# Prediction for Risk Processes

Egbert Dettweiler

Universität Tübingen

**Abstract**

A risk process is defined as a marked point process $((T_n, X_n))_{n\geq 1}$ on a certain probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where the time points $T_1 < T_2 < \cdots$ are the claim arrival times of claims from a given portfolio of risks and the marks $X_n$ are the claim amounts at time $T_n$. If $N_t(B)$ denotes the number of claims up to time $t$ with claim amount in a Borel set $B$, then $((T_n, X_n))_{n\geq 1}$ can equivalently be described by the family of processes $(N_t(B))_{t\geq 0}$ with $B \in \mathcal{B}(\mathbb{R}_+)$. Suppose that $(T, \mathcal{T})$ is a measurable space, $\Theta$ a $T$-valued random variable, and that $(\mathcal{F}_t^\Theta)_{t\geq 0}$ is the filtration defined by $\mathcal{F}_t^\Theta = \sigma(\Theta) \vee \sigma(\{N_s(B) : s \leq t, B \in \mathcal{B}(\mathbb{R}_+\})$. Assume that there is a family of $(\mathcal{F}_t^\Theta)$-adapted processes $(\lambda_t(B))_{t\geq 0}$ $(B \in \mathcal{B}(\mathbb{R}_+))$ such that all processes $(N_t(B) - \int_0^t \lambda_s(B)ds)_{t\geq 0}$ are local $(\mathcal{F}_t^\Theta)$-martingales. Then $((T_n, X_n))_{n\geq 1}$ is called a $\Theta$-*mixed* risk process, and for a number of reasons the random variable $\Theta$ is called the *portfolio structure*.

Now suppose that $Z = (Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^\Theta)$-adapted process and that $(\mathcal{F}_t)_{t\geq 0}$ is a subfiltration of $(\mathcal{F}_t^\Theta)$. The *filtering problem for $Z$ given* $(\mathcal{F}_t)$ is just the problem to determine the process $(\mathbf{E}\{Z_t \,|\, \mathcal{F}_t\})_{t\geq 0}$, and the *prediction problem* is the problem to determine for a given $h > 0$ the process $(\mathbf{E}\{Z_{t+h} \,|\, \mathcal{F}_t\})_{t\geq 0}$. For a number of relevant processes $Z$ one can use a martingale property inherited from the martingale property of $((T_n, X_n))_{n\geq 1}$ to solve the filtering and the prediction problem. A typical example is the process $(S_t^B)_{t\geq 0}$ $(B \in \mathcal{B}(\mathbb{R}_+))$ given by $S_t^B = \sum_{n\geq 1} X_n 1_{\{T_n \leq t\}} 1_{\{X_n \in B\}}$. In this case the process $(S_t^B - \int_0^t \int_B x\lambda_s(dx)ds)_{t\geq 0}$ is a local $(\mathcal{F}_t^\Theta)$-martingale.

## 1 Mixed Risk Processes

Let $(E, \mathcal{E})$ be a measurable space and let $\Delta$ denote an artificial element outside of $E$. We set $E_\Delta := E \cup \{\Delta\}$ and provide $E_\Delta$ with the $\sigma$-algebra $\mathcal{E}_\Delta := \sigma(\mathcal{E} \cup \{\{\Delta\}\})$. Now suppose that $(T_n)_{n\geq 1}$ is a *claim arrival process* on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, i.e. that $\mathbf{P}$-a.s.

$$0 =: T_0 \leq T_1 \leq T_2 \leq \cdots \quad \text{with } \; T_{n-1} < T_n \;, \text{ if } \; T_{n-1} < \infty \;,$$

and that $(X_n)_{n\geq 1}$ is a sequence of $E_\Delta$-valued random variables such that the following condition holds:

$$T_n = \infty \iff X_n = \Delta \;. \tag{1}$$

Then the double sequence $((T_n, X_n))_{n\geq 1}$ is called a *risk process* with *claim space* $E$. For $n = 0$ we make the convention $X_0 := \epsilon$ with a fixed $\epsilon \in E$.

We will always assume in the following that $(E, \mathcal{E})$ is a polish space provided with its Borel field. In most applications $E = \mathbb{R}_+$ and $\Delta = \infty$, i.e. $E_\Delta = \overline{\mathbb{R}}_+$. Then every $X_n$ is interpreted as the claim size of the $n$-th claim and $(X_n)_{n\geq 1}$ will be called the *claim size process* or the *claim amount process* of the risk process.

Let us denote by $M_+^z(X, \mathcal{X})$ the space of all $\overline{\mathbb{Z}}_+$-valued measures on a measurable space $(X, \mathcal{X})$.Then a risk process $((T_n, X_n))_{n\geq 1}$ with claim space $E$ can be equivalently described by the *random measure*

$$\mathcal{N} : (\Omega, \mathcal{F}, \mathbf{P}) \longrightarrow M_+^z(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}) ,$$

defined by

$$\mathcal{N} \;:=\; \sum_{n\geq 1} \delta_{(T_n, X_n)} \;.$$

For $t \geq 0$ and $B \in \mathcal{E}$ we set

$$N_t(B) \;:=\; \mathcal{N}([0,t] \times B) \;=\; \sum_{n\geq 1} 1_{\{T_n \leq t\}} 1_{\{X_n \in B\}} \;.$$

Then every $N_t(B)$ is a random variable and we can identify the random measure $\mathcal{N}$ with the family $\big((N_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$ of stochastic processes. We will call

$$\mathcal{N} \;=\; \big((N_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$$

the *risk measure* of the risk process $((T_n, X_n))_{n\geq 1}$.

Since $(N_t(E))_{t\geq 0}$ is just the *claim number process* of $(T_n)_{n\geq 1}$, we will also write $N_t$ instead of $N_t(E)$.

Now let $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ be a given right continuous filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. A family $\Lambda = \big((\lambda_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$ of $\mathbb{R}_+$-valued stochastic processes is called an $\mathbf{F}$-*intensity measure*, if the following properties hold:
(i) for every fixed $B \in \mathcal{E}$ the process $(\lambda_t(B))_{t\geq 0}$ is an $\mathbf{F}$-progressively measurable process with values in $\mathbb{R}_+$.
(ii) for every fixed $t \geq 0$,

$$B \longmapsto \lambda_t(B)$$

is a finite measure on $(E, \mathcal{E})$, and
(iii) for every $t \geq 0$,

$$\int_0^t \lambda_s(E)\, ds \;<\; \infty \quad \mathbf{P}\text{-a.s.}.$$

In the following we will just write $\lambda_t$ instead of $\lambda_t(E)$.

Now suppose that $((T_n, X_n))_{n\geq 1}$ is a risk process with associated risk measure $\mathcal{N} = \big((N_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$ and assume that $\mathcal{N}$ is $\mathbf{F}$-adapted. Then we say that $\mathcal{N}$ (or $((T_n, X_n))_{n\geq 1}$) has the $\mathbf{F}$-*intensity measure* $\Lambda = \big((\lambda_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$, if

$$\Big(N_{T_n\wedge t}(B) - \int_0^{T_n\wedge t} \lambda_s(B)\,ds\Big)_{t\geq 0} \tag{2}$$

is an $\mathbf{F}$-martingale for every $n \geq 1$ and every $B \in \mathcal{E}$. If the random variables $N_t := N_t(E)$ $(t > 0)$ are *integrable* (in this case we will also say that the risk process $((T_n, X_n))_{n\geq 1}$ is *integrable*), then (2) just means that for all $B \in \mathcal{E}$ the processes

$$\Big(N_t(B) - \int_0^t \lambda_s(B)\,ds\Big)_{t\geq 0} \tag{3}$$

are $\mathbf{F}$-martingales.

