On the Construction of Point Processes

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Abstract

Suppose that $(N_t)_{t\geq 0}$ is a counting process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and that Ω is provided with a filtration $(\mathcal{F}_t)_{t\geq 0}$ of the form $\mathcal{F}_t = \mathcal{G} \vee \sigma(\{N_s : s \leq t\})$. If $C = (C_t)_{t\geq 0}$ is the $(\mathcal{F}_t)$-compensator of (N_t), then (N_t) is called a *mixed (or conditioned) counting process with* $(\mathcal{F}_t)$*-compensator* C. The construction of such mixed counting processes for a given compensator is due to Jacod. By the structure of the filtration, the mixing takes place at time 0. In the present paper we study the problem how to construct point processes, where the mixing takes place "continuously" at every time point $t \geq 0$. The main result is the following: Suppose that $\mathbf{P}^C$ is the probability measure obtained from the Jacod construction - with mixing at time 0. Then it is shown that a certain Girsanov transformation of $\mathbf{P}^C$ can be interpreted as a probability measure, for which (N_t) is the result of continuous mixing.

1 Introduction

A point process on $\mathbb{R}_+$ is defined as a sequence $(T_n)_{n\geq 1}$ of $\overline{\mathbb{R}}_+$-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that (i) $0 < T_1$, (ii) $T_n \leq T_{n+1}$, and (iii) $T_n < T_{n+1}$, if $T_n < \infty$. Such a point process can be equivalently described by a counting process $(N_t)_{t\geq 0}$ related to $(T_n)_{n\geq 1}$ by

$$N_t = \sum_{n\geq 1} 1_{[0,t]}(T_n) \quad .$$

If $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a given standard filtration on Ω, such that $(N_t)_{t\geq 0}$ is $\mathbf{F}$-adapted, then by the theorem of Doob-Meyer there exists an increasing, right-continuous, $\mathbf{F}$-predictable process $(C_t)_{t\geq 0}$, called the $\mathbf{F}$-*compensator* of $(N_t)_{t\geq 0}$, such that $\big(N_{t\wedge T_n} - C_{t\wedge T_n}\big)_{n\geq 1}$ is an $\mathbf{F}$-martingale for every $n \geq 1$.

The compensator plays an essential rôle for the construction of point processes. Define now $\Omega = S^\infty$ to be the space of all sequences $(t_n)_{n\geq 1}$ in $\overline{\mathbb{R}}_+$ such that (i) $0 < t_1$, (ii) $t_n \leq t_{n+1}$, and (iii) $t_n < t_{n+1}$, if $t_n < \infty$. For $\omega = (t_k)_{k\geq 1} \in S^\infty$ and $n \geq 1$ we set $T_n(\omega) = t_n$ and introduce on $\Omega = S^\infty$ the filtration $\mathbf{F}^0 = (\mathcal{F}^0_t)_{t\geq 0}$ defined by

$$\mathcal{F}^0_t := \sigma\big(\{T_n \wedge t \,|\, n \geq 1\}\big)$$

We further set $\mathcal{F}^0 := \mathcal{F}^0_\infty$, and define $N = (N_t)_{t\geq 0}$ by $(T_n)_{n\geq 1}$ as above. Now suppose that $C = (C_t)_{t\geq 0}$ is a given increasing, right continuous, and $\mathbf{F}^0$-predictable process on Ω. Then a classical construction (due to Jacod, cf. Liptser, Shiryayev [1978]) proves that there exists a probability measure $\mathbf{P}^C$ on $(\Omega, \mathcal{F}^0)$ such that relative to $\mathbf{P}^C$ the given process C is just the $\mathbf{F}^0$-compensator of N.

There is a rather immediate generalization. Let $(D, \mathcal{D}, \mathbf{Q})$ be a further probability space and set $(\Omega, \mathcal{F}) := (S^\infty \times D, \mathcal{F}^0 \otimes \mathcal{D})$. Suppose that $(T_n)_{n\geq 1}$ and $N = (N_t)_{t\geq 0}$ are canonically extended to this larger Ω. We consider the filtration $\mathbf{F}^1 = (\mathcal{F}^1_t)_{t\geq 0}$ given by $\mathcal{F}^1_t := \mathcal{F}^0_t \otimes \mathcal{D}$ and suppose again that $C = (C_t)_{t\geq 0}$ is a given increasing, right continuous, but now $\mathbf{F}^1$-predictable process on Ω. Then $C(y) = (C_t(\cdot, y))_{t\geq 0}$ is $\mathbf{F}^0$-predictable for every $y \in D$, and hence there exists by the above mentioned construction a probability measure $\mathbf{P}^{C(y)}$ on S^∞ for every $y \in D$, such that relative to $\mathbf{P}^{C(y)}$ the process $C(y)$ is the $\mathbf{F}^0$-compesator of N. Now define

$$\mathbf{P}^C(d((t_n)_{n\geq 1}, y)) \;\; := \;\; \mathbf{Q}(dy)\mathbf{P}^{C(y)}(d(t_n)_{n\geq 1}) \;\; .$$

Then relative to $\mathbf{P}^C$ the process C is the $\mathbf{F}^1$-compensator of N. With this generalization of the first construction one gets among others the existence of the double-stochastic Poisson processes.

The just described construction method means that at the starting time 0 there is mixing according to the probability measure $\mathbf{Q}$ on $(D, \mathcal{D})$. Let us give a slightly different view of this mixing. Denote by Y the canonical projection from Ω onto D, and let $\mathbf{F}^N = (\mathcal{F}^N_t)_{t\geq 0}$ denote the natural filtration generated by the counting process N. Then $\mathcal{F}^N_t = \mathcal{F}^0_t \otimes \{\emptyset, D\}$ and $\mathcal{F}^1_t = \mathcal{F}^N_t \vee \sigma(Y)$, i.e. every σ-algebra of the natural filtration of N is enlarged by $\sigma(Y)$, $\mathcal{F}^1_0 = \sigma(Y)$, and

$$\mathbf{P}^C\big\{(T_n)_{n\geq 1} \in \cdot \,\big|\, Y = y\big\} \;\; = \;\; \mathbf{P}^{C(y)} \;\; .$$

Now we can start to explain the problem we will consider in this paper. Let first $\pi = (s_j)_{j\geq 1} \in S^\infty$ be fixed, i.e. $(s_j)_{j\geq 1}$ is an infinite (or finite, if $s_m = 0$ for some $m \geq 1$), strictly increasing sequence $0 =: s_0 < s_1 < s_2 < \cdots$ of given time points. For every $s_j < \infty$ $(j \geq 0)$ we choose a measurable space $(D^j, \mathcal{D}^j)$ and set

$$(\Omega, \mathcal{F}) := \Big(\Omega \times \prod_{s_j<\infty} D^j, \mathcal{F}^0 \otimes \bigotimes_{s_j<\infty} \mathcal{D}^j\Big) \;\; .$$

Let $Y^j : \Omega \to D^j$ denote the canonical projection. Then we define the following filtration $\mathbf{F}^\pi = (\mathcal{F}^\pi_t)_{t\geq 0}$ on Ω. We set

$$\mathcal{F}^\pi_t \;\; := \;\; \mathcal{F}^N_t \vee \sigma(\{Y^j \,|\, j \leq k-1\}) \;\text{ for }\; s_{k-1} \leq t < s_k \;\; .$$

Suppose that $\mathbf{Q}_0$ is a probability measure on D^0 and that for $k \geq 1$ $\mathbf{Q}_k$ is a stochastic kernel from $\big(\Omega, \mathcal{F}^N_{s_k} \vee \sigma(Y_0, \cdots, Y_{k-1})\big)$ to $(D^k, \mathcal{D}^k)$.

Now let $C = (C_t)_{t\geq 0}$ be a given increasing, right continuous, and $\mathbf{F}^\pi$-predictable process on Ω. Then for $s_{k-1} \leq t < s_k$, C_t is just a function on $S^\infty \times D^0 \times \cdots \times D^{k-1}$, and for every fixed $y_{k-1} \in D^{k-1}$ the process

$$\Big(C_t(\cdot, y_{k-1}) - C_{s_{k-1}}(\cdot, y_{k-1})\Big)_{s_{k-1}\leq t<s_k}$$

is predictable for the filtration $\big(\mathcal{F}_t^N \vee \sigma(\{Y_0, \cdots, Y_{k-2}\})\big)_{s_{k-1} \le t < s_k}$. It follows from the classical construction method, that there is a stochastic kernel $\mathbf{P}^{C_{s_{k-1}}(\cdot, y_{k-1})}$ from $\big(S^\infty \times D^0 \times \cdots \times D^{k-2}, \mathcal{F}_{s_{k-1}}^N \vee \sigma(\{Y_0, \cdots, Y_{k-2}\})\big)$ to $(\Omega, \mathcal{F}_{s_k}^N)$, such that $\big(C_t - C_{s_{k-1}}\big)_{s_{k-1} \le t < s_k}$ is the $(\mathcal{F}_t^\pi)$-compensator of $(N_t - N_{s_{k-1}})_{s_{k-1} \le t < s_k}$ relative to $\mathbf{P}^{C_{s_{k-1}}(\cdot, y_{k-1})}(d\omega)\mathbf{Q}_{k-1}(\cdot, dy_{k-1})$. Therefore, the composition

$$\mathbf{P}^\pi(d\omega) \;=\; \big(\mathbf{Q}_0(dy_0)\mathbf{P}^{C_0(\cdot, y_0)}(d\omega_0)\big) \circ \cdots \circ \big(\mathbf{Q}_n(\cdot, dy_n)\mathbf{P}^{C_{s_n}(\cdot, y_n)}(d\omega_n)\big) \circ \cdots$$

of all the above kernels gives a probability measure on Ω such that $C = (C_t)_{t \ge 0}$ is the $\mathbf{F}^\pi$-compensator of $N = (N_t)_{t \ge 0}$ relative to $\mathbf{P}^\pi$.

The roughly outlined ideas of conditioning or mixing at the time points $0 = s_0 < s_1 < s_2 < \cdots$, where the mixing may depend on the history of the point process up to the times s_j, are the content of the next paragraph. There we will prove the details even for marked point processes. At the same time, we will replace the above sequence of measurable spaces $(D^j, \mathcal{D}^j)$ $(j \ge 0)$ by a single function space. The reason is that later on we will consider the question, what happens in case that the mesh $|\pi|$ of $\pi = (s_j)_{j \ge 1}$ tends to zero. This means the question, whether there exists a kind of "permanent" or "continuous" mixing and not only mixing at fixed, discrete time points.

This problem is the content of the last paragraph. There we start with a fixed measure $\mathbf{P}^C = \mathbf{P}^{C(y)}Q(dy)$ obtained by the classical construction with mixing at time 0 (see above). We prove that a special type of Girsanov transformation of $\mathbf{P}^C$ can indeed be interpreted as a certain permanent mixing.

2 A Generalization of the Classical Construction

Let $(E, \mathcal{B}(E))$ be a polish space with its Borel field $\mathcal{B}(E)$. For an element $\triangle$ outside of E we set $E_\triangle = E \cup \{\triangle\}$. Then we define

$$\begin{aligned} S^\infty(E) \;:=\; \Big\{ ((t_k, x_k))_{k \ge 1} \in (\overline{\mathbb{R}}_+ \times E_\triangle)^{\mathbb{N}} \;\Big|\; & (i)\;\; 0 < t_1 \le t_2 \le \cdots, \\ & (ii)\;\; t_k < \infty \;\Rightarrow\; x_k \in E \text{ and } t_k < t_{k+1}, \\ & (iii)\;\; t_k = \infty \;\Rightarrow\; x_k = \triangle \text{ and } t_{k+1} = \infty \Big\}. \end{aligned}$$

If $\mathbf{t} = (t_k)_{k \ge 1} \in \overline{\mathbb{R}}_+^{\mathbb{N}}$ and $\mathbf{x} = (x_k)_{k \ge 1} \in E_\triangle^{\mathbb{N}}$ are such that $((t_k, x_k))_{k \ge 1} \in S^\infty(E)$, then we also write $(\mathbf{t}, \mathbf{x})$ instead of $((t_k, x_k))_{k \ge 1}$, and if $(\mathbf{t}, \mathbf{x})$ has the property that $t_k = \infty$ for $k > n$, then we will also write $((t_k, x_k))_{1 \le k \le n}$ for $(\mathbf{t}, \mathbf{x})$ and $(t_k)_{1 \le k \le n}$ for $\mathbf{t}$. It will often be convenient to define $t_0 = 0$ for a given $((t_k, x_k))_{k \ge 1}$.

For every $k \ge 1$ we will denote by T_k and X_k resp. the projections on $S^\infty(E)$ defined by

$$T_k(\omega) \;=\; t_k \;\text{ and }\; X_k(\omega) \;=\; x_k$$

resp. for $\omega = (\mathbf{t}, \mathbf{x}) = ((t_k, x_k))_{k \ge 1} \in S^\infty(E)$.

On $S^\infty(E)$ we will consider the filtration $(\mathcal{C}_t)_{t\geq 0}$ defined by

$$\mathcal{C}_t \ := \ \sigma\big(\big\{\{T_k \leq s\} \cap \{X_k \in B\} \,\big|\, s \leq t \text{ and } B \in \mathcal{B}(E)\big\}\big) \ ,$$

and $S^\infty(E)$ is provided with the σ-algebra $\mathcal{C} = \mathcal{C}_\infty$.

We also need the space

$$S^\infty \ := \ \Big\{ ((t_k))_{k\geq 1} \in \overline{\mathbb{R}}_+^{\mathbb{N}} \,\Big|\, \begin{array}{l} (i)\ \ 0 < t_1 \leq t_2 \leq \cdots \ , \\ (ii)\ \ t_k < \infty \ \Rightarrow \ \ t_k < t_{k+1} \ , \\ (iii)\ \ t_k = \infty \ \Rightarrow \ \ t_{k+1} = \infty \end{array} \Big\} \ .$$

If $x_0 \in E$ is fixed, then S^∞ can be identified with $S^\infty(\{x_0\})$ and hence as a subspace of $S^\infty(E)$. For this reason we use the same notation T_k for the projection defined on S^∞ onto the k-th coordinate. Moreover, $T_0 : S^\infty \to \mathbb{R}_+$ denotes the function $T_0 \equiv 0$.

Now let $(F, \mathcal{B}(F))$ be a second polish space with its Borel field. By $D_0(\mathbb{R}_+, F)$ we denote the space of all cadlag functions $f : \mathbb{R}_+ \to F$ with $f(0) = 0$. For $f \in D_0(\mathbb{R}_+, F)$ we use the notation f^t for the stopped function $f^t := f(\cdot \wedge t)$, and we define the projections Y_t and Y^t resp. on $D_0(\mathbb{R}_+, F)$ by

$$Y_t(f) := f(t) \ \text{ and } \ Y^t(f) := f^t$$

for $f \in D_0(\mathbb{R}_+, F)$. Furthermore, we introduce on $D_0(\mathbb{R}_+, F)$ the filtration $(\mathcal{D}_t)_{t\geq 0}$ given by

$$\mathcal{D}_t \ := \ \sigma\big(\{Y_s \,|\, s \leq t\}\big) \ = \ \sigma(\{Y^t\}) \ ,$$

and set $\mathcal{D} := \mathcal{D}_\infty$.

Let us already remark that

$$(\Omega, \mathcal{F}) \ \ := \ \ (S^\infty(E) \times D_0(\mathbb{R}_+, F), \mathcal{C} \otimes \mathcal{D})$$

will be the measurable space on which we are going to construct probability measures from certain data to obtain point processes.

There are two types of data involved in our construction.

(I) The first type is a kernel $\mathbf{C}$ from $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ to $\mathbb{R}_+ \times E$ which is predictable in a sense, we will define now.

