On the Covariance of Monotone Functions of a Random Variable

Klaus D. Schmidt

Lehrstuhl für Versicherungsmathematik
Technische Universität Dresden

Abstract

It is generally taken for granted that the covariance of two increasing functions of a random variable is positive. The present paper contains an elementary proof of this fact.

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space, let $X : \Omega \to \mathbf{R}$ be a random variable, and let $J \in \mathcal{B}(\mathbf{R})$ be a Borel set satisfying $P[\{X \in J\}] = 1$.

Intuition based on the usual interpretation of the covariance of two random variables suggests that the inequality

$$\mathrm{cov}[f(X), g(X)] \ \geq \ 0$$

should hold for any two increasing functions $f, g : J \to \mathbf{R}$. The inequality holds indeed, but a proof is difficult to find in the literature; an exception is the book by Schürger [1998; Aufgabe 4.22] who suggests a proof based on the independent product of two probability spaces. In the present paper we propose an elementary proof of the inequality.

In our proof, we shall need the following definition and the subsequent lemma:

A set $C \in \mathcal{B}(\mathbf{R})$ is said to be *increasing* if its indicator function χ_C is increasing. Thus, a set $C \in \mathcal{B}(\mathbf{R})$ is increasing if and only if either $C = \mathbf{R}$ or there exists some $c \in \mathbf{R}$ such that $C = [c, \infty)$ or $C = (c, \infty)$.

Lemma. *The inequality*

$$P[\{X \in A \cap B\}] \ \geq \ P[\{X \in A\}] \cdot P[\{X \in B\}]$$

holds for any two increasing sets $A, B \in \mathcal{B}(\mathbf{R})$.

Proof. Since A and B are increasing, we have $A \cap B = A$ or $A \cap B = B$. The assertion follows. □

We can now prove the main result of this paper:

Theorem. *The inequality*

$$\mathrm{cov}[f(X), g(X)] \geq 0$$

holds for any two increasing functions $f, g : J \to \mathbf{R}$ for which $f(X)$ and $g(X)$ have a finite second moment.

Proof. Assume first that $f(X)$ and $g(X)$ are positive. For $n, k \in \mathbf{N}$, define

$$
\begin{aligned}
A_{n,k} &:= f^{-1}((k2^{-n}, \infty)) \\
B_{n,k} &:= g^{-1}((k2^{-n}, \infty))
\end{aligned}
$$

Since f and g are assumed to be increasing, each of these sets is increasing. For $n \in \mathbf{N}$, define $f_n, g_n : \mathbf{R} \to \mathbf{R}_+$ by letting

$$f_n(x) := 2^{-n} \sum_{k=1}^{\infty} \chi_{A_{n,k}}(x)$$

$$g_n(x) := 2^{-n} \sum_{k=1}^{\infty} \chi_{B_{n,k}}(x)$$

By the Lemma, we have

$$
\begin{aligned}
E\left[f_n(X) \cdot g_n(X)\right] &= E\left[2^{-n} \sum_{k=1}^{\infty} \chi_{A_{n,k}}(X) \cdot 2^{-n} \sum_{l=1}^{\infty} \chi_{B_{n,l}}(X)\right] \\
&= 2^{-2n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} E[\chi_{A_{n,k} \cap B_{n,l}}(X)] \\
&= 2^{-2n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P[\{X \in A_{n,k} \cap B_{n,l}\}] \\
&\geq 2^{-2n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P[\{X \in A_{n,k}\}]\, P[\{X \in B_{n,l}\}] \\
&= 2^{-n} \sum_{k=1}^{\infty} P[\{X \in A_{n,k}\}] \cdot 2^{-n} \sum_{l=1}^{\infty} P[\{X \in B_{n,l}\}] \\
&= E[f_n(X)] \cdot E[g_n(X)]
\end{aligned}
$$

for all $n \in \mathbf{N}$. Since the sequence $\{f_n(X)\}_{n \in \mathbf{N}}$ increases to $f(X)$ and the sequence $\{g_n(X)\}_{n \in \mathbf{N}}$ increases to $g(X)$, the monotone convergence theorem yields

$$E\left[f(X) \cdot g(X)\right] \geq E[f(X)] \cdot E[g(X)]$$

Assume now that f and g are bounded below by some constant $c \in \mathbf{R}$. Then we have $f(x) - c \geq 0$ and $g(x) - c \geq 0$ for all $x \in J$, hence

$$E\left[\left(f(X) - c\right) \cdot \left(g(X) - c\right)\right] \geq E\left[f(X) - c\right] \cdot E\left[g(X) - c\right]$$

and thus

$$E\left[f(X) \cdot g(X)\right] \geq E[f(X)] \cdot E[g(X)]$$

Assume finally that f and g are arbitrary. Then we have

$$E\Big[\big(f(X) \vee (-n)\big) \cdot \big(g(X) \vee (-n)\big)\Big] \;\geq\; E\Big[f(X) \vee (-n)\Big] \cdot E\Big[g(X) \vee (-n)\Big]$$

for all $n \in \mathbf{N}$. Since $\sup_{n \in \mathbf{N}} |h(X) \vee (-n)| \leq |h(X)|$ and $\lim_{n \to \infty} h(X) \vee (-n) = h(X)$, the dominated convergence theorem yields

$$E\Big[f(X) \cdot g(X)\Big] \;\geq\; E[f(X)] \cdot E[g(X)]$$

This proves the assertion. $\qquad\square$

The construction used in the first part of the preceding proof is inspired by Milgrom and Weber [1982]. To complete the discussion, we present an even simpler proof (of a slightly weaker result) in the case where f is the identity function:

Theorem. *Assume that J is convex and that X has a finite second moment. Then the inequality*

$$\mathrm{cov}[X, g(X)] \;\geq\; 0$$

holds for every increasing function $g : J \to \mathbf{R}$ for which $g(X)$ has a finite second moment.

Proof. Since J is convex, we have $E[X] \in J$ and hence

$$
\begin{aligned}
\mathrm{cov}[X, g(X)] \;&=\; E\Big[\big(X - E[X]\big)\big(g(X) - E[g(X)]\big)\Big] \\
&=\; E\Big[\big(X - E[X]\big)\big(g(X) - g(E[X])\big)\Big] \\
&\quad + E\Big[\big(X - E[X]\big)\big(g(E[X]) - E[g(X)]\big)\Big] \\
&=\; E\Big[\big(X - E[X]\big)\big(g(X) - g(E[X])\big)\Big]
\end{aligned}
$$

Now the assertion follows from the assumption that g is increasing. $\qquad\square$

It appears, however, that this type of proof, which is taken from Schmidt [2002; Satz 10.2.7], cannot be extended to the general case.

References

Milgrom, P. R., and Weber, J. R. [1982] *A theory of auctions and competitive bidding.* Econometrica **50**, 1089–1122.

Schmidt, K. D. [2002]: *Versicherungsmathematik.* Springer, Berlin – Heidelberg – New York.

Schürger, K. [1998]: *Wahrscheinlichkeitstheorie.* Oldenbourg, München – Wien.

Klaus D. Schmidt
Lehrstuhl für Versicherungsmathematik
Technische Universität Dresden
D–01062 Dresden

e–mail: schmidt@math.tu-dresden.de

2nd July 2003