For $B \in \mathcal{E}$ the measure $\lambda_t(B)\,(dt\otimes d\mathbf{P})$ is obviously absolutely continuous relative to $\lambda_t(E)\,(dt\otimes d\mathbf{P})$. Hence there exists a Radon-Nikodym-density $\gamma_t(B)$ relative to $\lambda_t(E)\,(dt\otimes d\mathbf{P})$, and it is not difficult to prove that these densities can be chosen in such a way that

$$B \mapsto \gamma_t(B)$$

is a probability measure for all $\omega \in \Omega$. In case that always $\lambda_t(E) > 0$, one can just set

$$\gamma_t(B) \;:=\; \frac{\lambda_t(B)}{\lambda_t(E)}\,.$$

In the following we assume always that

$$\lambda_t(B) \;=\; \gamma_t(B)\lambda_t\,,$$

where $\gamma_t$ is a probability measure on $(E, \mathcal{E})$.

In this paper we will consider essentially two filtrations. The first filtration is the canonical filtration $\mathbf{F}^{\mathcal{N}} = (\mathcal{F}_t^{\mathcal{N}})_{t\geq 0}$ of $\mathcal{N}$, defined by

$$\mathcal{F}_t^{\mathcal{N}} \;:=\; \sigma\big(\{N_s(B)\,\big|\, s\leq t\,,\ B\in\mathcal{E}\}\big)\,.$$

For the second filtration we take a measurable space $(T, \mathcal{T})$, a measurable map

$$\Theta : (\Omega, \mathcal{F}, \mathbf{P}) \to (T, \mathcal{T})\,,$$

and define the filtration $\mathbf{F}^{\Theta} = (\mathcal{F}_t^{\Theta})_{t\geq 0}$ by

$$\mathcal{F}_t^{\Theta} \;:=\; \sigma(\Theta)\vee\mathcal{F}_t^{\mathcal{N}}\,.$$

$\Theta$ will shortly be called the *portfolio structure*, and if $((T_n, X_n))_{n\geq 1}$ is a risk process with a $\mathbf{F}^{\Theta}$-intensity measure $\Lambda = \big((\lambda_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$, then $((T_n, X_n))_{n\geq 1}$ is called a $\Theta$-*mixed risk process*.

Suppose that $G = (G_t)_{t\geq 0}$ is a $\mathbf{F}^\Theta$-adapted process. Then it is easily shown that on the set $\{T_{n-1} \leq t < T_n\}$ $G_t$ only depends on $(T_k, X_k)_{k\leq n-1}$ and on $\Theta$ and we express this dependence by the notation

$$G_t \quad = \quad G_t(\Theta, (T_k, X_k)_{k\leq n-1}) \ .$$

We will also often make use of the following convention: If a function $f$ depends on $(t_j, x_j)_{j\leq n}$, then we will use freely the different notations

$$f(t_1, \cdots, t_n, x_1, \cdots, x_n) \ , \quad f((t_j, x_j)_{j\leq n}) \ , \quad \text{or}$$

$$f((t_i, x_i)_{1\leq i\leq k-1}, (t_j, x_j)_{k\leq j\leq n}) \quad \text{for } 1 \leq k \leq n \ .$$

For the purposes of this paper we need some further regularity properties of the $\mathbf{F}^\Theta$-intensity measure

$$\Lambda \quad = \quad \big((\lambda_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$$

of a $\Theta$-mixed risk process $((T_n, X_n))_{n\geq 1}$.

**1.1 Definition**. The $\mathbf{F}^\Theta$-intensity measure $\Lambda$ is said to be *regular*, if the following condition holds:
There exists a $\sigma$-finite measure $\gamma$ on $(E, \mathcal{E})$ such that

$$\gamma_t \quad = \quad g_t(\cdot; \Theta, T_1, \cdots, T_{n-1}, X_1, \cdots, X_{n-1})\gamma$$

on $\{T_{n-1} \leq t < T_n\}$, and such that

$$(t, x, \theta, t_1, \cdots, t_{n-1}, x_1, \cdots, x_{n-1}) \mapsto g_t(x; \theta, t_1, \cdots, t_{n-1}, x_1, \cdots, x_{n-1})$$

is measurable.

In case that $\Lambda$ is regular, the distribution of

$$(\Theta, T_1, T_2, \cdots, X_1, X_2, \cdots)$$

can easily be computed. Denote by $\beta$ the distribution of $\Theta$ and suppose that $u_1, \cdots, u_n \in \mathbb{R}_+$, $B_1, \cdots, B_n \in \mathcal{E}$, and $C \in \mathcal{T}$ are given. Then

$$\begin{aligned} &\mathbf{P}\big\{T_1 \leq u_1, \cdots, T_n \leq u_n, X_1 \in B_1, \cdots, X_n \in B_n, \Theta \in C\big\} \\ &= \int_C \int_0^{u_1} \int_{B_1} \cdots \int_{t_{n-1}\wedge u_n}^{u_n} \int_{B_n} G^{(n)}(y, (t_i, x_i)_{i\leq n}) \\ &\qquad\qquad \gamma(dx_n)dt_n \cdots \gamma(dx_1)dt_1\beta(dy) \ , \end{aligned} \tag{4}$$

where the integrand $G^{(n)}(y, (t_i, x_i)_{i\leq n})$ is given by

$$\begin{aligned} G^{(n)}(y, (t_i, x_i)_{i\leq n}) \quad = \quad &\prod_{i=1}^n \big\{g_{t_i}(x_i; y, (t_j, x_j)_{j\leq i-1}) \\ &\cdot \lambda_{t_i}(y, (t_j, x_j)_{j\leq i-1})e^{-\int_{t_{i-1}}^{t_i} \lambda_s(y,(t_j,x_j)_{j\leq i-1})\,ds}\big\} \ . \end{aligned} \tag{5}$$

There are also explicit formulas for a number of important conditional distributions. Define

$$
\begin{aligned}
& G^{(n,k)}_{\Theta,(T_i,X_i)_{i\le n}}((t_j,x_j)_{1\le j\le k}) \\
&:= \prod_{l=1}^{k} \big\{ g_{t_l}(\Theta,(T_i,X_i)_{i\le n},(t_j,x_j)_{1\le j\le l-1}) \\
&\qquad \cdot \lambda_{t_l}(\Theta,(T_i,X_i)_{i\le n},(t_j,x_j)_{1\le j\le l-1}) e^{-\int_{t_{l-1}}^{t_l} \lambda_s(\Theta,(T_i,X_i)_{i\le n},(t_j,x_j)_{1\le j\le l-1})\,ds} \big\}
\end{aligned}
\tag{6}
$$

with the convention $t_0 = T_n$. Then for $n \ge 1$, $k \ge 1$, $u_1,\cdots,u_k > 0$, and $B_1,\cdots,B_k \in \mathcal{E}$ we have

$$
\begin{aligned}
& \mathbf{P}\big\{T_{n+1} \le u_1,\cdots,T_{n+k} \le u_k, X_{n+1} \in B_1,\cdots,X_{n+k} \in B_k \,\big|\, \mathcal{F}^{\Theta}_{T_n}\big\} \\
&= \int_{T_n\wedge u_1}^{u_1} \int_{B_1} \cdots \int_{t_{k-1}\wedge u_k}^{u_k} \int_{B_k} G^{(n,k)}_{\Theta,(T_i,X_i)_{i\le n}}((t_j,x_j)_{1\le j\le k}) \\
&\qquad\qquad \gamma(dx_k)dt_k \cdots \gamma(dx_1)dt_1 \;.
\end{aligned}
\tag{7}
$$