For every $t \geq 0$ let $\varphi_t : S^\infty(E) \to S^\infty(E)$ be given by

$$\varphi_t\big(((t_k, x_k))_{k\geq 1}\big) \ \ := \ \ ((u_k, y_k))_{k\geq 1} \ ,$$

where $u_k = t_k$, $y_k = x_k$ for $t_k \leq t$ and $u_k = \infty$, $y_k = \triangle$ for $t_k > t$. If $\theta = ((s_k, y_k))_{k\geq 1}$ is the element in $S^\infty(E)$ with $s_k = \infty$ and $y_k = \triangle$ for all $k \geq 1$, then

$$\varphi_t\big(((t_k, x_k))_{k\geq 1}\big) = \theta \quad \text{for} \quad t < t_1 \ . \tag{1}$$

Later on, we will also use a kind of complementary function $\psi_t : S^\infty(E) \to S^\infty(E)$ defined by

$$\psi_t\big(((t_k, x_k))_{k\geq 1}\big) \quad := \quad ((u_k, y_k))_{k\geq 1} \ ,$$

where $u_k = t_{k+n_t} - t$ and $y_k = x_{k+n_t}$ for $k \geq 1$ in case that

$$n_t := n_t\big(((t_k, x_k))_{k\geq 1}\big) := \min\{k \geq 0 | t_{k+1} > t\} < \infty \ ,$$

and $u_k = \infty$, $y_k = \triangle$ for all $k \geq 1$ in case that $n_t = \infty$.

The functions φ_t and ψ_t are connected with another function

$$\vartheta_t : S^\infty(E) \times S^\infty(E) \to S^\infty(E) \ \ ,$$

which will be used later. Let $(\mathbf{t}, \mathbf{x}) = ((t_k, x_k))_{k\geq 1}$ and $(\mathbf{u}, \mathbf{z}) = ((u_k, z_k))_{k\geq 1}$ be elements of $S^\infty(E)$. Then $\vartheta_t\big((\mathbf{t}, \mathbf{x}), (\mathbf{u}, \mathbf{z})\big)$ is defined to be that element $((s_k, y_k))_{k\geq 1} \in S^\infty(E)$, which is defined by $(s_k, y_k) = (t_k, x_k)$ for $t_k \leq t$ and by $(s_{n_t+k}, y_{n_t+k}) = (u_k + t, z_k)$ for $k \geq 1$ and $n_t := n_t((\mathbf{t}, \mathbf{x}))$ as defined above. It is easy to see that the following relations hold:

$$\begin{aligned} \varphi_t\big(\vartheta_t\big((\mathbf{t}, \mathbf{x}), (\mathbf{u}, \mathbf{z})\big)\big) \quad &= \quad \varphi_t\big((\mathbf{t}, \mathbf{x})\big) \ \ , \\ \psi_t\big(\vartheta_t\big((\mathbf{t}, \mathbf{x}), (\mathbf{u}, \mathbf{z})\big)\big) \quad &= \quad (\mathbf{u}, \mathbf{z}) \ \ , \\ \vartheta_t\big(\varphi_t\big((\mathbf{t}, \mathbf{x})\big), \psi_t\big((\mathbf{t}, \mathbf{x})\big)\big) \quad &= \quad (\mathbf{t}, \mathbf{x}) \ \ . \end{aligned} \tag{2}$$

Now the kernel $\mathbf{C}$ is called a *predictable kernel* from $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ to $\mathbb{R}_+ \times E$, if for all $(\mathbf{t}, \mathbf{x}) \in S^\infty(E)$, $f \in D_0(\mathbb{R}_+, F)$, $t \geq 0$, and $B \in \mathcal{B}(E)$,

$$\mathbf{C}\big((\mathbf{t}, \mathbf{x}), f; [0, t] \times B\big) \quad = \quad \mathbf{C}\big(\varphi_{t-}((\mathbf{t}, \mathbf{x})), f; [0, t] \times B\big) \ .$$

Later on, we will also write

$$\mathbf{C}_t\big((\mathbf{t}, \mathbf{x}), f; B\big) \quad := \quad \mathbf{C}\big((\mathbf{t}, \mathbf{x}), f; [0, t] \times B\big) \ .$$

2.1 Definition. Let $\mathbf{C}$ be a predictable kernel with the following additional properties:

(i) $\quad \mathbf{C}\big((\mathbf{t}, \mathbf{x}), f; \{0\} \times E\big) \ = \ 0$,
(ii) $\quad \mathbf{C}\big((\mathbf{t}, \mathbf{x}), f; \{t\} \times E\big) \ \leq 1$, and
(iii) $\quad \mathbf{C}\big((\mathbf{t}, \mathbf{x}), f; [0, t] \times E\big) \ = \ \infty \ \Rightarrow \ \mathbf{C}\big((\mathbf{t}, \mathbf{x}), f;]t, \infty] \times E\big) \ = \ 0$

for $(\mathbf{t}, \mathbf{x}) \in S^\infty(E)$, $f \in D_0(\mathbb{R}_+, F)$, and $t \geq 0$. Then $\mathbf{C}$ will be called a *compensator kernel.*

(II) A second type of data we need for our construction is given by a family $\mathbf{Q} = (\mathbf{Q}_{s,t})_{0\leq s<t\leq\infty}$ of stochastic kernels from $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ to $D_0(\mathbb{R}_+, F)$.

2.2 Definition. Suppose that the family $\mathbf{Q}$ has the properties

(i) $\quad \mathbf{Q}_{s,t}\big((\mathbf{t}, \mathbf{x}), f; A\big) \ = \ \mathbf{Q}_{s,t}\big(\varphi_s(\mathbf{t}, \mathbf{x}), f^s; A\big)$ and
(ii) $\quad \mathbf{Q}_{s,t}\big((\mathbf{t}, \mathbf{x}), f; \{g \in D_0(\mathbb{R}_+, F) | \ g^s = f^s \ and \ g = g^t\}\big) \ = \ 1$

for $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$, $f \in D_0(\mathbb{R}_+, F)$, $s < t \leq \infty$ and $A \in \mathcal{D}$. Then $\mathbf{Q}$ is called a *family of structural kernels.*

For the formulation of our first construction result we need some further preparations. Suppose first that the maps T_k, X_k, Y_t, and Y^t are canonically extended to $(\Omega, \mathcal{F}) := (S^\infty(E) \times D_0(\mathbb{R}_+, F), \mathcal{C} \otimes \mathcal{D})$. Assume for the moment that $\mathbf{P}$ is already a probability measure on $(\Omega, \mathcal{F})$ and that $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a given right continuous filtration. The process $((T_k, X_k))_{k\geq 1}$ will be called a *(marked) point process.* A marked point process can be eqivalently described by a random measure

$$N : (\Omega, \mathcal{F}, \mathbf{P}) \to M_+(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E))$$

($M_+(X, \mathcal{X})$:= space of non-negative measures on the measurable space $(X, \mathcal{X})$): For $t \geq 0$ and $B \in \mathcal{B}(E)$ we set

$$\begin{aligned} N_t(B) &= N([0,t] \times B) \\ &:= \Big(\sum_{k\geq 1} \delta_{(T_k, X_k)}\Big)([0,t] \times B) \;=\; \sum_{k\geq 1} 1_{\{T_k \leq t\} \cap \{X_k \in B\}} \quad . \end{aligned}$$

N will be called the *counting measure* of $((T_k, X_k))_{k\geq 1}$. Suppose that

$$C : (\Omega, \mathcal{F}, \mathbf{P}) \to M_+(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E))$$

is a second random measure with the properties that for every $B \in \mathcal{B}(E)$
(i) the process $t \mapsto C_t(B) := C(]0,t] \times B)$ is $\mathbf{F}$-predictable, and
(ii) $\big(N_{T_n \wedge t}(B) - C_{T_n \wedge t}(B)\big)_{t\geq 0}$ is an $\mathbf{F}$-martingale for every $n \geq 1$.
Then C is called the *compensator measure* of N (or of $((T_k, X_k))_{k\geq 1}$).

Suppose that $\pi = (s_k)_{k\geq 1}$ is a fixed given element in S^∞ such that $\sup_{k\geq 1} s_k = \infty$. Then we define

$$\mathbf{F}^\pi \;=\; (\mathcal{F}_t^\pi)_{t\geq 0}$$

to be the following filtration on Ω. For $t \in [s_{j-1}, s_j[$ $(j \geq 1)$ we set

$$\mathcal{F}_t^\pi \;:=\; \sigma\Big(\big\{\{T_k \leq s\} \cap \{X_k \in B\} \;\big|\; s \leq t,\ B \in \mathcal{B}(E)\big\}\Big) \;\vee\; \sigma\big(\{Y^{s_j}\}\big) \;.$$

Now we can state our first construction result.

2.3 Theorem. *Suppose that a compensator kernel* $\mathbf{C}$*, a family* $\mathbf{Q}$ *of structural kernels and a* $\pi \in S^\infty$ *as above are given. Then there exists a probability measure* $\mathbf{P}^\pi$ *on* $\Omega = S^\infty(E) \times D_0(\mathbb{R}_+, F)$ *with the following properties:*
(1) The marked point process $(\overline{\mathbf{T}}, \overline{\mathbf{X}}) = ((T_k, X_k))_{k\geq 1}$ *has the* $\mathbf{F}^\pi$*-compensator measure* $C^\pi(dt, dx)$ *given by*

$$\begin{aligned} C_t^\pi(B)(\omega) &= C^\pi([0,t] \times B)(\omega) \qquad (3) \\ &:= \sum_{j\geq 1} \Big(\mathbf{C}_{t\wedge s_j}\big((\overline{\mathbf{T}}(\omega), \overline{\mathbf{X}}(\omega)), Y^{s_j}(\omega); B\big) \\ &\qquad - \mathbf{C}_{t\wedge s_{j-1}}\big((\overline{\mathbf{T}}(\omega), \overline{\mathbf{X}}(\omega)), Y^{s_j}(\omega); B\big)\Big) \;. \end{aligned}$$

(2) For $s_{j-1} < \infty$ and every $B \in \mathcal{D}$ the map

$$\omega = ((\mathbf{t},\mathbf{x}), f) \mapsto \int 1_B(g)\mathbf{Q}_{s_{j-1},s_j}\big(\varphi_{s_{j-1}}((\mathbf{t},\mathbf{x})), f; dg\big) \tag{4}$$

is a version of the conditional probability

$$\mathbf{P}^{\pi}\big\{Y^{s_j} \in B | \mathcal{C}_{s_{j-1}} \vee \sigma(\{Y^{s_{j-1}}\})\big\} \ .$$

Proof. For $\pi = (s_j)_{j\geq 1}$ define $\pi_n := (s_j)_{1\leq j\leq n}$. Then the theorem is proved by induction in n.

(A) The case $n = 1$ is just the classical construction, which can be found e.g. in Last, Brandt [1995; theorem 8.2.1]. In this case, the filtration $\mathbf{F}^{\pi_1}$ is given by

$$\mathcal{F}_t^{\pi_1} \quad = \quad \mathcal{C}_t \vee \sigma(\{Y^{s_1}\}) \ ,$$

and we have to construct a probability measure $\mathbf{P}^1 = \mathbf{P}^{\pi_1}$ such that the point process $((T_k, X_k))_{k\geq 1}$ has the $\mathbf{F}^{\pi_1}$-compensator measure C^1 given by

$$C_t^1(A)(\omega) = C^1([0,t] \times A)(\omega) := \mathbf{C}_t\Big(\big((T_k(\omega), X_k(\omega))\big), Y^{s_1}(\omega); A\Big)$$

for $t \geq 0$ and $A \in \mathcal{B}(E)$, and such that

$$\mathbf{P}^1\{Y^{s_1} \in B\} \quad = \quad \int 1_B(g)\, \mathbf{Q}_{0,s_1}(dg)$$

for $B \in \mathcal{D}$ (observe that $\mathbf{Q}_{0,s_1}$ does not depend on ω).

We outline the main steps of the construction proof of $\mathbf{P}^1$ not only for completeness but also for a better understanding of the induction step (B) below.

For every fixed $f \in D_0(\mathbb{R}_+, F)$ we will construct by induction a projective sequence $(\mathbf{R}_n^1)_{n\geq 0}$ of probability measures on the spaces $(S^\infty(E), \mathcal{C}_{T_n})$. Since $\mathcal{C}_{T_0} = \mathcal{C}_0 = \{\emptyset, S^\infty(E)\}$, the definition of $\mathbf{R}_0^1$ is clear.

Now suppose that $\mathbf{R}_{n-1}^1$ ($n \geq 1$) is already constructed. First, we define a kernel F_n from $(S^\infty(E), \mathcal{C}_{T_{n-1}})$ to $(\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+))$. For every $t \geq 0$ and $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$ we set

$$\begin{aligned} &F_n\big(\varphi_{T_{n-1}}((\mathbf{t},\mathbf{x})), f;]t,\infty]\big) \\ &\quad := \prod_{T_{n-1}<u\leq t} \Big(1 - \mathbf{C}\big(\varphi_{T_{n-1}}((\mathbf{t},\mathbf{x})), f; \{u\} \times E\big)\Big) \\ &\qquad\qquad\qquad \cdot \exp\big\{ - \mathbf{C}^c\big(\varphi_{T_{n-1}}((\mathbf{t},\mathbf{x})), f;]T_{n-1}, t] \times E\big)\big\} \ , \end{aligned}$$

where $\mathbf{C}^c$ denotes the continuous part of the increasing function

$$t \mapsto \mathbf{C}_t((\mathbf{t},\mathbf{x})), f; E) \ .$$

For every $A \in \mathcal{B}(\overline{\mathbb{R}}_+)$, every $B \in \mathcal{B}(E)$, and all $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$ we have

$$\mathbf{C}\big((\mathbf{t},\mathbf{x}), f; A \times B\big) \quad \leq \quad \mathbf{C}\big((\mathbf{t},\mathbf{x}), f; A \times E\big) \ ,$$

and hence there exists a Radon-Nikodym density $t \mapsto H_t((\mathbf{t},\mathbf{x}), f; B)$ of the measure $\mathbf{C}\big((\mathbf{t},\mathbf{x}), f; \cdot \times B\big)$ relative to $\mathbf{C}\big((\mathbf{t},\mathbf{x}), f; \cdot \times E\big)$. Following the ideas in Meyer [1966;ch.VIII.p.154] and using that both $\mathbb{R}_+$ and E are polish spaces, one can prove that H may be assumed to have the following addditional properties:
(i) $(\mathbf{t},\mathbf{x}) \mapsto H_t((\mathbf{t},\mathbf{x}), f; B)$ is measurable for all $t \geq 0$ and $B \in \mathcal{B}(E)$,
(ii) $t \mapsto H_t((\mathbf{t},\mathbf{x}), f; B)$ is measurable for all $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$ and $B \in \mathcal{B}(E)$,
(iii) $B \mapsto H_t((\mathbf{t},\mathbf{x}), f; B)$ is a probability measure on $(E, \mathcal{B}(E))$ for every $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$ and all $t \geq 0$,
(iv) For every stochastic kernel G from $(S^\infty(E), \mathcal{C})$ to $(\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+))$ such that $G((\mathbf{t},\mathbf{x}); \cdot)$ is absolutely continuous relative to $\mathbf{C}\big((\mathbf{t},\mathbf{x}), f; \cdot \times E\big)$,

$$(\mathbf{t},\mathbf{x}) \mapsto \int_A H_t((\mathbf{t},\mathbf{x}), f; B)\, G((\mathbf{t},\mathbf{x}); dt)$$

is measurable for $A \in \mathcal{B}(\overline{\mathbb{R}}_+)$ and $B \in \mathcal{B}(E)$, and

$$\int_A H_t((\mathbf{t},\mathbf{x}), f; B)\, G((\mathbf{t},\mathbf{x}); dt)$$

defines a probability measure on $\overline{\mathbb{R}}_+ \times E$ for every $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$.

