=1}^m \left\{ \sum_{i=\sum_{k=1}^{j-1} n_k+1}^{\sum_{k=1}^{j} n_k} \chi_{\{X_i = h\}} = n_j^{(h)} \right\} \cap \bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] \\
&= \prod_{j=1}^m \mathbf{P}\left[\bigcap_{h=1}^d \left\{ \sum_{i=\sum_{k=1}^{j-1} n_k+1}^{\sum_{k=1}^{j} n_k} \chi_{\{X_i = h\}} = n_j^{(h)} \right\}\right] \cdot \mathbf{P}\left[\bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] \\
&= \prod_{j=1}^m \mathbf{P}\left[\bigcap_{h=1}^d \left\{ \sum_{i=1}^{n_j} \chi_{\{X_i = h\}} = n_j^{(h)} \right\}\right] \cdot \mathbf{P}\left[\bigcap_{j=1}^m \{N_{t_j} - N_{t_{j-1}} = n_j\}\right] \\
&= \prod_{j=1}^m \left(\frac{n_j!}{\prod_{h=1}^d n_j^{(h)}!} \prod_{h=1}^d \eta_h^{n_j^{(h)}} \right) \cdot \frac{n!}{\prod_{j=1}^m n_j!} \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m} \right)^{n_j} \cdot \mathbf{P}[N_{t_m} = n] \\
&= \prod_{j=1}^m \left(\frac{n_j!}{\prod_{h=1}^d n_j^{(h)}!} \prod_{h=1}^d \eta_h^{n_j^{(h)}} \right) \cdot \frac{n!}{\prod_{j=1}^m n_j!} \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m} \right)^{n_j} \cdot \int_{\mathbf{R}} \mathrm{e}^{-\lambda t_m} \, \frac{(\lambda t_m)^n}{n!} \, d\mathbf{P}_\Lambda(\lambda) \\
&= \int_{\mathbf{R}} \mathrm{e}^{-\lambda t_m} \, \lambda^n \prod_{h=1}^d \prod_{j=1}^m \frac{(\eta_h(t_j - t_{j-1}))^{n_j^{(h)}}}{n_j^{(h)}!} \, d\mathbf{P}_\Lambda(\lambda) \\
&= \int_{\mathbf{R}} \prod_{h=1}^d \prod_{j=1}^m \mathrm{e}^{-\lambda \eta_h (t_j - t_{j-1})} \frac{(\lambda \eta_h(t_j - t_{j-1}))^{n_j^{(h)}}}{n_j^{(h)}!} \, d\mathbf{P}_\Lambda(\lambda) \\
&= \int_{\mathbf{R}^d} \prod_{h=1}^d \prod_{j=1}^m \mathrm{e}^{-\lambda_h (t_j - t_{j-1})} \frac{(\lambda_h(t_j - t_{j-1}))^{n_j^{(h)}}}{n_j^{(h)}!} \, d\mathbf{P}_{(\Lambda\eta_1 \dots \Lambda\eta_d)'}(\lambda_1, \dots, \lambda_d)
\end{aligned}
$$

Here the final identity follows from the transformation theorem; see Billingsley (1995, Theorem 16.13). Therefore $\{(N_t^{(1)} \dots N_t^{(d)})'\}_{t \in \mathbf{R}}$ is a multivariate mixed Poisson process with parameter $\Lambda(\eta_1 \ \dots \ \eta_d)'$.

Using the identity established before and Corollary 4.5, we see that the coordinates of the multivariate mixed Poisson process are independent if and only if Λ is constant. This proves the final assertion. □

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