We remark that

$$
\mathcal{F}^{\Theta}_{T_n} = \sigma(\Theta,(T_j,X_j)_{0\le j\le n}) \;.
$$

There is also an explicit formula for the conditional distribution relative to the $\sigma$-algebra $\mathcal{F}^{\Theta}_t$ for $t \ge 0$ (cf. Dettweiler [2004]). For every $n \ge 1$, $k \ge 1$, $u_1,\cdots,u_k > 0$, and $B_1,\cdots,B_k \in \mathcal{E}$ one has

$$
\begin{aligned}
& 1_{\{T_{n-1}\le t<T_n\}} \mathbf{P}\big\{T_n \le u_1,\cdots,T_{n+k-1} \le u_k, \\
&\qquad\qquad X_n \in B_1,\cdots,X_{n+k-1} \in B_k \,\big|\, \mathcal{F}^{\Theta}_t\big\} \\
&= 1_{\{T_{n-1}\le t<T_n\}} \int_{t\wedge u_1}^{u_1} \int_{B_1} \cdots \int_{s_{k-1}\wedge u_k}^{u_k} \int_{B_k} G^{t,(n-1,k)}_{\Theta,(T_i,X_i)_{i\le n-1}}((t_j,x_j)_{1\le j\le k}) \\
&\qquad\qquad \gamma(dx_k)dt_k \cdots \gamma(dx_1)dt_1 \;,
\end{aligned}
\tag{8}
$$

where the density $G^{t,(n-1,k)}_{\Theta,(T_i,X_i)_{i\le n-1}}$ is given by

$$
\begin{aligned}
& G^{t,(n-1,k)}_{\Theta,(T_i,X_i)_{i\le n-1}}((t_j,x_j)_{1\le j\le k}) \\
&:= \prod_{l=1}^{k} \big\{ g_{t_l}(\Theta,(T_i,X_i)_{i\le n-1},(t_j,x_j)_{1\le j\le l-1}) \\
&\qquad \cdot \lambda_{t_l}(\Theta,(T_i,X_i)_{i\le n-1},(t_j,x_j)_{j\le l-1}) e^{-\int_{t_{l-1}}^{t_l} \lambda_s(\Theta,(T_i,X_i)_{i\le n-1},(t_j,x_j)_{j\le l-1})\,ds} \big\}
\end{aligned}
\tag{9}
$$

with the convention $t_0 = t$.

In this paper we will consider - beside the general results - three classes of special risk processes.

**(a) Mixed (homogeneous) Poisson Risk Processes:** This is the case, if $(\lambda_t)_{t\ge 0}$ is a constant process only depending on $\Theta$ and if also the densities $g_t$ above only depend on $\Theta$ (and also not on $t$). This means that we assume

$$
\lambda_t = \lambda(\Theta) \quad \text{, and} \tag{10}
$$

$$g_t(\cdot;\Theta,T_1,\cdots,T_{n-1},X_1,\cdots,X_{n-1}) \;=\; g(\cdot;\Theta)\;. \tag{11}$$

**(b) Mixed (inhomogeneous) Poisson Risk Processes:** In this case we assume

$$\lambda_t \;=\; \lambda_t(\Theta) \quad \text{, and} \tag{12}$$

$$g_t(\cdot;\Theta,T_1,\cdots,T_{n-1},X_1,\cdots,X_{n-1}) \;=\; g_t(\cdot;\Theta)\;. \tag{13}$$

**(c) Mixed Markovian Risk Processes:** Here we suppose that we have on the sets $\{T_{n-1}\le t<T_n\}$ $(n\ge 1)$

$$\lambda_t \;=\; \lambda_t^{(n)}(\Theta) \quad \text{, and} \tag{14}$$

$$g_t(\cdot;\Theta,T_1,\cdots,T_{n-1},X_1,\cdots,X_{n-1}) \;=\; g_t^{(n)}(\cdot;\Theta)\;, \tag{15}$$

and the mixed Markovian risk process is said to be *homogeneous*, if $\lambda_t^{(n)}$ and $g_t^{(n)}$ do not depend on $t$.

# 2 Prediction

From now on we will always assume that $((T_n,X_n))_{n\ge 1}$ is a $\Theta$-mixed risk process with a regular $\mathbf{F}^\Theta$-intensity measure $\Lambda$ as described in definition 1.1.

Suppose that $Z=(Z_s)_{s\ge 0}$ is an integrable, $\mathbf{F}^\Theta$-adapted process, that $0\le t<u$ are two given fixed time points and that $\mathcal{G}_t$ is a fixed sub-$\sigma$-algebra of $\mathcal{F}_t^\Theta$. Then we will call $\mathbf{E}\{Z_u\,|\,\mathcal{G}_t\}$ the *prediction of $Z_u$ on the basis of the informations given by $\mathcal{G}_t$* (or more shortly: the *prediction of $Z_u$ given $\mathcal{G}_t$*). This *prediction problem* will be solved in two steps: In this section we will first consider the prediction of $Z_u$ given $\mathcal{F}_t^\Theta$. Then in the next section we will solve the prediction of $Z_u$ given $\mathcal{G}_t$ by *filtering* $\mathbf{E}\{Z_u\,|\,\mathcal{F}_t^\Theta\}$, which simply means that we use the iteration formula for conditional expectations: $\mathbf{E}\{Z_u\,|\,\mathcal{G}_t\}=\mathbf{E}\big\{\mathbf{E}\{Z_u\,|\,\mathcal{F}_t^\Theta\}\,\big|\,\mathcal{G}_t\big\}$. Since $Z=(Z_s)_{s\ge 0}$ is assumed to be $\mathbf{F}^\Theta$-adapted, we have

$$Z_u \;=\; \sum_{n\ge 1} 1_{\{T_{n-1}\le t<T_n\}} \sum_{k\ge 1} 1_{\{T_{n-1+k-1}\le u<T_{n-1+k}\}} Z_u(\Theta,(T_i,X_i)_{i\le n-1+k-1})\;.$$

Thus we obtain from (8) the general formula

$$\mathbf{E}\{Z_u\,|\,\mathcal{F}_t^\Theta\} \;=\; \sum_{n\ge 1} 1_{\{T_{n-1}\le t<T_n\}} \sum_{k\ge 1} I_{n,k}^{t,u}(\Theta,(T_i,X_i)_{i\le n-1})\;, \tag{16}$$

where

$$I_{n,k}^{t,u}(\Theta,(T_i,X_i)_{i\le n-1}) \;:=\; \mathbf{E}\big\{1_{\{T_{n-1+k-1}\le u<T_{n-1+k}\}} Z_u(\Theta,(T_i,X_i)_{i\le n-1+k-1})\,\big|\,\mathcal{F}_t^\Theta\big\}\;.$$

For $k=1$ we have

$$I_{n,1}^{t,u}(\Theta,(T_i,X_i)_{i\le n-1}) \;=\; Z_u(\Theta,(T_i,X_i)_{i\le n-1}e^{-\int_t^u \lambda_r(\Theta,(T_i,X_i)_{i\le n-1})dr}\;, \tag{17}$$

and for $k > 1$ we have (with the convention $t_0 = t$)

$$
\begin{aligned}
& I^{t,u}_{n,k}(\Theta, (T_i, X_i)_{i\le n-1}) && (18)\\
&= \int_t^u \int_E \cdots \int_{t_{k-2}}^u \int_E \Big\{ Z_u(\Theta, (T_i, X_i)_{i\le n-1}, (t_j, x_j)_{1\le j\le k-1}) \\
&\qquad \cdot e^{-\int_t^u \lambda_r(\Theta,(T_i,X_i)_{i\le n-1},(t_j,x_j)_{1\le j\le k-1})dr} G^{t,(n-1,k-1)}_{\Theta,(T_i,X_i)_{i\le n-1}}((t_j, x_j)_{1\le j\le k-1}) \Big\} \\
&\qquad\qquad \gamma(dx_{k-1})dt_{k-1} \cdots \gamma(dx_1)dt_1 \ .
\end{aligned}
$$

The double series in formula (16) reduces to a finite sum, if $Z_u$ does not fully depend on $((T_n, X_n))_{n\ge 1}$, i.e. if

$$
Z_u \;=\; Z_u(\Theta, (T_i, X_i)_{0\le i\le m})
$$

for some fixed $m \ge 0$. We omit the details.

We will restrict now to more special prediction problems. For this we assume that in the following always $E = \mathbb{R}_+$ (or at least $\mathbb{R}^d$), but we will still use the notation $E$ to distinguish the claim space from the time axis.