We apply property (iv) of H for the kernel F_n defined above, and obtain that the definition

$$\begin{aligned} &G_n\big((\mathbf{t},\mathbf{x}), f; A \times B\big) \\ &\quad = \int_{]T_{n-1},\infty[\cap A} H_s\big(\varphi_{T_{n-1}}((\mathbf{t},\mathbf{x})), f; B\big) F_n\big(\varphi_{T_{n-1}}((\mathbf{t},\mathbf{x})), f; ds\big) \\ &\qquad\qquad + 1_{A\times B}(\infty, \triangle) F_n\big(\varphi_{T_{n-1}}((\mathbf{t},\mathbf{x})), f; \{\infty\}\big) \end{aligned}$$

for every $A \in \mathcal{B}(\overline{\mathbb{R}}_+)$, every $B \in \mathcal{B}(E)$, and all $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$, defines a stochastic kernel from $S^\infty(E)$ to $\overline{\mathbb{R}}_+ \times E$. For $C \in \mathcal{C}_{T_n}$ we define now

$$\begin{aligned} &\mathbf{R}^1_n(f; C) \\ &\quad = \int \cdots \int 1_C\big(((t_k, x_k))_{1\leq k\leq n}\big) G_n\big(((t_k, x_k))_{1\leq k\leq n-1}, f; d(t_n, x_n)\big) \\ &\qquad\qquad \cdots G_1\big(f; d(t_1, x_1)\big) \ , \end{aligned}$$

where for $(\mathbf{t},\mathbf{x}) = ((t_k, x_k))_{k\geq 1}$, $((t_k, x_k))_{1\leq k\leq n}$ is just $\varphi_{T_n}((\mathbf{t},\mathbf{x}))$, and
$G_1\big(f; d(t_1, x_1)\big) := G_1\big(\varphi_{T_0}(\mathbf{t},\mathbf{x}), f; d(t_1, x_1)\big) = G_1\big(\theta, f; d(t_1, x_1)\big)$ (cf. (1)).

Then $\mathbf{R}^1_n(f; \cdot)$ is a probability measure on $(S^\infty(E), \mathcal{C}_{T_n})$, and it is rather easy to check that the sequence $(\mathbf{R}^1_n(f; \cdot))_{n\geq 1}$ is projective, and hence defines a unique probability measure $\mathbf{R}^1(f; \cdot)$ on $(S^\infty(E), \mathcal{C})$ with the characteristic property that for every non-negative, bounded, $\mathcal{C}$-measurable function $g : S^\infty(E) \to \mathbb{R}_+$,

$$\int g\big(\varphi_{T_n}((\mathbf{t},\mathbf{x}))\big) \mathbf{R}^1\big(f; d(\mathbf{t},\mathbf{x})\big) \ = \ \int g\big(\varphi_{T_n}((\mathbf{t},\mathbf{x}))\big) \mathbf{R}^1_n\big(f; d(\mathbf{t},\mathbf{x})\big) \ .$$

Now we just replace f by f^{s_1}. Then it is proved in Last, Brandt [1995; theorem 8.2.1] that relative to $\mathbf{R}^1(f^{s_1}; \cdot)$ the marked point process $((T_k, X_k))_{k\geq 1}$ on $S^\infty(E)$ has the asserted compensator measure.

Moreover, by the construction,

$$f \mapsto \mathbf{R}^1(f^{s_1}; A)$$

is $\mathcal{D}$-measurable for every $A \in \mathcal{C}$, and hence

$$A \times B \ \mapsto \ \mathbf{P}^1(A \times B) := \int_B \mathbf{R}^1(f^{s_1}; A)\mathbf{Q}_{0,s_1}(df)$$

($A \in \mathcal{C}$, $B \in \mathcal{D}$) defines a probability measure $\mathbf{P}^{\pi_1} := \mathbf{P}^1$ on $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ with the asserted properties.

(B) Suppose that we have already constructed

$$\mathbf{P}^{n-1} \ := \ \mathbf{P}^{\pi_{n-1}} \ = \ \mathbf{P}^{s_1,\cdots,s_{n-1}} \ .$$

If $s_n = \infty$, then the proof of the theorem is finished. Hence assume that $s_n < \infty$. Then we proceed as follows.

For every $(\mathbf{u}, \mathbf{z}), (\mathbf{t}, \mathbf{x}) \in S^\infty(E)$, every $f \in D_0(\mathbb{R}_+, F)$ and every $A \in \mathcal{B}(E)$ we define

$$\begin{aligned} &\mathbf{C}^n_t\big((\mathbf{t}, \mathbf{x}), f; A\big) \ := \ \mathbf{C}^{(s_{n-1},(\mathbf{u},\mathbf{z}))}_t\big((\mathbf{t}, \mathbf{x}), f; A\big) \\ &\quad := \ \mathbf{C}_{s_{n-1}+t}\Big(\vartheta_{s_{n-1}}\big((\mathbf{u}, \mathbf{z}), (\mathbf{t}, \mathbf{x})\big), f; A\Big) \ - \ \mathbf{C}_{s_{n-1}}\big((\mathbf{u}, \mathbf{z}), f; A\big) \ . \end{aligned}$$

Then $\mathbf{C}^n$ is a kernel from $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ to $\mathbb{R}_+ \times E$ and one obtains exactely as in step (A) of the proof an associated stochastic kernel $\mathbf{R}^n$ such that for every $(\mathbf{u}, \mathbf{z}) \in S^\infty(E)$ and every $f \in D_0(\mathbb{R}_+, F)$ the marked point process $((T_k, X_k))_{k \geq 1}$ has the compensator measure $\mathbf{C}^{(s_n,(\mathbf{u},\mathbf{z}))}$ relative to the probability measure $\mathbf{R}^n\big((\mathbf{u}, \mathbf{z}), f; d(\mathbf{t}, \mathbf{x})\big)$.

Now we can define $\mathbf{P}^{s_1,\cdots,s_n}$. For every non-negative, bounded, $\mathcal{C} \otimes \mathcal{D}$-measurable function $H : \Omega \to \mathbf{R}_+$ we set

$$\begin{aligned} &\int H(\omega)\mathbf{P}^{s_1,\cdots,s_n}(d\omega) \\ &\quad = \ \int_{S^\infty(E)\times D_0(\mathbb{R}_+,F)} \int_{D_0(\mathbb{R}_+,F)} \int_{S^\infty(E)} H\big(\vartheta_{s_{n-1}}((\mathbf{u}, \mathbf{z}), (\mathbf{t}, \mathbf{x})), g\big) \\ &\qquad\qquad \Big[\mathbf{R}^n\big((\mathbf{u}, \mathbf{z}), g; d(\mathbf{t}, \mathbf{x})\big)\mathbf{Q}_{s_{n-1},s_n}\big((\mathbf{u}, \mathbf{z}), f; dg\big) \\ &\qquad\qquad\qquad \mathbf{P}^{s_1,\cdots,s_{n-1}}\big(d((\mathbf{u}, \mathbf{z}), f)\big)\Big] \ . \end{aligned}$$

With this definition we compute first the compensator measure of the canonical marked point process relative to $\mathbf{P}^n := \mathbf{P}^{s_1,\cdots,s_n}$.

Let $\mathbf{E}_n$ denote the expectation relative to $\mathbf{P}^n$. Then we have to prove (cf. e.g. Lipster, Shiryayev [1978; lemma 18.6]) that for every $A \in \mathcal{B}(E)$ and for every stopping time T such that $N_T(A)$ is $\mathbf{P}^n$-integrable,

$$\mathbf{E}_n\{N_T(A)\} \ = \ \mathbf{E}_n\{C^\pi_T(A)\} \ ,$$

where for $\omega = ((\mathbf{t},\mathbf{x}),f)$,

$$C_T^\pi(A)(\omega) = \sum_{j=1}^n \Big(\mathbf{C}_{T\wedge s_j}\big((\mathbf{t},\mathbf{x}),f^{s_j};A\big) - \mathbf{C}_{T\wedge s_{j-1}}\big((\mathbf{t},\mathbf{x}),f^{s_j};A\big)\Big)\ .$$

Now we have

$$\begin{aligned} N_T(A) &= N_T(A) - N_{T\wedge s_{n-1}}(A) + N_{T\wedge s_{n-1}}(A) \\ &= \big(N_T(A) - N_{s_{n-1}}(A)\big)1_{\{T>s_{n-1}\}} + N_{T\wedge s_{n-1}}(A) \end{aligned}$$

and also

$$C_T^\pi(A) = \big(C_T^\pi(A) - C_{s_{n-1}}^\pi(A)\big)1_{\{T>s_{n-1}\}} + C_{T\wedge s_{n-1}}^\pi(A)\ .$$

The induction hypothesis implies that

$$\mathbf{E}_n\{N_{T\wedge s_{n-1}}(A)\} = \mathbf{E}_n\{C_{T\wedge s_{n-1}}^\pi(A)\}\ ,$$

and it remains to prove

$$\begin{aligned} &\mathbf{E}_n\big\{\big(N_T(A) - N_{s_{n-1}}(A)\big)1_{\{T>s_{n-1}\}}\big\} \\ &\quad = \mathbf{E}_n\big\{\big(C_T^\pi(A) - C_{s_{n-1}}^\pi(A)\big)1_{\{T>s_{n-1}\}}\big\}\ . \end{aligned} \tag{5}$$

Let us start with the left-hand side of (5).

$$\begin{aligned} &\mathbf{E}_n\big\{\big(N_T(A) - N_{s_{n-1}}(A)\big)1_{\{T>s_{n-1}\}}\big\} \\ &\quad = \mathbf{E}_n\big\{\big(N_{T\vee s_{n-1}}(A) - N_{s_{n-1}}(A)\big)1_{\{T>s_{n-1}\}}\big\} \\ &\quad = \mathbf{E}_n\Big\{1_{\{T>s_{n-1}\}}\mathbf{E}_n\big\{N_{T\vee s_{n-1}}(A) - N_{s_{n-1}}(A)\mid \mathcal{F}_{s_{n-1}}^\pi\big\}\Big\} \\ &\quad = \mathbf{E}_n\Big\{1_{\{T>s_{n-1}\}}\mathbf{E}_n\big\{\sum_{k\geq 1} 1_{\{s_{n-1}<T_k\leq T\vee s_{n-1}\}\cap\{X_k\in A\}}\mid \mathcal{C}_{s_{n-1}}\vee\sigma(\{Y^{s_n}\})\big\}\Big\}\ . \end{aligned}$$

For the computation of the conditional expectation in the last line we remark that every $\omega = ((\mathbf{s},\mathbf{y}),h) \in \Omega = S^\infty(E)\times D_0(\mathbb{R}_+,F)$ can be written in the form $\omega = \big(\vartheta_{s_{n-1}}((\mathbf{u},\mathbf{z}),(\mathbf{t},\mathbf{x})),h\big)$ with $(\mathbf{u},\mathbf{z}),(\mathbf{t},\mathbf{x})\in S^\infty(E)$. We obtain

$$\begin{aligned} &\mathbf{E}_n\big\{\sum_{k\geq 1} 1_{\{s_{n-1}<T_k\leq T\vee s_{n-1}\}\cap\{X_k\in A\}}\mid \mathcal{C}_{s_{n-1}}\vee\sigma(\{Y^{s_n}\})\big\} \\ &\quad = \int_{S^\infty(E)}\Big(\sum_{k\geq 1} 1_{]0,T(\omega)\vee s_{n-1}-s_{n-1}]}(t_k)1_A(x_k)\Big)\mathbf{R}^n\big((\mathbf{u},\mathbf{z}),h;d(\mathbf{t},\mathbf{x})\big) \\ &\quad = \int_{S^\infty(E)} \mathbf{C}_{T(\omega)\vee s_{n-1}-s_{n-1}}^{(s_{n-1},(\mathbf{u},\mathbf{z}))}\big((\mathbf{t},\mathbf{x}),h;A\big) \\ &\qquad\qquad\qquad\qquad \mathbf{R}^n\big((\mathbf{u},\mathbf{z}),h;d(\mathbf{t},\mathbf{x})\big) \\ &\quad = \int_{S^\infty(E)} \big[\mathbf{C}_{T(\omega)\vee s_{n-1}}\big(\vartheta_{s_{n-1}}\big((\mathbf{u},\mathbf{z}),(\mathbf{t},\mathbf{x})\big),h;A\big) - \mathbf{C}_{s_{n-1}}\big((\mathbf{u},\mathbf{z}),h;A\big)\big] \\ &\qquad\qquad\qquad\qquad \mathbf{R}^n\big((\mathbf{u},\mathbf{z}),h;d(\mathbf{t},\mathbf{x})\big) \end{aligned}$$

$$= \quad \mathbf{E}_n\big\{C^\pi_{T\vee s_{n-1}}(A) - C^\pi_{s_{n-1}}(A)\,|\,\mathcal{C}_{s_{n-1}} \vee \sigma(\{Y^{s_n}\})\big\} \ ,$$

and

$$\mathbf{E}_n\big\{\big(N_T(A) - N_{s_{n-1}}(A)\big)1_{\{T>s_{n-1}\}}\big\} \quad = \quad \mathbf{E}_n\big\{\big(C^\pi_T(A) - C^\pi_{s_{n-1}}(A)\big)1_{\{T>s_{n-1}\}}\big\}$$

is proved.

To prove the second assertion of the theorem for $\mathbf{P}^n$, we take an $A \in \mathcal{C}_{s_{n-1}}$ and a $B \in \sigma(\{Y^{s_{n-1}}\})$. Then we get

$$\begin{aligned}
&\int_{A\times B} \mathbf{P}^n\big\{Y^{s_n} \in C\,|\,\mathcal{C}_{s_{n-1}} \vee \sigma(\{Y^{s_{n-1}}\})\big\}\, d\mathbf{P}^n \\
&\quad = \int 1_{A\times B}(\omega) 1_C(Y^{s_n}(\omega))\, \mathbf{P}^n(d\omega) \\
&\quad = \int_{S^\infty(E)\times D_0(\mathbb{R}_+,F)} 1_{A\times B}(\omega) \int_{D_0(\mathbb{R}_+,F)} 1_C(g) \\
&\qquad\qquad\qquad\qquad \mathbf{Q}_{s_{n-1},s_n}(\varphi_{s_{n-1}}((\mathbf{s},\mathbf{y})), f; dg)\mathbf{P}^{n-1}(d\omega) \ ,
\end{aligned}$$

where $\omega = (\varphi_{s_{n-1}}((\mathbf{s},\mathbf{y})), f)$ in the last integral. Altogether we have now proved that for $\pi = (s_j)_{1\le j\le n}$ the assertions of the theorem are true.

(C) If $\pi = (s_j)_{j\ge 1}$ has the property that $s_m = \infty$ for some $m \ge 1$, then the proof is finished with the construction of $\mathbf{P}^{s_1,\cdots,s_n}$, where $n = \min\{j|s_j = \infty\}$. Hence we suppose now that $s_j < \infty$ for all $j \ge 1$.

For every $n \ge 1$ let $\rho_n : \Omega \to \Omega$ be the projection defined by

$$\rho_n(\omega) \quad := \quad \big(\varphi_{s_n}((\mathbf{t},\mathbf{x})), f^{s_n}\big)$$

for $\omega = ((\mathbf{t},\mathbf{x}), f)$. Then the construction of $\mathbf{P}^n := \mathbf{P}^{s_1,\cdots,s_n}$ shows that for every non-negative, bounded, measurable function $H : \Omega \to \mathbb{R}_+$ we have

$$\int H(\rho_n(\omega))\mathbf{P}^{n+1}(d\omega) \quad = \quad \int H(\rho_n(\omega))\mathbf{P}^n(d\omega) \ .$$

Hence the theorem of Kolmogorov implies that there exists a unique probability measure $\mathbf{P}^\pi$ on Ω such that

$$\int H(\rho_n(\omega))\mathbf{P}^\pi(d\omega) \quad = \quad \int H(\rho_n(\omega))\mathbf{P}^n(d\omega)$$

holds for every n.

Let us prove that $\mathbf{P}^\pi$ fulfills the properties (1) and (2) of the theorem.

For the proof of (1) we choose an arbitrary $\mathbf{F}^\pi$-stopping time T such that $N_T(A)$ is $\mathbf{P}^\pi$-integrable for every $A \in \mathcal{B}(E)$. Then

$$\begin{aligned}
\mathbf{E}\, N_{T\wedge s_n}(A) \quad &= \quad \mathbf{E}_n\, N_{T\wedge s_n}(A) \\
&= \quad \mathbf{E}_n\, C^\pi_{T\wedge s_n}(A) \ = \ \mathbf{E}\, C^\pi_{T\wedge s_n}(A)
\end{aligned}$$

for every n and hence

$$\mathbf{E}\, N_T(A) \quad = \quad \mathbf{E}\, C_T^\pi(A) \quad .$$

This proves that (1) holds.

For the proof of (2) we take an arbitrary set $D \in \mathcal{C}_{s_{n-1}} \vee \sigma(\{Y^{s_{n-1}}\})$ and a $C \in \mathcal{D}$. Then

$$\begin{aligned}
&\int_D \mathbf{P}^\pi\big\{Y^{s_n} \in C \mid \mathcal{C}_{s_{n-1}} \vee \sigma(\{Y^{s_{n-1}}\})\big\}\, d\mathbf{P}^\pi \\
&= \int 1_D\, 1_C(Y^{s_n})\, d\mathbf{P}^\pi \\
&= \int 1_D\, 1_C(Y^{s_n})\, d\mathbf{P}^n \\
&= \int_\Omega 1_D \int_{D_0(\mathbb{R}_+,F)} 1_C(g)\mathbf{Q}_{s_{n-1},s_n}(\rho_{s_{n-1}}(\omega); dg)\mathbf{P}^{n-1}(d\omega) \\
&= \int_D \int_{D_0(\mathbb{R}_+,F)} 1_C(g)\mathbf{Q}_{s_{n-1},s_n}(\rho_{s_{n-1}}(\omega); dg)\mathbf{P}^{\pi}(d\omega) \quad ,
\end{aligned}$$

and (2) follows.

This finishes the proof of the theorem. $\square$

3 Consequences from the Construction

In the definition 2.1 of a compensator kernel we only demanded predictability on $S^\infty(E)$, i.e. predictability relative to the filtration $(\mathcal{C}_t)_{t\geq 0}$. In this section we will assume that the compensator kernel $\mathbf{C}$ has the stronger property that $\mathbf{C}$ is predictable relative to the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0} := (\mathcal{C}_t \vee \mathcal{D}_t)_{t\geq 0}$, i.e. that for every $t \geq 0$ and $A \in \mathcal{B}(E)$,

$$S^\infty(E) \times D_0(\mathbb{R}_+, F) \ni ((\mathbf{t},\mathbf{x}), f) \mapsto \mathbf{C}\big((\mathbf{t},\mathbf{x}), f; [0,t] \times A\big)$$

is $\mathbf{F}$-predictable. Let us call $\mathbf{C}$ in this case a *strongly predictable compensator kernel.*

If $\mathbf{C}$ is strongly predictable, then we have especially

$$\mathbf{C}\big((\mathbf{t},\mathbf{x}), f; [0,t] \times A\big) \quad = \quad \mathbf{C}\big((\mathbf{t},\mathbf{x}), f^t; [0,t] \times A\big) \; ,$$

and from theorem 2.3 we get the following result.

3.1 Proposition. *Suppose that* $\mathbf{C}$ *is a strongly predictable compensator kernel. For every family* $\mathbf{Q}$ *of structural kernels and every* $\pi \in S^\infty$ *with* $\sup_{j\geq 1} s_j = \infty$ *let* $\mathbf{P}^{\pi,\mathbf{Q}}$ *denote the probability measure on* $\Omega = S^\infty(E) \times D_0(\mathbb{R}_+, F)$ *constructed in theorem 2.3. Then the marked point process* $(\overline{\mathbf{T}}, \overline{\mathbf{X}}) = ((T_k, X_k))_{k\geq 1}$ *on* Ω *has the same* $\mathbf{F}$*-compensator measure* C *for every probability measure* $\mathbf{P}^{\pi,\mathbf{Q}}$*.* C *is given by*

$$C_t(A)(\omega) \quad = \quad \mathbf{C}_t\big(((T_k(\omega), X_k(\omega)))_{k\geq 1}, Y(\omega); A\big) \tag{6}$$

for $t \geq 0$ and $A \in \mathcal{B}(E)$, and $\mathbf{P}^{\pi,\mathbf{Q}}$ and $\mathbf{Q}$ are related by

$$\begin{aligned}
&\mathbf{P}^{\pi,\mathbf{Q}}\big\{Y^{s_n} \in B \,\big|\, \mathcal{C}_{s_{n-1}} \vee \sigma(\{Y^{s_{n-1}}\})\big\} \\
&\quad = \int_{D_0(\mathbb{R}_+,F)} 1_B(\chi_{s_{n-1}}(f,g))\mathbf{Q}_{s_{n-1},s_n}(\,\cdot\,;dg)
\end{aligned} \tag{7}$$

for $n \geq 1$ and $B \in \mathcal{D}$.

In this section we will be mainly concerned with the following question. Let $\mathbf{C}$ be strongly predictable and let $\mathbf{Q}$ be a family of structural kernels. Then for every $\pi \in S^\infty$ we have the probability measure $\mathbf{P}^{\pi,\mathbf{Q}}$ on Ω. Now suppose that $(\pi_n)_{n\geq 1}$ is a sequence in S^∞ such that $\lim_{n\to\infty} |\pi_n| = 0$, where $|\pi| := \sup_{j\geq 1} |s_j - s_{j-1}|$ for $\pi = (s_j)_{j\geq 1} \in S^\infty$. Does there exist a limit probability measure $\mathbf{P}^{\mathbf{Q}}$ of the sequence $(\mathbf{P}^{\pi_n,\mathbf{Q}})_{n\geq 1}$ such that $C = (C_t(B))_{t\geq 0, B\in\mathcal{B}(E)}$ is again the compensator measure of the canonical marked point process on Ω?

Below we will give an answer to that question for an important special case. Before, we need some further consequences of the construction in the last section.

Suppose that the family $\mathbf{Q}$ of structural kernels has the special property that

$$\mathbf{Q}_{s,t}((\mathbf{t},\mathbf{x}), f; B) \;\; = \;\; \mathbf{Q}_{s,t}(f; B)$$

for all $t \geq 0$, $((\mathbf{t},\mathbf{x}), f) \in \Omega$, and $B \in \mathcal{D}$, i.e. every $\mathbf{Q}_{s,t}$ is just a kernel from $D_0(\mathbb{R}_+, F)$ to $D_0(\mathbb{R}_+, F)$. We will shortly say that $\mathbf{Q}$ *does not depend on $S^\infty(E)$*.

If $\mathbf{Q}$ does not depend on $S^\infty(E)$, then for every $\pi \in S^\infty$ the probability measure $\mathbf{P}^{\pi,\mathbf{Q}}$ turns out to be well known from the classical construction.

In the following it will write integrals often in the form $\int \mu(dx) f(x)$. The reason is simply that this notation is more natural for the composition of kernels: Suppose that $(X, \mathcal{X})$, $(Y, \mathcal{Y})$, and $(Z, \mathcal{Z})$ are measurable spaces, that K is a kernel from X to Y and L is a kernel from Y to Z. Then the composition $K \circ L$ of K and L is the kernel from X to Z, defined by

$$(K \circ L)(x, C) \;\; := \;\; \int_Y K(x, dy) L(y, B) \tag{8}$$

for $x \in X$ and $C \in \mathcal{Z}$.

Every strongly predictable compensator kernel $\mathbf{C}$ determines a family $(\mathbf{R}_{s,t})_{0\leq s<t\leq\infty}$ of stochastic kernels from $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ to $S^\infty(E)$. For $0 \leq s < t \leq \infty$, $A \in \mathcal{C}$, and $\omega = (((\mathbf{t},\mathbf{x}), f) \in S^\infty(E) \times D_0(\mathbb{R}_+, F)$ we define the compensator measure $C^{s,t;\omega}(dr, dx)$ by

$$\begin{aligned}
&C^{s,t;\omega}_r(A) = C^{s,t;\omega}([0,r] \times A) \\
&:= \;\; \mathbf{C}_{t\wedge(s+r)}(\vartheta_s((\mathbf{t},\mathbf{x}),(\mathbf{u},\mathbf{y}), f; A) - \mathbf{C}_s((\mathbf{t},\mathbf{x}), f; A)
\end{aligned} \tag{9}$$

for $r \geq 0$ and $A \in \mathcal{B}(E)$. This compensator measure determines a unique probability measure

$$\tilde{\mathbf{R}}_{s,t}\big((\mathbf{t},\mathbf{x}), f; d(\mathbf{u},\mathbf{y})\big)$$

on $S^\infty(E)$, such that the canonical marked point process has $C^{s,t;\omega}(dr, dx)$ as compensator measure, and

$$((\mathbf{t}, \mathbf{x}), f) \mapsto \tilde{\mathbf{R}}_{s,t}\big((\mathbf{t}, \mathbf{x}), f; d(\mathbf{u}, \mathbf{y})\big)$$

is a stochastic kernel from $S^\infty(E) \times D_0(\mathbb{R}_+, F)$ to $S^\infty(E)$, which we denote by $\tilde{\mathbf{R}}_{s,t}$. To every $\tilde{\mathbf{R}}_{s,t}$ we associate a new kernel $\mathbf{R}_{s,t}$, which is related to $\tilde{\mathbf{R}}_{s,t}$ in the following way: Let $H : S^\infty(E) \to \mathbb{R}_+$ be a bounded, $\mathcal{C}$-measurable function. Then we set for $((\mathbf{t}, \mathbf{x}), f) \in S^\infty(E) \times D_0(\mathbb{R}_+, F)$

$$\begin{aligned} &\int \mathbf{R}_{s,t}\big((\mathbf{t}, \mathbf{x}), f; d(\mathbf{v}, \mathbf{z})\big) H(\mathbf{v}, \mathbf{z}) \\ &:= \int \tilde{\mathbf{R}}_{s,t}\big((\mathbf{t}, \mathbf{x}), f; d(\mathbf{u}, \mathbf{y})\big)\, H\Big(\vartheta_s\big(\varphi_s(\mathbf{t}, \mathbf{x}), \varphi_{t-s}(\mathbf{u}, \mathbf{y})\big)\Big) . \end{aligned} \tag{10}$$

The thus defined family $(\mathbf{R}_{s,t})_{0\le s<t\le\infty}$ of kernels has the property that

$$\mathbf{R}_{s,t} \circ \mathbf{R}_{t,v} \;=\; \mathbf{R}_{s,v} \;, \tag{11}$$

for $s < t < v$, i.e. the family $(\mathbf{R}_{s,t})_{0\le s<t\le\infty}$ is a so-called *hemigroup* of kernels relative to the kernel composition introduced in (8).

From now on we will assume that the family $\mathbf{Q}$ of structural kernels does not depend on $S^\infty(E)$ and is also a hemigroup relative to the composition of kernels, i.e. we assume that

$$\mathbf{Q}_{s,t} \circ \mathbf{Q}_{t,v} \;=\; \mathbf{Q}_{s,v} \tag{12}$$

holds for all $0 \le s < t < v \le \infty$. Then the following result holds.

3.2 Proposition. *Suppose that* $\mathbf{C}$ *is a strongly predictable compensator kernel and that* $\mathbf{Q}$ *does not depend on* $S^\infty(E)$*. Then there exists a probability measure* $\hat{\mathbf{Q}}$ *on* $D_0(\mathbb{R}_+, F)$ *and a kernel* $\hat{\mathbf{R}}$ *from* $D_0(\mathbb{R}_+, F)$ *to* $S^\infty(E)$*, such that for every* $\pi = (s_j)_{j\ge1} \in S^\infty$ *with* $\sup_{j\ge1} s_j = \infty$,

$$\mathbf{P}^{\pi,\mathbf{Q}} \;=\; \hat{\mathbf{R}} \circ \hat{\mathbf{Q}} \;, \tag{13}$$

which means by (8) that for every non-negative, bounded, measurable function H *on* Ω

$$\begin{aligned} &\int H(\omega)\mathbf{P}^{\pi,\mathbf{Q}}(d\omega) \\ &= \int_{D_0(\mathbb{R}_+,F)} \int_{S^\infty(E)} \hat{\mathbf{Q}}(df)\hat{\mathbf{R}}(f; d(\mathbf{t}, \mathbf{x}))H((\mathbf{t}, \mathbf{x}), f) . \end{aligned} \tag{14}$$

The measure $\hat{\mathbf{Q}}$ *is related to the kernel family* $\mathbf{Q}$ *in the following way. For every* $n \ge 1$ *and every* $B \in \mathcal{D}$,

$$f \mapsto \int 1_B(g)\mathbf{Q}_{s_{n-1},s_n}(f; dg)$$

is a version of the conditional probability $\hat{\mathbf{Q}}\{Y^{s_n} \in B | \mathcal{D}_{s_{n-1}}\}$*. In the present case, the compensator measure* $C = (C_t(A))_{t\geq 0, A\in\mathcal{B}(E)}$ *of the canonical marked point process is even the compensator measure relative to the filtration* $(\mathcal{D} \vee \mathcal{C}_t)_{t\geq 0}$.

Proof. We set $\pi_n = (s_j)_{1\leq j\leq n}$. Then it follows from theorem 2.3 that

$$\mathbf{P}^{\pi_n,\mathbf{Q}} = \mathbf{Q}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{Q}_{s_{n-1},s_n} \circ \mathbf{R}_{s_{n-1},s_n} .$$

Since $\mathbf{Q}$ does not depend on $S^\infty(E)$, we can change the order of integration in such a way that

$$\mathbf{P}^{\pi_n,\mathbf{Q}} = \big(\mathbf{Q}_{0,s_1} \circ \cdots \circ \mathbf{Q}_{s_{n-1},s_n}\big) \circ \big(\mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{R}_{s_{n-1},s_n}\big) .$$

Thus we get from (11) and (12)

$$\mathbf{P}^{\pi_n,\mathbf{Q}} = \mathbf{Q}_{0,s_n} \circ \mathbf{R}_{0,s_n} .$$

With

$$\hat{\mathbf{Q}} := \mathbf{Q}_{0,\infty} \quad \text{and} \quad \hat{\mathbf{R}} := \mathbf{R}_{0,\infty}$$

assertion (13) is proved.
The assertion that

$$\hat{\mathbf{Q}}\{Y^{s_n} \in B \mid \mathcal{D}_{s_{n-1}}\} = \int 1_B(g)\mathbf{Q}_{s_{n-1},s_n}(\,\cdot\,; dg) \quad \hat{\mathbf{Q}}\text{-a.s.},$$

is immediate from the construction of $\hat{\mathbf{Q}}$. □

The next result is an essential step to a solution of the stated problem.