For a fixed Borel subset $B$ of $E$ and $t \ge 0$ we set

$$
S^B_t \;:=\; \sum_{n\ge 1} X_n 1_{\{X_n\in B\}} 1_{\{T_n\le t\}} \ . \tag{19}
$$

Then

$$
S^B_t \;=\; \int_0^t \int_B x\, N(ds, dx) \ .
$$

We will assume that for all $t \ge 0$

$$
\mathbf{E} \int_0^t \int_B |x|\, g_s(x)\lambda_s \gamma(dx)ds \;<\; \infty \ . \tag{20}
$$

Then one knows (cf. Dettweiler [2004]) that

$$
\big(S^B_t - \int_0^t \int_B x\, g_s(x)\lambda_s\gamma(dx)ds\big)_{t\ge 0}
$$

is an $\mathbf{F}^\Theta$-martingale. This implies

$$
\mathbf{E}\{S^B_u - S^B_t \,|\, \mathcal{F}^\Theta_t\} \;=\; \mathbf{E}\big\{ \int_t^u \int_B x\, g_s(x)\lambda_s\gamma(dx)ds \,\big|\, \mathcal{F}^\Theta_t \big\} \ . \tag{21}
$$

Thus the prediction of $S^B_u - S^B_t$ given $\mathcal{F}^\Theta_t$ is the same as the prediction of

$$
Z_u \;:=\; \int_t^u \int_B x\, g_s(x)\lambda_s\gamma(dx)ds \tag{22}
$$

given $\mathcal{F}^\Theta_t$.

If the density processes $g$ and $\lambda$ fully depend on $((T_n, X_n))_{n\geq 1}$, one has to use the general formula (16) for the $Z_u$ given by (22), which clearly is not an easy and practical task. But there are two important special cases, where the prediction is quite easy, since $g$ and $\lambda$ only depend on $\Theta$.

**2.1 Proposition**. *Suppose that $((T_n, X_n))_{n\geq 1}$ is a mixed inhomogeneous Poisson risk process, for which the integrability assumption (20) holds. Then*

$$\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^\Theta\} = \int_t^u \int_B x g_s(x;\Theta)\lambda_s(\Theta)\gamma(dx)ds \ . \tag{23}$$

*If especially $((T_n, X_n))_{n\geq 1}$ is a mixed homogeneous Poisson risk process, then*

$$\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^\Theta\} \;=\; (u-t)\lambda(\Theta)\int_B x g(x;\Theta)\gamma(dx) \ . \tag{24}$$

The prediction problem for general mixed risk processes is getting more simple in the following situation. We introduce the stopping time

$$T_t := \inf\{v > t \,|\, N_v - N_t \geq 1\} \ ,$$

and consider the prediction problem for the increment $S_{T_t\wedge u}^B - S_t^B$. Since $T_t = T_n$ on $\{T_{n-1} \leq t < T_n\}$, we have

$$\mathbf{E}\{S_{T_t\wedge u}^B - S_t^B \,|\, \mathcal{F}_t^\Theta\} \;=\; \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}} \mathbf{E}\{1_{\{T_n\leq u\}}1_{\{X_n\in B\}}X_n \,|\, \mathcal{F}_t^\Theta\} \ , \tag{25}$$

and one obtains

$$\mathbf{E}\{S_{T_t\wedge u}^B - S_t^B \,|\, \mathcal{F}_t^\Theta\} \;=\; \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}} J_n^{t,u}(\Theta, (T_i, X_i)_{i\leq n-1}) \ , \tag{26}$$

where

$$\begin{aligned} &J_n^{t,u}(\Theta, (T_i, X_i)_{i\leq n-1}) \\ &\quad= \int_t^u \Big(\int_B x g_s(x;\Theta,(T_i,X_i)_{i\leq n-1})\gamma(dx)\Big) \\ &\qquad\qquad \lambda_s(\Theta,(T_i,X_i)_{i\leq n-1}) e^{-\int_t^s \lambda_r(\Theta,(T_i,X_i)_{i\leq n-1})\,dr}\,ds \ . \end{aligned} \tag{27}$$

Especially, we have the following result:

**2.2 Proposition**. *Suppose that the integrability condition (20) holds.*
*(a) If $((T_n, X_n))_{n\geq 1}$ is a mixed Markovian risk process (see (14) and (15), then*

$$\begin{aligned} &\mathbf{E}\{S_{T_t\wedge u}^B - S_t^B \,|\, \mathcal{F}_t^\Theta\} \\ &\quad= \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}} \int_t^u \Big(\int_B x g_s^{(n)}(x;\Theta)\gamma(dx)\Big)\lambda_s^{(n)}(\Theta) e^{-\int_t^s \lambda_r^{(n)}(\Theta)\,dr}\,ds \ . \end{aligned} \tag{28}$$

*Thus - in case that $((T_n, X_n))_{n\geq 1}$ is homogeneous -*

$$\begin{aligned}
&\mathbf{E}\{S^B_{T_t\wedge u} - S^B_t \,|\, \mathcal{F}^\Theta_t\} \qquad (29)\\
&= \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}}\Big(1 - e^{-(u-t)\lambda^{(n)}(\Theta)}\Big)\Big(\int_B x g^{(n)}(x;\Theta)\gamma(dx)\Big)\ .
\end{aligned}$$

*(b) If* $((T_n, X_n))_{n\geq 1}$ *is a mixed inhomogeneous Poisson risk process, then*

$$\mathbf{E}\{S^B_{T_t\wedge u} - S^B_t \,|\, \mathcal{F}^\Theta_t\} \;=\; \int_t^u \Big(\int_B x g_s(x;\Theta)\gamma(dx)\Big)\lambda_s(\Theta)e^{-\int_t^s \lambda_r(\Theta)\,dr}\, ds\ , \quad (30)$$

*and in case of a homogeneous Poisson risk process*

$$\mathbf{E}\{S^B_{T_t\wedge u} - S^B_t \,|\, \mathcal{F}^\Theta_t\} \;=\; \Big(1 - e^{-(u-t)\lambda(\Theta)}\Big)\Big(\int_B x g(x;\Theta)\gamma(dx)\Big)\ . \quad (31)$$

In the next (and last) prediction problem we replace the deterministic time points $t$ and $u$ by the stopping times $T_{n-1}$ and $T_n$, and consider the prediction of $X_n 1_{\{X_n\in B\}}$ given $\mathcal{F}^\Theta_{T_{n-1}}$. This means that we determine

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^\Theta_{T_{n-1}}\} \;=\; \mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \Theta, (T_i, X_i)_{i\leq n-1}\}\ .$$

From (7) we obtain

$$\begin{aligned}
&\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^\Theta_{T_{n-1}}\}\\
&= \int_{T_{n-1}}^\infty \int_B x\, G^{(n-1,1)}_{\Theta,(T_i,X_i)_{i\leq n-1}}(t,x)\gamma(dx)\,dt\\
&= \int_{T_{n-1}}^\infty \Big(\int_B x\, g_t(x;\Theta,(T_i,X_i)_{i\leq n-1})\gamma(dx)\Big) \qquad (32)\\
&\qquad\qquad \lambda_t(\Theta,(T_i,X_i)_{i\leq n-1})e^{-\int_{T_{n-1}}^t \lambda_r(\Theta,(T_i,X_i)_{i\leq n-1})dr}\, dt\ .
\end{aligned}$$

This implies for our examples of mixed risk processes the following proposition:

**2.3 Proposition**. *Suppose that the integrability condition (20) holds.*
*(a) If* $((T_n, X_n))_{n\geq 1}$ *is a mixed Markovian risk process, then*

$$\begin{aligned}
&\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^\Theta_{T_{n-1}}\} \qquad (33)\\
&= \int_{T_{n-1}}^\infty \Big(\int_B x g^{(n)}_t(x;\Theta)\gamma(dx)\Big)\lambda^{(n)}_t(\Theta)e^{-\int_{T_{n-1}}^t \lambda^{(n)}_r(\Theta)\,dr}\, dt\ .
\end{aligned}$$