3.3 Proposition. *Suppose that* $\mathbf{Q}$ *does not depend on* $S^\infty(E)$ *and that* $\mathbf{G} = (\mathbf{G}_{s,t})_{0\leq s<t\leq\infty}$ *is a second hemigroup of structural kernels. Let further as before* $\pi = (s_j)_{j\geq 1} \in S^\infty$ *with* $\sup_{j\geq 1} s_j = \infty$ *be given, and denote by* $\hat{\mathbf{P}}$ *the measure* $\mathbf{P}^{\pi,\mathbf{Q}} = \hat{\mathbf{R}} \circ \hat{\mathbf{Q}}$ *of proposition 3.2.*
(1) Suppose that for every $0 \leq s < t \leq \infty$, $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$, *and* $f \in D_0(\mathbb{R}_+, F)$, *the probability measures* $\mathbf{G}_{s,t}\big((\mathbf{t},\mathbf{x}), f; dg\big)$ *and* $\mathbf{Q}_{s,t}(f; dg)$ *are equivalent. Then there exists a positive process* $X = (X_{s_n})_{n\geq 0}$ *on* Ω *with the following properties:*
(i) For every $n \geq 1$ *the random variable* X_{s_n} *is* $\mathcal{C}_{s_{n-1}} \vee \mathcal{D}_{s_n}$*-measurable.*
(ii) $(X_{s_n})_{n\geq 1}$ *is a* $(\mathcal{C}_{s_n} \vee \mathcal{D}_{s_n})_{n\geq 0}$*-martingale such that* $X_0 = 1$.
(iii) For every $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$ *the process*

$$\big(X_{s_n}((\mathbf{t},\mathbf{x}),\,\cdot\,)\big)_{n\geq 0}$$

is a martingale relative to the filtration $(\mathcal{D}_{s_n})_{n\geq 0}$ *and the probability measure* $\hat{\mathbf{Q}}$.
(iv) For every $A \in \mathcal{C}$, $B \in \mathcal{D}$ *(as* σ*-algebras on* $S^\infty(E)$ *and* $D_0(\mathbb{R}_+, F)$ *resp.), and every* $n \geq 1$,

$$\begin{aligned} &\mathbf{P}^{\pi,\mathbf{G}}\big\{\varphi_{s_n}\big((\mathbf{T},\mathbf{X})\big) \in A, Y^{s_n} \in B\big\} \qquad (15)\\ &\quad = \int 1_A(\varphi_{s_n}(\mathbf{t},\mathbf{x}))1_B(f^{s_n})\, X_{s_n}((\mathbf{t},\mathbf{x}), f)\hat{\mathbf{P}}(d((\mathbf{t},\mathbf{x}), f)) . \end{aligned}$$

(v) For every $n \geq 1$,

$$\mathbf{G}_{s_{n-1},s_n}\big((\mathbf{t},\mathbf{x}),f;dg\big) \;=\; \frac{X_{s_n}\big((\mathbf{t},\mathbf{x}),g\big)}{X_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),f^{s_{n-1}}\big)}\mathbf{Q}_{s_{n-1},s_n}(f,dg)\;. \tag{16}$$

(vi) For every $B \in \mathcal{D}$ (as a σ-algebra on $D_0(\mathbb{R}_+,F)$) and $m < n$ the map

$$((\mathbf{t},\mathbf{x}),f) \mapsto \int_{D_0(\mathbb{R}_+,F)} 1_B(g)\Phi^{\pi}_{m,n}\big(\varphi_{s_m}((\mathbf{t},\mathbf{x})),g\big)\mathbf{Q}_{s_m,s_n}(f,dg)\;, \tag{17}$$

where

$$\Phi^{\pi}_{m,n}\big(\varphi_{s_m}((\mathbf{t},\mathbf{x})),g\big) \;=\; \int_{S^\infty(E)} \frac{X_{s_n}\big((\mathbf{s},\mathbf{y}),g\big)}{X_{s_m}\big((\mathbf{s},\mathbf{y}),g\big)}\mathbf{R}_{s_m,s_n}((\mathbf{t},\mathbf{x}),g;d(\mathbf{s},\mathbf{y}))\;, \tag{18}$$

is a version of the conditional probability

$$\mathbf{P}^{\pi,\mathbf{G}}\big\{Y^{s_n} \in B \,\big|\, \mathcal{C}_{s_m} \vee \mathcal{D}_{s_m}\big\}\;.$$

(2) Conversely, suppose that $X = (X_{s_n})_{n\geq 0}$ is a process on Ω with the properties (i), (ii) and (iii) of (1). Then the sequence $(X_{s_n} \cdot \hat{\mathbf{P}})_{n\geq 0}$ is a consistent sequence of probability measures on Ω, and the limit measure $\mathbf{P}^X$ can be given the following interpretation. We set

$$\mathbf{G}_{s_{n-1},s_n}\big((\mathbf{t},\mathbf{x}),f;dg\big) \;=\; \frac{X_{s_n}\big((\mathbf{t},\mathbf{x}),g\big)}{X_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),f\big)}\mathbf{Q}_{s_{n-1},s_n}(f,dg)\;.$$

Then $\mathbf{G}_{s_{n-1},s_n}\big((\mathbf{t},\mathbf{x}),f;dg\big)$ is a probability measure on $D_0(\mathbb{R}_+,F)$ for all $((\mathbf{t},\mathbf{x}),f) \in S^\infty(E)$ and $\mathbf{G}_{s_{n-1},s_n}$ is a stochastic kernel from Ω to $D_0(\mathbb{R}_+,F)$. Obviosly, for every $((\mathbf{t},\mathbf{x}),f) \in \Omega$ the probability measure $\mathbf{G}_{s_{n-1},s_n}\big((\mathbf{t},\mathbf{x}),f;dg\big)$ is equivalent to $\mathbf{Q}_{s_{n-1},s_n}(f,dg)$ with the density

$$\frac{X_{s_n}\big((\mathbf{t},\mathbf{x}),g\big)}{X_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),f\big)}\;.$$

Now define the probability measure $\mathbf{P}^{\pi,\mathbf{G}}$ as in section 2 by

$$\begin{aligned}&\mathbf{P}^{\pi,\mathbf{G}}\big\{\varphi_{s_n}\big((\mathbf{T},\mathbf{X})\big) \in A, Y^{s_n} \in B\big\} \\ &\quad= \int \big(\mathbf{G}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{G}_{s_{n-1},s_n} \circ \mathbf{R}_{s_{n-1},s_n}\big)(d\omega)\,1_{A\times B}(\omega)\;.\end{aligned} \tag{19}$$

Then $\mathbf{P}^X = \mathbf{P}^{\pi,\mathbf{G}}$.

Proof. By assumption and definition 2.2 we have for every $n \geq 1$,

$$\mathbf{G}_{s_{n-1},s_n}\big((\mathbf{t},\mathbf{x}),f;dg\big) \;=\; Z_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),g\big)\mathbf{Q}_{s_{n-1},s_n}(f;dg)\;,$$

where the Radon-Nikodym density $Z_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),\cdot\big)$ of $\mathbf{G}_{s_{n-1},s_n}\big((\mathbf{t},\mathbf{x}),f;\cdot\big)$ relative to $\mathbf{Q}_{s_{n-1},s_n}(f;\cdot)$ has this special form because of definition 2.2,(ii). Moreover, it follows from that definition also that $Z_{s_{n-1}}$ is $\mathcal{C}_{s_{n-1}} \vee \mathcal{D}_{s_n}$-measurable. Now we define for every $n \geq 1$,

$$X_{s_n}\big((\mathbf{t},\mathbf{x}), f^{s_n}\big) \;:=\; \prod_{j=0}^{n-1} Z_{s_j}\big((\mathbf{t},\mathbf{x}), f^{s_{j+1}}\big) \;\; ,$$

and set $X_0 := 1$. Then every X_{s_n} is clearly positive and $\mathcal{C}_{s_{n-1}} \vee \mathcal{D}_{s_n}$-measurable for $n \geq 1$, i.e. (i) holds.

For the proof of property (ii) we take arbitrary fixed $A \in \mathcal{C}_{s_{n-1}}$ and $B \in \mathcal{D}_{s_{n-1}}$, and get

$$\begin{aligned}
&\int_{A\cap B} X_{s_n}\, d\,\hat{\mathbf{P}} \\
&\quad= \int_{A\cap B} Z_{s_{n-1}} \cdots Z_0\, d\,\hat{\mathbf{P}} \\
&\quad= \int_{A\cap B} Z_{s_{n-1}} \cdots Z_0\, d\big(\mathbf{Q}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{Q}_{s_{n-2},s_{n-1}} \circ \mathbf{R}_{s_{n-2},s_{n-1}} \circ \mathbf{Q}_{s_{n-1},s_n}\big) \\
&\quad= \int_{\Omega} \Big(\int_{D_0(\mathbb{R}_+,F)} Z_{s_{n-1}}\, d\,\mathbf{Q}_{s_{n-1},s_n} \Big) (Z_{s_{n-2}} \cdots Z_0)\, 1_{A\cap B} \\
&\qquad\qquad\qquad\qquad d\big(\mathbf{Q}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{Q}_{s_{n-2},s_{n-1}} \circ \mathbf{R}_{s_{n-2},s_{n-1}}\big) \\
&\quad= \int_{\Omega} (Z_{s_{n-2}} \cdots Z_0)\, 1_{A\cap B}\, d\big(\mathbf{Q}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{Q}_{s_{n-2},s_{n-1}} \circ \mathbf{R}_{s_{n-2},s_{n-1}}\big) \\
&\quad= \int_{A\cap B} X_{s_{n-1}}\, d\,\hat{\mathbf{P}} \;\; ,
\end{aligned}$$

where we used that by assumption

$$\int_{D_0(\mathbb{R}_+,F)} Z_{s_{n-1}}\, d\,\mathbf{Q}_{s_{n-1},s_n} \;=\; \int_{D_0(\mathbb{R}_+,F)} d\,\mathbf{G}_{s_{n-1},s_n} \;=\; 1 \;\; . \tag{20}$$

It follows from the just derived equation that $(X_{s_n})_{n\geq 1}$ is a $(\mathcal{C}_{s_n} \vee \mathcal{D}_{s_n})_{n\geq 0}$-martingale and (ii) is proved.

The proof of (iii) is an immediate consequence of (20).

For the proof of (iv) we take $A \in \mathcal{C}$ and $B \in \mathcal{D}$ (as σ-algebras on $S^\infty(E)$ and $D_0(\mathbb{R}_+, F)$ resp.). Then

$$\begin{aligned}
&\mathbf{P}^{\pi,\mathbf{G}}\big\{\varphi_{s_n}\big((\mathbf{T},\mathbf{X})\big) \in A, Y^{s_n} \in B\big\} \\
&\quad= \int 1_{A\times B}\, d\big(\mathbf{G}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{G}_{s_{n-1},s_n} \circ \mathbf{R}_{s_{n-1},s_n}\big) \\
&\quad= \int 1_{A\times B}\, d\big((Z_0\mathbf{Q}_{0,s_1}) \circ \mathbf{R}_{0,s_1} \circ \cdots \circ ((Z_{s_{n-1}}\mathbf{Q}_{s_{n-1},s_n}) \circ \mathbf{R}_{s_{n-1},s_n}\big) \\
&\quad= \int 1_{A\times B} X_{s_n}\, d\big(\mathbf{Q}_{0,s_1} \circ \mathbf{R}_{0,s_1} \circ \cdots \circ \mathbf{Q}_{s_{n-1},s_n} \circ \mathbf{R}_{s_{n-1},s_n}\big) \\
&\quad= \int 1_A((\mathbf{t},\mathbf{x}))1_B(f)\, X_{s_n}((\mathbf{t},\mathbf{x}), f)\hat{\mathbf{P}}(d((\mathbf{t},\mathbf{x}), f)) \;\; ,
\end{aligned}$$

and (iv) is proved.

The proof of (v) follows from

$$\frac{X_{s_n}\big((\mathbf{t},\mathbf{x}),f^{s_n}\big)}{X_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),f^{s_{n-1}}\big)} \quad = \quad Z_{s_{n-1}}\big((\mathbf{t},\mathbf{x}),f^{s_n}\big) \ .$$

For the proof of (vi) we take a $C\in\mathcal{C}$ and a $D\in\mathcal{D}$ (again as σ-algebras on $S^\infty(E)$ and $D_0(\mathbb{R}_+,F)$ resp.). Then we get

$$\begin{aligned}
&\int_{C\times D} \mathbf{P}^{\pi,\mathbf{G}}\big\{Y^{s_n}\in B\,\big|\,\mathcal{C}_{s_m}\vee\mathcal{D}_{s_m}\big\}\,d\mathbf{P}^{\pi,\mathbf{G}}\\
&= \int 1_C(\varphi_{s_m}((\mathbf{T},\mathbf{X})))1_B(Y^{s_n})1_D(Y^{s_m})\,d\mathbf{P}^{\pi,\mathbf{G}} \qquad (21)\\
&= \int 1_C(\varphi_{s_m}((\mathbf{T},\mathbf{X}))1_B(Y^{s_n})1_D(Y^{s_m})\\
&\qquad\qquad d\big(\mathbf{G}_{0,s_1}\circ\mathbf{R}_{0,s_1}\circ\cdots\circ\mathbf{R}_{s_{n-2},s_{n-1}}\circ\mathbf{G}_{s_{n-1},s_n}\big)\\
&= \int\Big\{\int 1_B(Y^{s_n})d\big(\mathbf{G}_{s_m,s_{m+1}}\circ\mathbf{R}_{s_m,s_{m+1}}\circ\cdots\circ\mathbf{G}_{s_{n-1},s_n}\big)\Big\}1_C(\varphi_{s_m}((\mathbf{T},\mathbf{X})))\\
&\qquad\cdot\ 1_D(Y^{s_m})\,d\big(\mathbf{G}_{0,s_1}\circ\mathbf{R}_{0,s_1}\circ\cdots\circ\mathbf{G}_{s_{m-1},s_m}\circ\mathbf{R}_{s_{m-1},s_m}\big)\ .
\end{aligned}$$

For the inner integral in the last line of (21) we get more precisely

$$\begin{aligned}
&\int 1_B(Y^{s_n})d\big(\mathbf{G}_{s_m,s_{m+1}}\circ\mathbf{R}_{s_m,s_{m+1}}\circ\cdots\circ\mathbf{G}_{s_{n-1},s_n}\big)\\
&= \int 1_B(Y^{s_n})Z_{s_m}\cdots Z_{s_{n-1}}\,d(\mathbf{Q}_{s_m,s_n}\circ\mathbf{R}_{s_m,s_n})\\
&= \int 1_B(g)\Big\{\int\frac{X_{s_n}((\mathbf{s},\mathbf{y}),g)}{X_{s_n}((\mathbf{s},\mathbf{y}),g)}\\
&\qquad\qquad \mathbf{R}_{s_m,s_n}((\mathbf{T},\mathbf{X}),g;d(\mathbf{s},\mathbf{y}))\Big\}\mathbf{Q}_{s_m,s_n}(Y^{s_m};dg)\\
&= \int 1_B(g)\Phi^\pi_{m,n}(\varphi_{s_m}((\mathbf{T},\mathbf{X}),g)\mathbf{Q}_{s_m,s_n}(Y^{s_m};dg)\ .
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{C\times D} \mathbf{P}^{\pi,\mathbf{G}}\big\{Y^{s_n}\in B\,\big|\,\mathcal{C}_{s_m}\vee\mathcal{D}_{s_m}\big\}\,d\mathbf{P}^{\pi,\mathbf{G}}\\
&= \int\Big\{\int 1_B(g)\Phi^\pi_{m,n}(\varphi_{s_m}((\mathbf{T},\mathbf{X}),g)\mathbf{Q}_{s_m,s_n}(Y^{s_m};dg)\Big\}1_C(\varphi_{s_m}((\mathbf{T},\mathbf{X})))\\
&\qquad\cdot\ 1_D(Y^{s_m})\,d\big(\mathbf{G}_{0,s_1}\circ\mathbf{R}_{0,s_1}\circ\cdots\circ\mathbf{G}_{s_{m-1},s_m}\circ\mathbf{R}_{s_{m-1},s_m}\big)\ ,
\end{aligned}$$

and (vi) is proved.

(2): Since $(X_{s_n})_{n\ge1}$ is a $(\mathcal{C}_{s_n}\vee\mathcal{D}_{s_n})_{n\ge0}$-martingale by property (ii), we have

$$\int_{A\cap B} X_{s_n}d\,\hat{\mathbf{P}} \quad = \quad \int_{A\cap B} X_{s_{n-1}}d\,\hat{\mathbf{P}}$$

for every $A \in \mathcal{C}_{s_{n-1}}$ and $B \in \mathcal{D}_{s_{n-1}}$. This shows that

$$X_{s_n} \cdot \hat{\mathbf{P}}|_{\mathcal{C}_{s_{n-1}} \vee \mathcal{D}_{s_{n-1}}} = X_{s_{n-1}} \cdot \hat{\mathbf{P}} .$$

Thus the sequence $(X_{s_n} \cdot \hat{\mathbf{P}})_{n \geq 1}$ is a consistent family of probability measures, and there exists a probability measure $\mathbf{P}^X$ on Ω such that

$$\mathbf{P}^X|_{\mathcal{C}_{s_n} \vee \mathcal{D}_{s_n}} = X_{s_n} \cdot \hat{\mathbf{P}}$$

for every $n \geq 0$. By property (iii) every G_{s_{n-1},s_n} is a stochastic kernel and

$$\begin{aligned} & X_n \hat{\mathbf{P}}|_{\mathcal{C}_{s_n} \vee \mathcal{D}_{s_n}} \\ &= (X_{s_1} \mathbf{Q}_{0,s_1}) \circ \mathbf{R}_{0,s_1} \circ \cdots \circ (\frac{X_{s_n}}{X_{s_{n-1}}} \mathbf{Q}_{s_{n-1},s_n}) \circ \mathbf{R}_{s_{n-1},s_n} . \end{aligned}$$

The assertion $\mathbf{P}^X = \mathbf{P}^{\pi,\mathbf{G}}$ is then obvious. $\square$

Now we can formulate the main result of this section.

3.4 Theorem. *Let $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ denote the filtration on $\Omega = S^\infty(E) \times D_0(\mathbf{R}_+, F)$, defined by $\mathcal{F}_t := \mathcal{C}_t \vee \mathcal{D}_t$, and suppose that $X = (X_t)_{t \geq 0}$ is a process on Ω with the properties*
(1) X is a positive, continuous $\mathbf{F}$-martingale relative to $\hat{\mathbf{P}} = \hat{\mathbf{R}} \circ \hat{\mathbf{Q}}$ with $X_0 = 1$, and
(2) for every $(\mathbf{t}, \mathbf{x}) \in S^\infty(E)$ the process $\big(X_t((\mathbf{t}, \mathbf{x}), \cdot)\big)_{t \geq 0}$ is a $(\mathcal{D}_t)$-martingale relative to $\hat{\mathbf{Q}}$.
Then $\big(X_t \cdot \hat{\mathbf{P}}\big)_{t \geq 0}$ is a consistent family of probability measures on the measurable spaces $(\Omega, \mathcal{F}_t)$, and the limit $\mathbf{P}^X$ is a probability measure on $(\Omega, \mathcal{F})$ with the properties:
(i) The canonical marked point process $((T_n, X_n))_{n \geq 1}$ has the $\mathbf{F}$-compensator measure C given by

$$C_t(B)(\omega) = \mathbf{C}_t\big(((T_k(\omega), X_k(\omega)))_{k \geq 1}, Y(\omega); B\big) \tag{22}$$

for $t \geq 0$ and $B \in \mathcal{B}(E)$,
(ii) for every $C \in \mathcal{C}$, $D \in \mathcal{D}$ (again as σ-algebras on $S^\infty(E)$ and $D_0(\mathbb{R}_+, F)$ resp.) and $s \geq 0$ the probability measures $\mathbf{P}^X$ and $\hat{\mathbf{P}}$ are related by

$$\begin{aligned} & \mathbf{P}^X\big\{\varphi_s\big((\mathbf{T}, \mathbf{X})\big) \in C, Y^s \in D\big\} \qquad (23) \\ &= \int 1_C(\varphi_s(\mathbf{t}, \mathbf{x})) 1_D(f^s)\, X_s((\mathbf{t}, \mathbf{x}), f) \hat{\mathbf{P}}(d((\mathbf{t}, \mathbf{x}), f)) , \end{aligned}$$

(iii) For every $D \in \mathcal{D}$ (as a σ-algebra on $D_0(\mathbb{R}_+, F)$) and $0 \leq s < t$ the map

$$((\mathbf{t}, \mathbf{x}), f) \mapsto \int_{D_0(\mathbb{R}_+, F)} 1_D(g) \Phi_{s,t}\big((\mathbf{t}, \mathbf{x}), g\big) \mathbf{Q}_{s,t}(f, dg) , \tag{24}$$

where

$$\Phi_{s,t}\big((\mathbf{t}, \mathbf{x}), g\big) = \int_{S^\infty(E)} \frac{X_t\big((\mathbf{s}, \mathbf{y}), g\big)}{X_s\big((\mathbf{s}, \mathbf{y}), g\big)} \mathbf{R}_{s,t}((\mathbf{t}, \mathbf{x}), g; d(\mathbf{s}, \mathbf{y})) , \tag{25}$$

is a version of the conditional probability

$$\mathbf{P}^X\big\{Y^t \in D \,\big|\, \mathcal{F}_s\big\} \ .$$

(iv) Furthermore, if we set

$$\mathbf{G}_{s,t}\big((\mathbf{t},\mathbf{x}), f; dg\big) \ = \ \frac{X_t\big(\varphi_s(\mathbf{t},\mathbf{x}), g\big)}{X_s\big((\mathbf{t},\mathbf{x}), f\big)} \mathbf{Q}_{s,t}(f, dg) \ , \tag{26}$$

we can define for every $\pi \in S^\infty$ *the probability measure* $\mathbf{P}^{\pi,\mathbf{G}}$ *as in (19). Then for every sequence* $(\pi_n)_{n\geq 1}$ *in* S^∞ *with* $\lim_{n\to\infty} |\pi_n| = 0$ *we have*

$$\lim_{n\to\infty} \mathbf{P}^{\pi_n,\mathbf{G}} \ = \ \mathbf{P}^X \tag{27}$$

in the sense that for every $t \geq 0$

$$\lim_{n\to\infty} \sup_{A\in\mathcal{F}_t} |\mathbf{P}^X\{A\} - \mathbf{P}^{\pi_n,\mathbf{G}}\{A\}| \ = \ 0 \ . \tag{28}$$

Proof. The idea of the proof is to approximate the given process X by a sequence $(X^n)_{n\geq 1}$ of discrete time processes as considered in proposition 3.3. Thus we start with the proof of assertion (iv).

We take a fixed sequence $(\pi^n)_{n\geq 1}$ of elements $\pi^n = (s_j^n)_{j\geq 1} \in S^\infty$ such that $\pi^n \subset \pi^{n+1}$ for every $n \geq 1$ and $\lim_{n\to\infty} |\pi^n| = 0$.

Then we define for every $n \geq 1$ the discrete time process $(X^n_{s_k^n})_{k\geq 1}$ by

$$X^n_{s_k^n}((\mathbf{t},\mathbf{x}), f) \ := \ \prod_{j=1}^k \frac{X_{s_j^n}\big(\varphi_{s_{j-1}^n}((\mathbf{t},\mathbf{x})), f\big)}{X_{s_{j-1}^n}\big(\varphi_{s_{j-1}^n}((\mathbf{t},\mathbf{x})), f\big)} \ , \tag{29}$$

and set $X_0^n = 1$. By definition and assumption (2) it follows immediately that every process $X^n = (X^n_{s_k^n})_{k\geq 1}$ has the properties (i), (ii) and (iii) of part (1) of proposition 3.3. Now the main part of the proof of (iv) consists in showing that $\lim_{n\to\infty} X_t^n = X_t$ in L^1 for every $t > 0$, where $X_t^n := X^n_{s_k^n}$ for $s_{k-1}^n < t \leq s_k^n$.

For the following we assume w.l.o.g. that the given $t > 0$ belongs to every partition π^n, i.e. $t = s_{m_n}^n$ for some $m_n \geq 1$. We will further assume first that $c \leq X_s \leq C$ for all $s \leq t$, where $0 < c < C < \infty$ are constants. As usual, we denote by $[X]$ the quadratic variation of X. Then the Ito-formula gives

$$\log X_t \ = \ \int_0^t \frac{1}{X_s}\, dX_s - \frac{1}{2}\int_0^t \frac{1}{X_s^2}\, d[X]_s \ . \tag{30}$$

Using this formula, we get

$$\begin{aligned} &\log X_t^n - \log X_t \\ &\quad = \ R_n^1 + R_n^2 - \frac{1}{2}R_n^3 + R_n^4 - \frac{1}{2}R_n^5 \ , \end{aligned}$$

where

$$R_n^1 \;:=\; \log X^n_{s^n_{m_n}} - \sum_{j=1}^{m_n} \frac{X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)} + \frac{1}{2}\sum_{j=1}^{m_n} \frac{\Big(X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)\Big)^2}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)^2} \quad ,$$

$$R_n^2 \;:=\; \sum_{j=1}^{m_n}\Big[\frac{X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)} - \frac{X_{s^n_j}(\cdot,\cdot) - X_{s^n_{j-1}}(\cdot,\cdot)}{X_{s^n_{j-1}}(\cdot,\cdot)}\Big] \quad ,$$

$$R_n^3 \;:=\; \sum_{j=1}^{m_n}\Big[\frac{\Big(X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)\Big)^2}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)^2} - \frac{\big(X_{s^n_j}(\cdot,\cdot) - X_{s^n_{j-1}}(\cdot,\cdot)\big)^2}{X_{s^n_{j-1}}(\cdot,\cdot)^2}\Big] \quad ,$$

$$R_n^4 \;:=\; \sum_{j=1}^{m_n} \frac{X_{s^n_j}(\cdot,\cdot) - X_{s^n_{j-1}}(\cdot,\cdot)}{X_{s^n_{j-1}}(\cdot,\cdot)} - \int_0^t \frac{1}{X_s}\, dX_s \quad ,$$

$$R_n^5 \;:=\; \sum_{j=1}^{m_n} \frac{\big(X_{s^n_j}(\cdot,\cdot) - X_{s^n_{j-1}}(\cdot,\cdot)\big)^2}{X_{s^n_{j-1}}(\cdot,\cdot)^2} - \int_0^t \frac{1}{X_s^2}\, d[X]_s \quad .$$

We will prove that $\lim_{n\to\infty} R_n^i = 0$ for $i = 1, \cdots, 5$ in probability.
(α) $\lim R_n^1 = 0$: From the Taylor expansion

$$\log(1+x) \;=\; x - \frac{1}{2}x^2 + \frac{1}{3}\frac{1}{(1+\theta x)^3}x^3 \quad ,$$

with $0 \le \theta \le 1$, we get

$$|R_n^1| \le \frac{1}{3}\sum_{j=1}^{m_n} \frac{1}{\Big(1+\theta_j \frac{X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)}\Big)^3} \cdot \frac{\Big|X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)\Big|^3}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)^3} \quad .$$

From the assumption that $c \le X_s \le C$ for all $s \ge 0$ one derives that

$$1 + \theta_j \frac{X_{s^n_j}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big) - X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)}{X_{s^n_{j-1}}\big(\varphi_{s^n_{j-1}}(\cdot),\cdot\big)} \;\ge\; \frac{c}{C}$$

and hence we get

$$
\begin{aligned}
|R_n^1| &\leq \frac{1}{3}\frac{C^3}{c^6}\sum_{j=1}^{m_n}\left|X_{s_j^n}\left(\varphi_{s_{j-1}^n}(\cdot),\cdot\right) - X_{s_{j-1}^n}\left(\varphi_{s_{j-1}^n}(\cdot),\cdot\right)\right|^3 \\
&\leq \frac{4}{3}\frac{C^3}{c^6}(S_n^1 + S_n^2) ,
\end{aligned}
$$

with

$$
S_n^1 := \sum_{j=1}^{m_n}\left|X_{s_j^n}(\cdot,\cdot) - X_{s_{j-1}^n}(\cdot,\cdot)\right|^3
$$

and

$$
S_n^2 := \sum_{j=1}^{m_n}\left|X_{s_j^n}(\cdot,\cdot) - X_{s_j^n}\left(\varphi_{s_{j-1}^n}(\cdot),\cdot\right)\right|^3 .
$$

Since

$$
S_n^1 \leq \max_{1\leq j\leq m_n}\left|X_{s_j^n}(\cdot,\cdot) - X_{s_{j-1}^n}(\cdot,\cdot)\right|\sum_{j=1}^{m_n}\left|X_{s_j^n}(\cdot,\cdot) - X_{s_{j-1}^n}(\cdot,\cdot)\right|^2 ,
$$

and

$$
\lim_{n\to\infty}\sum_{j=1}^{m_n}\left|X_{s_j^n}(\cdot,\cdot) - X_{s_{j-1}^n}(\cdot,\cdot)\right|^2 = [X]_t
$$

in L^1, the assumption of the continuity of X implies that

$$
\lim_{n\to\infty} S_n^1 = 0
$$

in probability.

For S_n^2 we consider a fixed $\omega = ((\mathbf{t},\mathbf{x}),f) \in S^\infty(E)\times D_0(\mathbb{R}_+,F)$ with $(\mathbf{t},\mathbf{x}) = ((t_k,x_k))_{k\geq 1}$. If $t < t_1$, then obviously $S_n^2 = 0$. Hence suppose $t \geq t_1$ and set $m := \max\{k\geq 1 | t_k \leq t\}$. If $|\pi^n| < \max_{1\leq k\leq m}|t_k - t_{k-1}|$, and $j(k) := \min\{s_j^n | s_j^n > t_k\}$ for $k = 1,\cdots,m$, then

$$
S_n^2((\mathbf{t},\mathbf{x}),f) = \sum_{k=1}^{m}\left|X_{s_{j(k)}^n}((\mathbf{t},\mathbf{x}),f) - X_{s_{j(k)}^n}(\varphi_{s_{j(k)-1}^n}(\mathbf{t},\mathbf{x}),f)\right|^3 , \tag{31}
$$

and hence

$$
\lim_{n\to\infty} S_n^2((\mathbf{t},\mathbf{x}),f) = \sum_{k=1}^{m}\left|X_{t_k+}((\mathbf{t},\mathbf{x}),f) - X_{t_k+}(\varphi_{t_k-}(\mathbf{t},\mathbf{x}),f)\right|^3 .
$$

The continuity assumption on X now implies $\lim_{n\to\infty} S_n^2 = 0$ $\hat{\mathbf{P}}$-a.s.. Altogether, $\lim_{n\to\infty} R_n^1 = 0$ in probability is proved.

(β) $\lim R_n^2 = 0$: As in the last part of the proof of (α) let $\omega = ((\mathbf{t},\mathbf{x}), f) \in S^\infty(E) \times D_0(\mathbb{R}_+, F)$ with $(\mathbf{t},\mathbf{x}) = ((t_k, x_k))_{k\geq 1}$ be fixed. Then we get with the notations above for the non-trivial case $t \geq t_1$

$$\begin{aligned}
|R_n^2((\mathbf{t},\mathbf{x}), f)| &\leq c^{-1} \sum_{j=1}^{m_n} \left| X_{s_j^n}((\mathbf{t},\mathbf{x}), f) - X_{s_j^n}(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f) \right| \\
&= c^{-1} \sum_{k=1}^{m} \left| X_{s_{j(k)}^n}((\mathbf{t},\mathbf{x}), f) - X_{s_{j(k)}^n}(\varphi_{s_{j(k)-1}^n}(\mathbf{t},\mathbf{x}), f) \right| ,
\end{aligned}$$

and hence

$$\limsup_{n\to\infty} |R_n^2((\mathbf{t},\mathbf{x}), f)| \leq c^{-1} \sum_{k=1}^{m} \left| X_{t_k+}((\mathbf{t},\mathbf{x}), f) - X_{t_k+}(\varphi_{t_k-}(\mathbf{t},\mathbf{x}), f) \right| = 0 ,$$

and $\lim_{n\to\infty} R_n^2 = 0$ in probability follows.

(γ) $\lim R_n^3 = 0$: Now we have for the fixed given ω

$$\begin{aligned}
&|R_n^3((\mathbf{t},\mathbf{x}), f)| \\
&\leq c^{-2} \sum_{j=1}^{m_n} \Big[\Big(X_{s_j^n}(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f) - X_{s_{j-1}^n}(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f) \Big)^2 \\
&\qquad\qquad - \Big(X_{s_j^n}((\mathbf{t},\mathbf{x}), f) - X_{s_{j-1}^n}((\mathbf{t},\mathbf{x}), f) \Big)^2 \Big] \\
&\leq c^{-2} \big(A_n^1((\mathbf{t},\mathbf{x}), f) + 2A_n^2((\mathbf{t},\mathbf{x}), f) \big) ,
\end{aligned}$$

with

$$A_n^1((\mathbf{t},\mathbf{x}), f) = \sum_{j=1}^{m_n} \big(X_{s_j^n}((\mathbf{t},\mathbf{x}), f) - X_{s_j^n}(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f) \big)^2$$

and

$$\begin{aligned}
A_n^2((\mathbf{t},\mathbf{x}), f) = \sum_{j=1}^{m_n} &\left| X_{s_j^n}((\mathbf{t},\mathbf{x}), f) - X_{s_j^n}(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f) \right| \\
&\cdot \left| X_{s_j^n}\big(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f\big) - X_{s_{j-1}^n}\big(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f\big) \right| .
\end{aligned}$$

Now

$$\lim_{n\to\infty} A_n^1((\mathbf{t},\mathbf{x}), f) = 0$$

follows as $\lim_{n\to\infty} S_n^2 = 0$ in (α). For $A_n^2((\mathbf{t},\mathbf{x}), f)$ we have

$$A_n^2((\mathbf{t},\mathbf{x}), f) \leq 2C \sum_{j=1}^{m_n} \left| X_{s_j^n}((\mathbf{t},\mathbf{x}), f) - X_{s_j^n}(\varphi_{s_{j-1}^n}(\mathbf{t},\mathbf{x}), f) \right| ,$$

and

$$\lim_{n\to\infty} A_n^2((\mathbf{t},\mathbf{x}), f) = 0 .$$

follows. Thus $\lim_{n\to\infty} R_n^3((\mathbf{t},\mathbf{x}),f) = 0$ for all $((\mathbf{t},\mathbf{x}),f)$, and especially, $\lim_{n\to\infty} R_n^3 = 0$ in probability.