*Thus - in case that* $((T_n, X_n))_{n\geq 1}$ *is homogeneous -*

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^\Theta_{T_{n-1}}\} \;=\; \int_B x g^{(n)}(x;\Theta)\gamma(dx)\ . \quad (34)$$

*(b) If* $((T_n, X_n))_{n\geq 1}$ *is a mixed inhomogeneous Poisson risk process, then*

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^\Theta_{T_{n-1}}\} \;=\; \int_{T_{n-1}}^\infty \Big(\int_B x g_t(x;\Theta)\gamma(dx)\Big)\lambda_t(\Theta)e^{-\int_{T_{n-1}}^t \lambda_r(\Theta)\,dr}\, dt\ , \quad (35)$$

*and in case of a mixed homogeneous Poisson risk process*

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^\Theta_{T_{n-1}}\} \;=\; \int_B x g(x;\Theta)\gamma(dx)\ . \quad (36)$$

# 3 Filtering

In this section we filter the prediction formulas of the foregoing section relative to sub-$\sigma$-algebras $\mathcal{G}_t$ of $\mathcal{F}_t^{\mathcal{N}}$ (resp. sub-$\sigma$-algebras $\mathcal{G}_{T_{n-1}}$ of $\mathcal{F}_{T_{n-1}}^{\mathcal{N}}$). First we consider filtering relative to $\mathcal{F}_t^{\mathcal{N}}$.

Let us make the following **convention:** Below there will often occur - in connection with conditional expectations - quotients of the form

$$\frac{F(x)}{G(x)} \ ,$$

where it may happen that the denominator $G(x)$ is zero. In that case the value of that quotient is defined to be zero.

The following lemma will be the basis of most of the results in this section.

**3.1 Lemma**. *Suppose that $F = F(\Theta, (T_i, X_i)_{i \le n-1})$ is integrable. Then for every $t \ge 0$ and every $n \ge 1$ the following filtering formula holds on the set $\{T_{n-1} \le t < T_n\}$:*

$$\mathbf{E}\{F(\Theta, (T_i, X_i)_{i \le n-1}) \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \frac{\Phi_t^{n,F}((T_i, X_i)_{i \le n-1})}{\Psi_t^n((T_i, X_i)_{i \le n-1})} \ , \tag{37}$$

*with*

$$\begin{aligned} &\Phi_t^{n,F}((T_i, X_i)_{i \le n-1}) \\ &= \int_T \Big\{ F(y, (T_i, X_i)_{i \le n-1}) e^{-\int_{T_{n-1}}^t \lambda_r(y, (T_i, X_i)_{i \le n-1})\, dr} \\ &\qquad\qquad G^{(n-1)}(y, (T_i, X_i)_{i \le n-1}) \Big\} \beta(dy) \end{aligned} \tag{38}$$

*and*

$$\begin{aligned} &\Psi_t^n((T_i, X_i)_{i \le n-1}) \\ &= \int_T \Big\{ e^{-\int_{T_{n-1}}^t \lambda_r(y, (T_i, X_i)_{i \le n-1})\, dr} \\ &\qquad\qquad G^{(n-1)}(y, (T_i, X_i)_{i \le n-1}) \Big\} \beta(dy) \ . \end{aligned} \tag{39}$$

**Proof.** Let $H$ be an arbitrary bounded, $\mathcal{F}_t^{\mathcal{N}}$-measurable function. Since $H = H((T_i, X_i)_{i \le n-1})$ on $\{T_{n-1} \le t < T_n\}$ (cf. Dettweiler [2004]), we obtain from (4)

$$\begin{aligned} &\int_\Omega 1_{\{T_{n-1} \le t < T_n\}} \, H \, F(\Theta, (T_i, X_i)_{i \le n-1}) \, d\mathbf{P} \\ &= \int_T \Big( \int_0^t \int_E \cdots \int_{t_{n-2}}^t \int_E \int_t^\infty \Big\{ H((t_i, x_i)_{i \le n-1}) F(y, (t_i, x_i)_{i \le n-1}) \\ &\quad G^{(n-1)}(y, (t_i, x_i)_{i \le n-1}) \lambda_{t_n}(y, (t_i, x_i)_{i \le n-1}) \\ &\quad e^{-\int_{t_{n-1}}^{t_n} \lambda_r(y, (t_i, x_i)_{i \le n-1})\, dr} \Big\} dt_n \gamma(dx_{n-1}) dt_{n-1} \cdots \gamma(dx_1) dt_1 \Big) \beta(dy) \end{aligned}$$

$$
\begin{aligned}
&= \int_T \Big( \int_0^t \int_E \cdots \int_{t_{n-2}}^t \int_E \Big\{ H((t_i,x_i)_{i\le n-1}) F(y,(t_i,x_i)_{i\le n-1}) \\
&\qquad\qquad G^{(n-1)}(y,(t_i,x_i)_{i\le n-1}) e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,x_i)_{j\le n-1})dr} \Big\} \\
&\qquad\qquad \gamma(dx_{n-1})dt_{n-1}\cdots\gamma(dx_1)dt_1 \Big) \beta(dy) \\
&= \int_0^t \int_E \cdots \int_{t_{n-2}}^t \int_E H((t_i,x_i)_{i\le n-1}) \\
&\qquad \Big( \int_T \Big\{ F(y,(t_i,x_i)_{i\le n-1}) e^{-\int_{t_{n-1}}^t \lambda_r(y,(t,x_i)_{i\le n-1})dr} \\
&\qquad G^{(n-1)}(y,(t_i,x_i)_{i\le n-1}) \Big\} \beta(dy) \Big) \gamma(dx_{n-1})dt_{n-1}\cdots\gamma(dx_1)dt_1 \\
&= \int_0^t \int_E \cdots \int_{t_{n-2}}^t \int_E \Big\{ H((t_i,x_i)_{i\le n-1}) \\
&\qquad\qquad \Phi_t^{n,F}((t_i,x_i)_{i\le n-1}) \Big\} \gamma(dx_{n-1})dt_{n-1}\cdots\gamma(dx_1)dt_1 \\
&= \int_0^t \int_E \cdots \int_{t_{n-2}}^t \int_E \Big\{ H((t_i,x_i)_{i\le n-1}) \frac{\Phi_t^{n,F}((t_i,x_i)_{i\le n-1})}{\Psi_t^n((t_i,x_i)_{i\le n-1})} \\
&\qquad\qquad \Psi_t^n((t_i,x_i)_{i\le n-1}) \Big\} \gamma(dx_{n-1})dt_{n-1}\cdots\gamma(dx_1)dt_1 \\
&= \int_T \Big( \int_0^t \int_E \cdots \int_{t_{n-2}}^t \int_E \int_t^\infty \Big\{ H((t_i,x_i)_{i\le n-1}) \\
&\quad \frac{\Phi_t^{n,F}((t_i,x_i)_{i\le n-1})}{\Psi_t^n((t_i,x_i)_{i\le n-1})} G^{(n-1)}(y,(t_i,x_i)_{i\le n-1}) \lambda_{t_n}(y,(t_i,x_i)_{i\le n-1}) \\
&\quad e^{-\int_{t_{n-1}}^{t_n} \lambda_r(y,(t_i,x_i)_{i\le n-1})\,dr} \Big\}\, dt_n \gamma(dx_{n-1})dt_{n-1}\cdots\gamma(dx_1)dt_1 \Big) \beta(dy) \\
&= \int_\Omega 1_{\{T_{n-1}\le t<T_n\}}\, H\, \frac{\Phi_t^{n,F}((T_i,X_i)_{i\le n-1})}{\Psi_t^n((T_i,X_i)_{i\le n-1})}\, d\mathbf{P}\ .
\end{aligned}
$$

Since

$$
1_{\{T_{n-1}\le t<T_n\}}\, \frac{\Phi_t^{n,F}((T_i,X_i)_{i\le n-1})}{\Psi_t^n((T_i,X_i)_{i\le n-1})}
$$

is $\mathcal{F}_t^{\mathcal{N}}$-measurable, the assertion of the lemma is proved. □

The lemma could be applied quite general to the prediction formula (16). Thus we would get

$$
\mathbf{E}\{Z_u \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \sum_{n\ge 1} 1_{\{T_{n-1}\le t<T_n\}} \sum_{k\ge 1} \frac{\Phi_t^{n,I_{n,k}^{t,u}}((T_i,X_i)_{i\le n-1})}{\Psi_t^n((T_i,X_i)_{i\le n-1})}\ . \tag{40}
$$

We will not pursue this general setting.