(δ) $\lim R_n^4 = 0$: By our assumption $s \mapsto X_s^{-1}$ is a continuous function bounded by c^{-1}. If we set

$$F_n \;:=\; \sum_{j\geq 1} 1_{]s_{j-1}^n, s_j^n]} X_{s_{j-1}^n}^{-1} \quad ,$$

then

$$\lim_{n\to\infty} F_n(s) \;=\; X_s^{-1}$$

and

$$\int_0^t F_n(s)dX_s \;=\; \sum_{j=1}^{m_n} \frac{X_{s_j^n}(\cdot,\cdot) - X_{s_{j-1}^n}(\cdot,\cdot)}{X_{s_{j-1}^n}(\cdot,\cdot)} \quad ,$$

and $\lim_{n\to\infty} R_n^4 = 0$ in probability (even in L^2) follows easily.

(ϵ) $\lim R_n^5 = 0$: We have

$$R_n^5 \;=\; B_n^1 + B_n^2$$

with

$$B_n^1 \;:=\; \sum_{j=1}^{m_n} \frac{\big(X_{s_j^n}(\cdot,\cdot) - X_{s_{j-1}^n}(\cdot,\cdot)\big)^2 \;-\; \big([X]_{s_j^n}(\cdot,\cdot) - [X]_{s_{j-1}^n}(\cdot,\cdot)\big)}{X_{s_{j-1}^n}(\cdot,\cdot)^2} \quad ,$$

and

$$B_n^2 \;:=\; \sum_{j=1}^{m_n} \frac{[X]_{s_j^n}(\cdot,\cdot) - [X]_{s_{j-1}^n}(\cdot,\cdot)}{X_{s_{j-1}^n}(\cdot,\cdot)^2} - \int_0^t \frac{1}{X_s^2} d[X]_s \quad .$$

Since

$$B_n^1 \;\leq\; c^{-2}\Big(\sum_{j=1}^{m_n} \big(X_{s_j^n} - X_{s_{j-1}^n}\big)^2 \;-\; [X]_t\Big) \quad ,$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{m_n} \big(X_{s_j^n} - X_{s_{j-1}^n}\big)^2 \;=\; [X]_t$$

in L^1, $\lim_{n\to\infty} B_n^1 = 0$ in L^1. For B_n^2 we set (similar as in (δ))

$$G_n \;:=\; \sum_{j\geq 1} 1_{]s_{j-1}^n, s_j^n]} X_{s_{j-1}^n}^{-2} \quad .$$

Then

$$\lim_{n\to\infty} G_n(s) \;=\; X_s^{-2} \;,\; \text{and}$$

$$\sum_{j=1}^{m_n} \frac{[X]_{s_j^n}(\cdot,\cdot) - [X]_{s_{j-1}^n}(\cdot,\cdot)}{X_{s_{j-1}^n}(\cdot,\cdot)^2} \quad = \quad \int_0^t G_n(s) d[X]_s \ ,$$

and hence

$$\lim_{n\to\infty} B_n^2 \quad = \quad 0 \quad \hat{\mathbf{P}}\text{-a.s.}.$$

This proves (ϵ).

Altogether we have proved

$$\lim_{n\to\infty} \log X_t^n \quad = \quad \log X_t$$

in probability for all $t \geq 0$ under the special assumption $c \leq X_t \leq C$ for all $t \geq 0$. If this assumption does not hold, we define the stopping times

$$S_m \quad := \quad \inf\left\{t \geq 0 \middle| X_t < \frac{1}{m} \ \ or \ \ X_t > m\right\}$$

$(m \geq 1)$. Since X is a continuous, positive process, $\lim_{m\to\infty} S_m = \infty$ $\hat{\mathbf{P}}$-a.s.. Therefore,

$$\begin{aligned}
&\hat{\mathbf{P}}\left\{ \left|\log X_t^n - \log X_t\right| > \delta\right\} \\
&\quad \leq \ \hat{\mathbf{P}}\left\{S_m \leq t\right\} + \hat{\mathbf{P}}\left(\left\{S_m > t\right\} \cap \left\{\left|\log X_t^n - \log X_t\right| > \delta\right\}\right) \\
&\quad \leq \ \hat{\mathbf{P}}\left\{S_m \leq t\right\} + \hat{\mathbf{P}}\left\{\left|\log X_{S_m\wedge t}^n - \log X_{S_m\wedge t}\right| > \delta\right\} \ .
\end{aligned}$$

For m large enough, we have $\hat{\mathbf{P}}\left\{S_m \leq t\right\} < \frac{\varepsilon}{2}$. For that m there exists an n_0 such that

$$\hat{\mathbf{P}}\left\{\left|\log X_{S_m\wedge t}^n - \log X_{S_m\wedge t}\right| > \delta\right\} \quad < \quad \frac{\varepsilon}{2}$$

for all $n \geq n_0$, and it follows that

$$\lim_{n\to\infty} \log X_t^n \quad = \quad \log X_t$$

in probability without any boundedness restriction. Thus we have finally obtained that

$$\lim_{n\to\infty} X_t^n \quad = \quad X_t$$

in probability.

Since $\mathbf{E}X_t^n = \mathbf{E}X_t = 1$ for all $n \geq 1$ and since the random variables X_t^n are positive, the sequence $(X_t^n)_{n\geq 1}$ is uniformly integrable, which now implies that

$$\lim_{n\to\infty} X_t^n \quad = \quad X_t$$

even in L^1.

Now consider the family $(X_t \cdot \hat{\mathbf{P}})_{t\geq 0}$ of probability measures on the measurable spaces $(\Omega, \mathcal{F}_t)$. The consistency of this family is an immediate consequence of the martingale property of X. Thus there exists a unique probability measure $\mathbf{P}^X$ on $(\Omega, \mathcal{F})$ such that

$$\mathbf{P}^X|_{\mathcal{F}_t} \;=\; X_t \cdot \hat{\mathbf{P}}$$

for every $t \geq 0$. Similarly, for the discrete time process X^n defined in (29) there exists a unique probability measure $\mathbf{P}^{X^n}$ such that

$$\mathbf{P}^{X^n}|_{\mathcal{F}_t} \;=\; X_t^n \cdot \hat{\mathbf{P}}$$

for every $t \geq 0$. Furthermore, it follows from proposition 3.3 that

$$\mathbf{P}^{\pi,\mathbf{G}} \;=\; \mathbf{P}^{X^n} \,.$$

From $\lim_{n\to\infty} X_t^n = X_t$ in L^1 for $t \geq 0$ we get for every $A \in \mathcal{F}_t$ that

$$\begin{aligned}
|\mathbf{P}^X(A) - \mathbf{P}^{\pi,\mathbf{G}}(A)| &= |\mathbf{P}^X(A) - \mathbf{P}^{X^n}(A)| \\
&= \Big| \int 1_A \, X_t d\hat{\mathbf{P}} - \int 1_A \, X_t^n d\hat{\mathbf{P}} \Big| \\
&\leq \int |X_t - X_t^n| d\hat{\mathbf{P}} \,,
\end{aligned}$$

and (28) follows. Thus we have proved assertion (iv) of the theorem.

Proof of (i): We have to prove that relative to the limit measure $\mathbf{P}^X$ the point process $((T_n, X_n))_{n\geq 1}$ has the asserted $\mathbf{F}$-compensator measure C. For every $m \geq 1$ we have

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}^X}\{N_{T_m\wedge t}(B)\} &= \int N_{T_m\wedge t}(B) d\mathbf{P}^X \\
&= \int N_{T_m\wedge t}(B) X_t d\hat{\mathbf{P}} \\
&= \lim_{n\to\infty} \int N_{T_m\wedge t}(B) X_t^n d\hat{\mathbf{P}} \\
&= \lim_{n\to\infty} \int C_{T_m\wedge t}(B) X_t^n d\hat{\mathbf{P}} \\
&= \int C_{T_m\wedge t}(B) X_t d\hat{\mathbf{P}} \\
&= \int C_{T_m\wedge t}(B) d\mathbf{P}^X \,,
\end{aligned}$$

since by proposition 3.3 every process $(X_t^n)_{t\geq 0}$ defines a probability measure $\mathbf{P}^{X^n}$ for which $((T_n, X_n))_{n\geq 1}$ has the $\mathbf{F}$-compensator measure C. This finishes the proof of (i).

Proof of (ii): This follows from (15) and (28).

Proof of (iii): For $D \in \mathcal{D}$ and $A \in \mathcal{F}_s$ we obtain for $t > s$

$$\begin{aligned}
\int_A \mathbf{P}^X\{Y^t \in D \,|\, \mathcal{F}_s\} d\mathbf{P}^X &= \int_A 1_D(Y^t) d\mathbf{P}^X \\
&= \int 1_A 1_D(Y^t) \frac{X_t}{X_s} X_s \, d\hat{\mathbf{P}} \\
&= \int 1_A 1_D(Y^t) \frac{X_t}{X_s} X_s \, d\big((\mathbf{Q}_{0,s} \circ \mathbf{R}_{0,s}) \circ (\mathbf{Q}_{s,t} \circ \mathbf{R}_{s,t})\big) \\
&= \int 1_A \, X_s \Big(\int 1_D(Y^t) \frac{X_t}{X_s} \, d(\mathbf{Q}_{s,t} \circ \mathbf{R}_{s,t}) \Big) \, d(\mathbf{Q}_{0,s} \circ \mathbf{R}_{0,s}) \\
&= \int 1_A \Big(\int 1_D \Phi_{s,t} \, d\mathbf{Q}_{s,t} \Big) X_s \, d(\mathbf{Q}_{0,s} \circ \mathbf{R}_{0,s}) \\
&= \int_A \Big(\int 1_D \Phi_{s,t} \, d\mathbf{Q}_{s,t} \Big) d\, \mathbf{P}^X \ ,
\end{aligned}$$

and (iii) follows. □

Remark. (1) Theorem 3.4 remains true, if we demand instead of condition (1) the weaker condition that X is a positive cadlag $\mathbf{F}$-martingale with $X_0 = 1$ such that X is predictable on $S^\infty(E)$ in the sense that

$$X_t((\mathbf{t}, \mathbf{x}), \cdot) \;=\; X_t(\varphi_{t-}((\mathbf{t}, \mathbf{x})), \cdot)$$

for every $(\mathbf{t}, \mathbf{x}) \in S^\infty(E)$. The proof makes a little bit more effort, since the Ito formula for $\log X_t$ (see (30)) is now a little bit more complicated because of the possible jumps of $(X_t)_{t \geq 0}$. If $(X_t^c)_{t \geq 0}$ denotes the continuous part of $(X_t)_{t \geq 0}$, one has to use now the formula

$$\log X_t \;=\; \int_0^t \frac{1}{X_s} dX_s^c - \frac{1}{2} \int_0^t \frac{1}{X_s^2} d[X^c]_s + \sum_{s \leq t} \Delta \log X_s \ .$$

(2) Condition (2) of the theorem follows in a certain sense from condition (1). More precisely, one can prove that condition (1) or the condition stated in remark (1) implies that there is a modification of $(X_t)_{t \geq 0}$ such that (2) holds for this modification.

Especially part (iv) of the last theorem shows in connection with proposition 3.3 that the probability measure $\mathbf{P}^X$ presents a model of a point process, where continuous mixing takes place. For this reason, we will call in the following the process Y the *mixing process.*

It seems not to be obviuos whether martingales of the type considered in the above theorem exist. Our next result is concerned with this problem.

3.6 Proposition. *We denote again by* $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ *the filtration on* $\Omega = S^\infty(E) \times D_0(\mathbf{R}_+, F)$*, defined by* $\mathcal{F}_t := \mathcal{C}_t \vee \mathcal{D}_t$.
(1) Suppose that $Z = (Z_t)_{t \geq 0}$ *is a continuous, square integrable* $(\mathcal{D}_t)$*-martingale on* $D_0(\mathbf{R}_+, F)$ *relative to* $\hat{\mathbf{Q}}$*. Then the extension of* Z *to* Ω *(which we denote again by*

Z) is an $\mathbf{F}$-martingale relative to $\hat{\mathbf{P}}$.
(2) Let now $F = (F_t)_{t\geq 0}$ be a given $\mathbf{F}$-progressively measurable process on Ω such that

$$\int_0^t F_s^2 \, d\,[Z]_s$$

is $\hat{\mathbf{P}}$-integrable for all $t \geq 0$, and that

$$\int_0^t F_s\big((\mathbf{t},\mathbf{x}),\cdot\big)^2 \, d\,[Z]_s$$

is $\hat{\mathbf{Q}}$-integrable for every $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$. Define

$$M_t \;=\; \int_0^t F_s \, d\, Z_s \tag{32}$$

for $t \geq 0$. Then $M = (M_t)_{t\geq 0}$ is a continuous, square integrable $\mathbf{F}$-martingale relative to $\hat{\mathbf{P}}$, which clearly has the quadratic variation $[M]$ given by

$$[M]_t \;=\; \int_0^t F_s^2 \, d\,[Z]_s \tag{33}$$

for all $t \geq 0$. Define $X = (X_t)_{t\geq 0}$ by

$$X_t \;:=\; \exp\big\{M_t - \frac{1}{2}[M]_t\big\}$$

for $t \geq 0$, and suppose that the following conditions hold:

$$\mathbf{E}_{\hat{\mathbf{P}}} \exp\big(\frac{1}{2}[M]_t\big) \;<\; \infty \quad \textit{for all}\, t \geq 0 \tag{34}$$

and

$$\mathbf{E}_{\hat{\mathbf{Q}}} \exp\big(\frac{1}{2}[M]_t((\mathbf{t},\mathbf{x}),\cdot)\big) \;<\; \infty \quad \textit{for}\, t \geq 0 \textit{ and every } (\mathbf{t},\mathbf{x}) \in S^\infty(E). \tag{35}$$

Then X has properties (1) and (2) of theorem 3.5, and hence the assertions of the theorem hold for X.

Proof. (1): Let $0 < s < t$ be given. We know that $\mathbf{Q}_{s,t}$ is a regular conditional distribution of $\hat{\mathbf{Q}}\{Y^t \in \cdot \,|\, \mathcal{D}_s\}$. Since Z is a $(\mathcal{D}_t)$-martingale, we have

$$\int Z_t(g)\mathbf{Q}_{s,t}(\cdot; dg) \;=\; \mathbf{E}_{\hat{\mathbf{Q}}}\{Z_t \,|\, \mathcal{D}_s\} \;=\; Z_s \quad \hat{\mathbf{Q}}\text{-a.s.}\ .$$

Now we take an $A \in \mathcal{C}_s$ and a $B \in \mathcal{D}_s$. Then

$$\begin{aligned}
&\int_{A\times B} Z_t \, d\hat{\mathbf{P}} \\
&= \int\int 1_A((\mathbf{t},\mathbf{x}))1_B(f)Z_t(f)\hat{\mathbf{R}}(f; d(\mathbf{t},\mathbf{x})\hat{\mathbf{Q}}(df)
\end{aligned}$$

$$
\begin{aligned}
&= \int\int\int 1_A((\mathbf{t},\mathbf{x}))1_B(f^s)Z_t(g)\mathbf{Q}_{s,t}(f^s;dg)\mathbf{R}_{0,s}(f^s;d(\mathbf{t},\mathbf{x}))\mathbf{Q}_{0,s}(df) \\
&= \int\Big[\int Z_t(g)\mathbf{Q}_{s,t}(f^s;dg)\Big]\Big[\int 1_A((\mathbf{t},\mathbf{x}))\mathbf{R}_{0,s}(f^s;d(\mathbf{t},\mathbf{x}))\Big]1_B(f^s)\mathbf{Q}_{0,s}(df) \\
&= \int Z_s(f)\Big[\int 1_A((\mathbf{t},\mathbf{x}))\mathbf{R}_{0,s}(f^s;d(\mathbf{t},\mathbf{x}))\Big]1_B(f^s)\mathbf{Q}_{0,s}(df) \\
&= \int\int 1_A((\mathbf{t},\mathbf{x}))1_B(f)Z_s(f)\mathbf{R}_{0,s}(f;d(\mathbf{t},\mathbf{x})\mathbf{Q}_{0,s}(df) \\
&= \int_{A\times B} Z_s\, d\hat{\mathbf{P}} \ ,
\end{aligned}
$$

and it follows that Z is even an $\mathbf{F}$-martingale.

(2): By our assumptions, M is a continuous, square integrable $\mathbf{F}$-martingale. Now it is well known (cf. e.g. Karatzas, Shreve [1988]) that under the condition (34) of (2) the process X is an $\mathbf{F}$-martingale, i.e. condition (1) of theorem 3.4 is proved. Similarly, condition (35) implies that for every $(\mathbf{t},\mathbf{x}) \in S^\infty(E)$ the process $(X_t((\mathbf{t},\mathbf{x}),\cdot))_{t\geq 0}$ is a $(\mathcal{D}_t)$-martingale relative to $\hat{\mathbf{Q}}$, and condition (2) of theorem 3.4 follows. □

The conditions (34) and (35) in the last proposition are not the most general conditions, which ensure that $\big(\exp\big\{M_t - \frac{1}{2}[M]_t\big\}\big)_{t\geq 0}$ has the martingale properties (1) and (2) of theorem 3.4. This is known from the theory of exponential martingales (cf. Liptser, Shiryayev [1978]). We have just chosen these two conditions to give an impression of the general problem to obtain martingales of the above type. In many cases, condition (35) is the easier condition, e.g. if the process F does not depend on the elements in $D_0(\mathbb{R}_+, F)$.

We conclude this section with a general example, which shows the difference between point processes with continuous mixing and classical mixed point processes. To avoid an overburdening of the notations we restrict ourselves in this example to the case of non-marked point processes.

3.7 Proposition. *Suppose that the process Z of proposition 3.6 is a $\hat{\mathbf{Q}}$-Brownian motion $B = (B_t)_{t\geq 0}$ and that $\overline{\mathbf{T}} = (T_n)_{n\geq 1}$ is a point process such that the counting process $(N_t)_{t\geq 0}$ of $(T_n)_{n\geq 1}$ is a double-stochastic Poisson process with intensity $(\lambda_t(B))_{t\geq 0}$. Let the process $(F_t)_{t\geq 0}$ of proposition 3.6 be given by $F_t(\mathbf{t}) := N_t(\mathbf{t})^{\frac{1}{2}}$, and suppose further that*

$$
\mathbf{E}_{\hat{\mathbf{Q}}}\Big(\exp\Big\{\big(e^{\frac{1}{2}t} - 1\big)\int_0^t \lambda_s(B)ds\Big\}\Big) \ < \ \infty \tag{36}
$$

for all $t \geq 0$. Then the process $(X_t)_{t\geq 0}$ of proposition 3.6 has the properties (34) and (35), and (36) holds in case of the following two simple examples for the intensity $(\lambda_t(B))_{t\geq 0}$.
Example (1): Let $0 < \alpha < \beta$ be given constants and denote by T the stopping time $T = \inf\{t > 0 : |B_t| > 1\}$. Define

$$
\lambda_s(B) \ := \ \alpha 1_{\{T>s\}} + \beta 1_{\{T\leq s\}} \ . \tag{37}
$$

Example (2): Define

$$\lambda_s(B) \quad := \quad \log(1 + \max_{r \le s} |B_r|) \; . \tag{38}$$

The distribution of $\overline{\mathbf{T}}$ *relative to* $\mathbf{P}^X$ *is given as follows: For every* $k \ge 1$ *we set* $\gamma_k := \sqrt{k} - \sqrt{k-1}$*. Then for* $0 \le t_1 < t_2 < \cdots < t_{n+1} = t$ *one has*

$$\begin{aligned} &\mathbf{P}^X\big\{T_1 > t_1, T_2 > t_2, \cdots, T_{n+1} > t_{n+1}\big\} \\ &= \mathbf{E}_{\hat{\mathbf{Q}}}\Big(\exp\big\{ - \int_0^t \lambda_s(B)\, ds\big\} \Big[\int_{t_1}^\infty \int_{t_2}^\infty \cdots \int_{t_n}^\infty \\ &\qquad \exp\big\{ \sum_{k=1}^n \big(\gamma_k (B_{t\wedge s_k} - B_{s_k}) - \frac{1}{2}(t \wedge s_k - s_k)\big)\big\} \prod_{k=1}^n \lambda_{s_k}(B)\, ds_n \cdots ds_1 \Big]\Big) \end{aligned} \tag{39}$$

The distribution of the mixing process Y *is given by*

$$\begin{aligned} &\mathbf{P}^X\big\{Y^t \in D\big\} \\ &= \sum_{k\ge 1} \mathbf{E}_{\hat{\mathbf{Q}}}\Big(1_D(B^t) \exp\big\{ - \int_0^t \lambda_s(B)\, ds\big\} \Big[\int_0^t \int_{s_1}^t \cdots \int_{s_{k-2}}^t \\ &\qquad \exp\big\{ \sum_{j=1}^{k-1} \big(\gamma_j (B_t - B_{s_j}) - \frac{1}{2}(t - s_j)\big)\big\} \prod_{j=1}^{k-1} \lambda_{s_j}(B)\, ds_{k-1} \cdots ds_1 \Big]\Big) \end{aligned} \tag{40}$$

for every $t \ge 0$ *and* $D \in \mathcal{D}$*.*

Proof. Since

$$[M]_t(\omega) \quad = \quad \int_0^t N_s((\mathbf{t},\mathbf{x}))\, ds \quad \text{for } \; \omega = ((\mathbf{t},\mathbf{x}), y) \; ,$$

$[M]_t(\omega)$ does not depend on y, and hence condition (35) is trivially fulfilled. For the proof of (34) we use that N_t has the Poisson distribution with parameter $\int_0^t \lambda_s(B) ds$ relative to the probability measure $\hat{\mathbf{R}}(B) = \hat{\mathbf{R}}(B; d(\mathbf{t},\mathbf{x}))$. Hence we get first

$$\begin{aligned} \mathbf{E}_{\hat{\mathbf{R}}(B)}\big\{ \exp\big(\frac{1}{2}[M]_t\big)\big\} &= \mathbf{E}_{\hat{\mathbf{R}}(B)}\big\{ \exp\big(\frac{1}{2}\int_0^t N_s\, ds\big)\big\} \\ &\le \mathbf{E}_{\hat{\mathbf{R}}(B)}\big\{ \exp\big(\frac{1}{2} t N_t\big)\big\} \\ &= e^{-\int_0^t \lambda_s(B) ds} \sum_{k\ge 0} \frac{e^{\frac{1}{2}tk}\big(\int_0^t \lambda_s(B) ds\big)^k}{k!} \\ &= \exp\Big(\big(e^{\frac{1}{2}t} - 1\big) \int_0^t \lambda_s(B) ds \Big) \;, \end{aligned}$$

and assumption (36) implies

$$\begin{aligned} \mathbf{E}_{\hat{\mathbf{P}}}\big\{ \exp\big(\frac{1}{2}[M]_t\big)\big\} &= \mathbf{E}_{\hat{\mathbf{Q}}}\Big\{ \mathbf{E}_{\hat{\mathbf{R}}(B)}\big\{ \exp\big(\frac{1}{2}[M]_t\big)\big\}\Big\} \\ &\le \mathbf{E}_{\hat{\mathbf{Q}}}\Big\{ \exp\Big(\big(e^{\frac{1}{2}t} - 1\big) \int_0^t \lambda_s(B) ds \Big)\Big\} \; < \; \infty \; . \end{aligned}$$

Thus (34) holds and hence we know from proposition 3.6 that the process X defined by

$$X_t = \exp\left\{ \int_0^t N_s(\mathbf{t})^{\frac{1}{2}}\, dB_s - \frac{1}{2}\int_0^t N_s(\mathbf{t})\, ds \right\} \tag{41}$$

has the properties (1) and (2) of theorem 3.4. Thus there exists a probability measure $\mathbf{P}^X$ on $\Omega = S^\infty \times D_0(\mathbb{R}_+, F)$ with the properties stated in theorem 3.4.

Now let us show that for the examples (1) and (2) of intensities the condition (36) holds.

For example (1) we obtain

$$\int_0^t \lambda_s(B) ds = \alpha T + \beta(t - T) = \beta t - (\beta - \alpha) T \ ,$$

and (36) holds, since $\beta > \alpha$.

For example (2) we get

$$\int_0^t \log(1 + \max_{r\leq s} |B_r|)\, ds \leq t \log(1 + \max_{s\leq t} |B_s|) \ ,$$

and hence

$$\exp\Big((e^{\frac{1}{2}t} - 1) \int_0^t \lambda_s(B) ds \Big) \leq (1 + \max_{s\leq t} |B_s|)^{t(e^{\frac{1}{2}t}-1)} \ .$$

Now it follows from the theorem of Fernique (cf. Araujo, Giné [1980;theorem 6.5]) that for every $t \geq 0$ there exists an $r_0 > 0$ such for all $r < r_0$

$$\mathbf{E}_{\hat{\mathbf{Q}}} \exp\left\{ r \max_{s\leq t} |B_s|^2 \right\} < \infty \ .$$

This implies especially that

$$\mathbf{E}_{\hat{\mathbf{Q}}} \max_{s\leq t} |B_s|^n < \infty$$

for every $n \geq 1$, and

$$\mathbf{E}_{\hat{\mathbf{Q}}} \left\{ (1 + \max_{s\leq t} |B_s|)^{t(e^{\frac{1}{2}t}-1)} \right\} < \infty$$

follows. Thus we have proved (36) for example (2).

For the proof of the formulas (39) and (40) we use that

$$N_t = \sum_{k\geq 1} 1_{\{T_k \leq t\}} \quad \text{and}$$

$$N_t^{\frac{1}{2}} = \sum_{k\geq 1} \gamma_k 1_{\{T_k \leq t\}} \ .$$

This shows that the process X (see (41)) has the form

$$X_t \;=\; \exp\Big\{\sum_{k\geq 1} 1_{\{T_k\leq t\}}[\gamma_k(B_t-B_{T_k})-\frac{1}{2}(t-T_k)]\Big\}\ . \tag{42}$$

Using this formula, it is now easy to derive (39) and (40). First we prove (39). From the definition of $\mathbf{P}^X$ we have

$$\begin{aligned}
&\mathbf{P}^X\big\{T_1>t_1,T_2>t_2,\cdots,T_{n+1}>t_{n+1}\big\}\\
&\quad= \mathbf{E}_{\hat{\mathbf{P}}}\big\{1_{\{T_1>t_1,T_2>t_2,\cdots,T_{n+1}>t_{n+1}\}}X_t\big\}\\
&\quad= \mathbf{E}_{\hat{\mathbf{P}}}\Big\{1_{\{T_1>t_1,T_2>t_2,\cdots,T_{n+1}>t_{n+1}\}}\\
&\qquad\qquad\qquad\cdot\exp\Big\{\sum_{k=1}^{n}[\gamma_k(B_{t\wedge T_k}-B_{T_k})-\frac{1}{2}(t\wedge T_k-T_k)]\Big\}\\
&\quad= \mathbf{E}_{\hat{\mathbf{Q}}}\{G(B)\}\ ,
\end{aligned}$$

with

$$\begin{aligned}
G(B)=\mathbf{E}_{\hat{\mathbf{R}}(B)}\Big\{&1_{\{T_1>t_1,T_2>t_2,\cdots,T_{n+1}>t_{n+1}\}}\\
&\cdot\exp\Big\{\sum_{k=1}^{n}[\gamma_k(B_{t\wedge T_k}-B_{T_k})-\frac{1}{2}(t\wedge T_k-T_k)]\Big\}\ .
\end{aligned}$$

From the definition of $\hat{\mathbf{R}}$ we get with the shorter notation

$$F_k(s_k)\ :=\ \gamma_k(B_{t\wedge s_k}-B_{s_k})-\frac{1}{2}(t\wedge s_k-s_k)\quad\text{and}\quad \lambda_s\ =\ \lambda_s(B)$$

$(1\leq k\leq n)$

$$\begin{aligned}
G(B)\ &=\ \int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty}\int_{t}^{\infty}\exp\Big\{\sum_{k=1}^{n}F_k(s_k)\Big\}\\
&\qquad\qquad\cdot\prod_{k=1}^{n+1}\Big[\exp\Big\{-\int_{s_{k-1}}^{s_k}\lambda_r dr\Big\}\lambda_{s_k}\Big]\,ds_{n+1}\,ds_n\cdots ds_1\\
&=\ \int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty}\exp\Big\{\sum_{k=1}^{n}F_k(s_k)\Big\}\exp\{-\int_{s_k}^{t}\lambda_r\,dr\}\\
&\qquad\qquad\cdot\prod_{k=1}^{n}\Big[\exp\Big\{-\int_{s_{k-1}}^{s_k}\lambda_r\,dr\Big\}\lambda_{s_k}\Big]\,ds_n\cdots ds_1\\
&=\ \exp\{-\int_0^t\lambda_r\,dr\}\int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty}\exp\Big\{\sum_{k=1}^{n}F_k(s_k)\Big\}\prod_{k=1}^{n}\lambda_{s_k}ds_n\cdots ds_1\ ,
\end{aligned}$$

and integrating $G(B)$ relative to $\hat{\mathbf{Q}}$ gives the asserted formula (39).

For the proof of (40) we proceed similarly. First we have

$$\mathbf{P}^X\big\{Y^t\in D\big\}\ =\ \mathbf{E}_{\hat{\mathbf{Q}}}\big\{1_D(B^t)\mathbf{E}_{\hat{\mathbf{R}}(B)}\{X_t\}\big\}\ ,$$

and for the inner integral we get

$$\mathbf{E}_{\hat{\mathbf{R}}(B)}\{X_t\} \quad = \quad \sum_{k\geq 1}\mathbf{E}_{\hat{\mathbf{R}}(B)}\big\{1_{\{T_1\leq t,\cdots,T_{k-1}\leq t,T_k>t\}}X_t\big\}\ . \tag{43}$$

For every term under the sum of equation (43) we compute further

$$\begin{aligned}
&\mathbf{E}_{\hat{\mathbf{R}}(B)}\big\{1_{\{T_1\leq t,\cdots,T_{k-1}\leq t,T_k>t\}}X_t\big\}\\
&= \int_0^t\int_{t_1}^t\cdots\int_{t_{k-2}}^t\int_t^\infty \exp\big\{\sum_{j=1}^{k-1}F_j(s_j)\big\}\\
&\qquad\qquad\cdot\prod_{j=1}^{k}\big[\exp\big\{-\int_{s_{j-1}}^{s_j}\lambda_r dr\big\}\lambda_{s_j}\big]\,ds_k\cdots ds_1\\
&= \exp\{-\int_0^t\lambda_r\,dr\}\int_0^t\cdots\int_{t_{k-2}}^t \exp\big\{\sum_{j=1}^{k-1}F_k(s_k)\big\}\prod_{j=1}^{k-1}\lambda_{s_j}ds_{k-1}\cdots ds_1\ ,
\end{aligned}$$

where for $k=1$ the last "iterated" integral is understood to have the value 1. Now the asserted equation (40) follows from (43). □

Remark: Formula (39) proves especially that the probability measures $\mathbf{P}^X$ define new classes of point processes not covered by the classical construction, and formula (40) shows the mutual effect between the mixing of the point process and the influence of the point process itself on the structural data.

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30th October 2005