As a first concrete application, we compute the prediction of $S_u^B - S_t^B$ given $\mathcal{F}_t^{\mathcal{N}}$ for mixed Poisson processes(cf. proposition 2.1). We remark that for a mixed inhomogeneous Poisson process the density $G^{(n-1)}$ is given by

$$
G^{(n-1)}(y,(t_i,x_i)_{i\le n-1}) \;=\; e^{-\int_0^{t_{n-1}} \lambda_r(y)dr} \prod_{i=1}^{n-1} \big( g_{t_i}(x_i;y) \lambda_{t_i}(y) \big)\ , \tag{41}
$$

and for a mixed homogeneous Poisson process we have

$$G^{(n-1)}(y,(t_i,x_i)_{i\le n-1}) \;=\; \lambda(y)^{n-1}e^{-t_{n-1}\lambda(y)}\prod_{i=1}^{n-1} g(x_i;y)\;. \tag{42}$$

Thus the following results follow immediately from lemma 3.1

**3.2 Proposition**. *Suppose that* $((T_n,X_n))_{n\ge1}$ *is a mixed inhomogeneous Poisson risk process, for which the integrability assumption (20) holds. Then*

$$\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \sum_{n\ge1} 1_{\{T_{n-1}\le t<T_n\}} \frac{\Phi_{t,u}^{n,B}((T_i,X_i)_{i\le n-1})}{\Psi_t^n((T_i,X_i)_{i\le n-1})}\;, \tag{43}$$

*with*

$$\begin{aligned}\Phi_{t,u}^{n,B}((T_i,X_i)_{i\le n-1}) \;=\; & \int_T \Big\{\Big(\int_t^u\int_B x g_s(x;y)\lambda_s(y)\gamma(dx)ds\Big) e^{-\int_0^t \lambda_r(y)\,dr} \\ & \prod_{i=1}^{n-1}\big(g_{T_i}(X_i;y)\lambda_{T_i}(y)\big)\Big\}\beta(dy)\;,\end{aligned} \tag{44}$$

*and*

$$\Psi_t^n((T_i,X_i)_{i\le n-1}) \;=\; \int_T e^{-\int_0^t \lambda_r(y)\,dr}\prod_{i=1}^{n-1}\big(g_{T_i}(X_i;y)\lambda_{T_i}(y)\big)\Big\}\beta(dy)\;. \tag{45}$$

*If* $((T_n,X_n))_{n\ge1}$ *is a mixed homogeneous Poisson risk process, then*

$$\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \sum_{n\ge1} 1_{\{T_{n-1}\le t<T_n\}} \frac{\Phi_{t,u}^{n,B}((X_i)_{i\le n-1})}{\Psi_t^n((X_i)_{i\le n-1})}\;, \tag{46}$$

*with*

$$\begin{aligned}&\Phi_{t,u}^{n,B}((X_i)_{i\le n-1}) \\ &= \;(u-t)\int_T\Big\{\big(\int_B xg(x;y)\,\gamma(dx)\big)\lambda(y)^n e^{-t\lambda(y)}\prod_{i=1}^{n-1} g(X_i;y)\Big\}\beta(dy)\end{aligned} \tag{47}$$

*and*

$$\Psi_t^n((X_i)_{i\le n-1}) \;=\; \int_T \Big\{\lambda(y)^{n-1}e^{-t\lambda(y)}\prod_{i=1}^{n-1} g(X_i;y)\Big\}\beta(dy)\;. \tag{48}$$

**Remark.** Suppose that the measure $\gamma$ is a probability measure. Then the intensity measure $\Lambda^*$, defined by

$$\lambda_t^*(B) \;:=\; \gamma(B)\;, \tag{49}$$

is extremely simple, and it is not difficult to see that there is a probability measure $\mathbf{P}^*$ on $(\Omega,\mathcal{F})$, such that relative to $\mathbf{P}^*$ the risk process $(T_n,X_n)_{n\ge1}$ has the $\mathbf{F}^\Theta$-intensity measure $\Lambda^*$. It is easily proved (cf. also Brémaud [1981]) that the restriction of the original probability measure $\mathbf{P}$ to $\mathcal{F}_t^\Theta$ is absolutely continuous relative to the restriction of $\mathbf{P}^*$ to $\mathcal{F}_t^\Theta$ and has the Radon-Nikodym-density $L_t$, given by

$$L_t \;=\; e^t e^{-\int_{t_{n-1}}^t \lambda_r(\Theta,(T_i,X_i)_{i\le n-1})dr} G^{(n-1)}(\Theta,(T_i,X_i)_{i\le n-1}) \tag{50}$$

on $\{T_{n-1}\le t<T_n\}$. It follows that (43) is just the formula

$$\mathbf{E}_{\mathbf{P}}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \frac{\mathbf{E}_{\mathbf{P}^*}\{(S_u^B - S_t^B)L_t \,|\, \mathcal{F}_t^{\mathcal{N}}\}}{\mathbf{E}_{\mathbf{P}^*}\{L_t \,|\, \mathcal{F}_t^{\mathcal{N}}\}}\;, \tag{51}$$

which is well known in the literature (cf. Brémaud [1981]). We will not pursue this idea (i.e. using a type of Girsanov transformation for prediction), since our formulas are immediate consequences from the construction of marked point processes.

In connection with the above theorem we consider a related prediction problem, which occurs, if for the given time point $t$ there is only the information on

$$\{T_{n-1}\le t<T_n\}\;,\ \text{and } (X_i)_{i\le n-1} \quad (n\ge 1)\;.$$

available. To model this situation we set

$$\begin{aligned}\mathcal{G}_t^n &:= \sigma(\{1_{\{T_{n-1}\le t<T_n\}}, X_1,\cdots,X_{n-1}\}) \qquad (52)\\ &= \sigma\Big(\Big\{\{T_{n-1}\le t<T_n\}\cap\bigcap_{j=1}^{n-1}\{X_j\in B_j\}\,\Big|\,B_j\in\mathcal{E}\ (1\le j\le n-1)\Big\}\Big)\end{aligned}$$

and

$$\mathcal{G}_t \;:=\; \bigvee_{n\ge 1}\mathcal{G}_t^n\;. \tag{53}$$

For the filtering relative to $\mathcal{G}_t$ the following lemma is proved similarly as lemma 3.1.

**3.4 Lemma**. *Suppose that $F=F(\Theta,(T_i,X_i)_{i\le n-1})$ is integrable. Then for every $t\ge 0$ and every $n\ge 1$ the following filtering formula holds on $\{T_{n-1}\le t<T_n\}$:*

$$\begin{aligned}\mathbf{E}\{F(\Theta,(T_i,X_i)_{i\le n-1})\,|\,\mathcal{G}_t\} &= \mathbf{E}\{F(\Theta,(T_i,X_i)_{i\le n-1})\,|\,\mathcal{G}_t^n\} \qquad (54)\\ &= \frac{\overline{\Phi}_t^{n,F}(X_i)_{i\le n-1})}{\overline{\Psi}_t^n((X_i)_{i\le n-1})}\;,\quad \textit{with}\end{aligned}$$

$$\begin{aligned}\overline{\Phi}_t^{n,F}((X_i)_{i\le n-1}) \;=\; \int_T\int_0^t\cdots\int_{t_{n-2}}^t \Big\{&F(y,(t_i,X_i)_{i\le n-1})e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,X_i)_{i\le n-1})\,dr}\\ &G^{(n-1)}(y,(t_i,X_i)_{i\le n-1})\Big\}dt_{n-1}\cdots dt_1\beta(dy)\end{aligned}$$

*and*

$$\overline{\Psi}_t^n((X_i)_{i\leq n-1}) = \int_T \int_0^t \cdots \int_{t_{n-2}}^t \Big\{ e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,X_i)_{i\leq n-1})\,dr} \\ G^{(n-1)}(y,(t_i,X_i)_{i\leq n-1}) \Big\} dt_{n-1}\cdots dt_1 \beta(dy) \; .$$

If we apply this lemma to the filtering of $\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^\Theta\}$ relative to $\mathcal{G}_t$ for mixed Poisson risk processes we get:

**3.5 Proposition**. *Suppose that $((T_n,X_n))_{n\geq 1}$ is a mixed inhomogeneous Poisson risk process, for which the integrability assumption (20) holds. Then*

$$\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{G}_t\} = \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}} \frac{\overline{\Phi}_{t,u}^B((X_i)_{i\leq n-1})}{\overline{\Psi}_t((X_i)_{i\leq n-1})} \; , \tag{55}$$

*with*

$$\overline{\Phi}_{t,u}^{n,B}((X_i)_{i\leq n-1}) = \int_T \int_0^t \cdots \int_{t_{n-2}}^t \Big\{ \Big( \int_t^u \int_B x g_s(x;y)\lambda_s(y)\gamma(dx)ds \Big) e^{-\int_0^t \lambda_r(y)\,dr} \\ \prod_{i=1}^{n-1} \big( g_{t_i}(X_i;y)\lambda_{t_i}(y) \big) \Big\} dt_{n-1}\cdots dt_1 \beta(dy) \; ,$$

*and*

$$\overline{\Psi}_t^n((X_i)_{i\leq n-1}) \\ = \int_T \int_0^t \cdots \int_{t_{n-2}}^t e^{-\int_0^t \lambda_r(y)\,dr} \prod_{i=1}^{n-1} \big( g_{t_i}(X_i;y)\lambda_{t_i}(y) \big) \Big\} dt_{n-1}\cdots dt_1 \beta(dy) \; .$$

*If $((T_n,X_n))_{n\geq 1}$ is a mixed homogeneous Poisson risk process, then (cf. proposition 3.2)*

$$\mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{F}_t^{\mathcal{N}}\} = \mathbf{E}\{S_u^B - S_t^B \,|\, \mathcal{G}_t\} \; . \tag{56}$$

Now we consider the filtering of $\mathbf{E}\{S_{T_t\wedge u}^B - S_t^B \,|\, \mathcal{F}_t^\Theta\}$ relative to $\mathcal{F}_t^{\mathcal{N}}$ and also to $\mathcal{G}_t$. Using the formula (26) we get from lemma 3.1 and lemma 3.4 the following general result:

**3.6 Proposition**. *Let $((T_n,X_n))_{n\geq 1}$ be a mixed risk process with regular $\mathbf{F}^\Theta$-intensity measure $\Lambda$ and suppose that $S_{T_t\wedge u}^B - S_t^B$ is integrable. If $J_n^{t,u}$ is defined by (27), then*

$$\mathbf{E}\{S_{T_t\wedge u}^B - S_t^B \,|\, \mathcal{F}_t^{\mathcal{N}}\} = \sum_{n\geq 1} 1_{[T_{n-1},T_n[}(t) \frac{\Phi_{t,u}^n((T_i,X_i)_{i\leq n-1})}{\Psi_t^n((T_i,X_i)_{i\leq n-1})} \; , \tag{57}$$

*where* $\Phi^n_{t,u}$ *and* $\Psi^n_t$ *are given by*

$$\Phi^n_{t,u}((T_i,X_i)_{i\le n-1}) = \int_T \Big\{ J^n_{t,u}(y,(T_i,X_i)_{i\le n-1}) e^{-\int_{t_{n-1}}^t \lambda_r(y,(T_i,X_i)_{i\le n-1})dr} G^{(n-1)}(y,(T_i,X_i)_{i\le n-1})\Big\}\beta(dy)$$

*and*

$$\Psi^n_t((T_i,X_i)_{i\le n-1}) = \int_T \Big\{ e^{-\int_{t_{n-1}}^t \lambda_r(y,(T_i,X_i)_{i\le n-1})dr} G^{(n-1)}(y,(T_i,X_i)_{i\le n-1})\Big\}\beta(dy)\ .$$

*For the filtering relative to* $\mathcal{G}_t$ *we have*

$$\mathbf{E}\{S^B_{T_t\wedge u} - S^B_t \,|\, \mathcal{G}_t\} = \sum_{n\ge 1} 1_{[T_{n-1},T_n[}(t) \frac{\overline{\Phi}^n_{t,u}((X_i)_{i\le n-1})}{\overline{\Psi}^n_t((X_i)_{i\le n-1})}\ , \tag{58}$$

*where* $\overline{\Phi}^n_{t,u}$ *and* $\overline{\Psi}^n_t$ *are given by*

$$\overline{\Phi}^n_{t,u}((X_i)_{i\le n-1}) = \int_T\int_0^t \cdots \int_{t_{n-2}}^t \Big\{ J^n_{t,u}(y,(t_i,X_i)_{i\le n-1}) e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,X_i)_{i\le n-1})dr} G^{(n-1)}(y,(t_i,X_i)_{i\le n-1})\Big\} dt_{n-1}\cdots dt_1\beta(dy)$$

*and*

$$\overline{\Psi}^n_t((X_i)_{i\le n-1}) = \int_T\int_0^t \cdots \int_{t_{n-2}}^t \Big\{ e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,X_i)_{i\le n-1})dr} G^{(n-1)}(y,(t_i,X_i)_{i\le n-1})\Big\} dt_{n-1}\cdots dt_1\beta(dy)\ .$$

The above general result can easily be applied to more special mixed risk processes. We just give two examples.

**3.7 Proposition**. *Let* $((T_n,X_n))_{n\ge1}$ *be a mixed risk process such that* $S^B_{T_t\wedge u} - S^B_t$ *is integrable.*
*(a) If* $((T_n,X_n))_{n\ge1}$ *is a mixed homogeneous Markovian risk process, then*

$$\mathbf{E}\{S^B_{T_t\wedge u} - S^B_t \,|\, \mathcal{F}^{\mathcal{N}}_t\} = \sum_{n\ge 1} 1_{[T_{n-1},T_n[}(t) \frac{\Phi^n_{t,u}((T_i,X_i)_{i\le n-1})}{\Psi^n_t((T_i,X_i)_{i\le n-1})}\ , \tag{59}$$

*with*

$$\Phi^n_{t,u}((T_i,X_i)_{i\le n-1}) = \int_T \Big\{ \big((1-e^{-(u-t)\lambda^{(n)}(y)}\big)\int_B x g^{(n)}(x,y)\gamma(dx)\big) e^{-(t-T_{n-1})\lambda^{(n)}(y)} e^{-\sum_{i=1}^{n-1}(T_i-T_{i-1})\lambda^{(i)}(y)} \prod_{i=1}^{n-1}\big(g^{(i)}(x;y)\lambda^{(i)}(y)\big)\Big\}\beta(dy)$$

*and*

$$\Psi^n_{t,u}((T_i,X_i)_{i\le n-1}) \;=\; \int_T \Big\{ e^{-(t-T_{n-1})\lambda^{(n)}(y)} e^{-\sum_{i=1}^{n-1}(T_i-T_{i-1})\lambda^{(i)}(y)} \prod_{i=1}^{n-1}\big(g^{(i)}(x;y)\lambda^{(i)}(y)\big)\Big\}\beta(dy)\ .$$

*Similarly,*

$$\mathbf{E}\{S^B_{T_t\wedge u}-S^B_t\,|\,\mathcal{G}_t\} \;=\; \sum_{n\ge 1} 1_{[T_{n-1},T_n[}(t)\frac{\overline{\Phi}^n_{t,u}((X_i)_{i\le n-1})}{\overline{\Psi}^n_t((X_i)_{i\le n-1})}\ , \qquad (60)$$

*with*

$$\begin{aligned}&\overline{\Phi}^n_{t,u}((X_i)_{i\le n-1})\\ &= \int_T \Big\{ \big((1-e^{-(u-t)\lambda^{(n)}(y)}\big)\int_B xg^{(n)}(x,y)\gamma(dx)\big)\prod_{i=1}^{n-1}\big(g^{(i)}(x;y)\lambda^{(i)}(y)\big)\\ &\quad\Big(\int_0^t\cdots\int_{t_{n-2}}^t e^{-(t-t_{n-1})\lambda^{(n)}(y)-\sum_{i=1}^{n-1}(t_i-t_{i-1})\lambda^{(i)}(y)}dt_{n-1}\cdots dt_1\Big)\Big\}\beta(dy)\end{aligned}$$

*and*

$$\begin{aligned}&\overline{\Psi}^n_{t,u}((X_i)_{i\le n-1})\\ &= \int_T \Big\{ \prod_{i=1}^{n-1}\big(g^{(i)}(x;y)\lambda^{(i)}(y)\big)\\ &\quad\Big(\int_0^t\cdots\int_{t_{n-2}}^t e^{-(t-t_{n-1})\lambda^{(n)}(y)-\sum_{i=1}^{n-1}(t_i-t_{i-1})\lambda^{(i)}(y)}dt_{n-1}\cdots dt_1\Big)\Big\}\beta(dy)\ .\end{aligned}$$

*(b) If $((T_n,X_n))_{n\ge1}$ is a mixed homogeneous Poisson risk process, then*

$$\begin{aligned}\mathbf{E}\{S^B_{T_t\wedge u}-S^B_t\,|\,\mathcal{F}^{\mathcal{N}}_t\} &= \mathbf{E}\{S^B_{T_t\wedge u}-S^B_t\,|\,\mathcal{G}_t\} \qquad (61)\\ &= \sum_{n\ge1} 1_{\{T_{n-1}\le t<T_n\}}\frac{\Phi^n_{t,u}((X_i)_{i\le n-1})}{\Psi^n_t((X_i)_{i\le n-1})}\ ,\end{aligned}$$

*with*

$$\Phi^n_{t,u}((X_i)_{i\le n-1}) \;=\; \int_T \Big\{\big(1-e^{-(u-t)\lambda(y)}\big) \int_B xg(x;y)\,\gamma(dx)\lambda(y)^{n-1}e^{-t\lambda(y)}\prod_{i=1}^{n-1} g(X_i;y)\Big\}\beta(dy)$$

*and*

$$\Psi_t((X_i)_{i\le n-1}) \;=\; \int_T \Big\{\lambda(y)^{n-1}e^{-t\lambda(y)}\prod_{i=1}^{n-1} g(X_i;y)\Big\}\beta(dy)\ .$$

Now we consider the prediction of $X_n 1_{\{X_n \in B\}}$ given

$$\mathcal{G}_{T_{n-1}} \;=\; \sigma(\{X_1, \cdots, X_{n-1}\}) \;,$$

i.e. the filtering of $\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, \mathcal{F}^{\Theta}_{T_{n-1}}\}$ relative to $\mathcal{G}_{T_{n-1}}$. If we write

$$J^n(\Theta, (T_i, X_i)_{i\le n-1})$$

for the right hand side of equation (32), we have the following general result:

**3.8 Theorem**. *Let $((T_n, X_n))_{n\ge 1}$ be a mixed risk process with regular $\mathbf{F}^{\Theta}$-intensity measure and suppose that $X_n 1_{\{X_n\in B\}}$ is integrable. Then*

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, (X_i)_{i\le n-1}\} \;=\; \frac{\overline{\Phi}_n((X_i)_{i\le n-1})}{\overline{\Psi}_n((X_i)_{i\le n-1})} \tag{62}$$

*with*

$$\overline{\Phi}_n((X_i)_{i\le n-1}) \;=\; \int_T \int_0^\infty \cdots \int_{t_{n-2}}^\infty \Big\{ J^n(y, (t_i, X_i)_{i\le n-1}) \\ G^{(n-1)}(y, (t_i, X_i)_{i\le n-1}) \Big\}\, dt_{n-1} \cdots dt_1 \beta(dy) \;,$$

*and*

$$\overline{\Psi}_n((X_i)_{i\le n-1}) \;=\; \int_T \int_0^\infty \cdots \int_{t_{n-2}}^\infty \Big\{ G^{(n-1)}(y, (t_i, X_i)_{i\le n-1}) \Big\}\, dt_{n-1} \cdots dt_1 \beta(dy) \;.$$

This result can easily be applied to special mixed risk processes. Thus we get e.g.:

**3.9 Proposition**. *(a) If $(T_n, X_n)_{n\ge 1}$ is a mixed Markovian risk process, then*

$$\begin{aligned} J^n(y, (t_i, x_i)_{i\le n-1}) \;&=\; J^n(y, t_{n-1}) \\ &=\; \int_{t_{n-1}}^\infty \Big( \int_B x g^{(n)}_{t_n}(x; y) \gamma(dx) \Big) \lambda^{(n)}_{t_n}(y) e^{-\int_{t_{n-1}}^{t_n} \lambda^{(n)}_s(y) ds}\, dt_n \;, \end{aligned} \tag{63}$$

*and*

$$\begin{aligned} &\mathbf{E}\{X_n 1_{\{X_n\in B\}} \,|\, (X_i)_{1\le i\le n-1}\} \\ &=\; \frac{\int_T \int_0^\infty \cdots \int_{t_{n-2}}^\infty J^n(y, t_{n-1}) G^{(n-1)}(y, (t_i, X_i)_{i\le n-1}) dt_{n-1} \cdots dt_1 \beta(dy)}{\int_T \int_0^\infty \cdots \int_{t_{n-2}}^\infty G^{(n-1)}(y, (t_i, X_i)_{i\le n-1}) dt_{n-1} \cdots dt_1 \beta(dy)} \;, \end{aligned} \tag{64}$$

*where*

$$G^{(n-1)}(y, (t_i, x_i)_{i\le n-1}) \;=\; \prod_{i=1}^{n-1} g^{(i)}_{t_i}(x; y) \lambda^{(i)}_{t_i}(y) e^{-\int_{t_{i-1}}^{t_i} \lambda^{(i)}_s(y) ds} \;.$$

*If $(T_n, X_n)_{n\ge 1}$ is homogeneous, then*

$$J^n(y, t_{n-1}) \;=\; J^n(y) \;=\; \int_B x g^{(n)}(x; y) \gamma(dx) \;,$$

*and*

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}}\,|\,(X_i)_{i\le n-1}\} \qquad (65)$$
$$= \frac{\int_T \left\{ \int_B x g^{(n)}(x;y)\gamma(dx) \prod_{i=1}^{n-1} g^{(i)}(X_i;y)\right\} \beta(dy)}{\int_T \left\{ \prod_{i=1}^{n-1} g^{(i)}(X_i;y)\right\} \beta(dy)} \ .$$

*(b)If $(T_n, X_n)_{n\ge 1}$ is a mixed homogeneous Poisson risk process, then*

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}}\,|\,(X_i)_{i\le n-1}\} \qquad (66)$$
$$= \frac{\int_T \left\{ \int_B x g^(x;y)\gamma(dx) \prod_{i=1}^{n-1} g(X_i;y)\right\} \beta(dy)}{\int_T \left\{ \prod_{i=1}^{n-1} g(X_i;y)\right\} \beta(dy)} \ .$$

**Proof.** Formula (64) follows immediately from theorem 3.8. We just prove (65), which implies (66). From (64) we have

$$\mathbf{E}\{X_n 1_{\{X_n\in B\}}\,|\,(X_i)_{i\le n-1}\} \;=\; \frac{\overline{\Phi}_n((X_i)_{i\le n-1})}{\overline{\Psi}_n((X_i)_{i\le n-1})}$$

with

$$\overline{\Phi}_n((X_i)_{i\le n-1}) \;=\; \int_T \Big\{ \Big( \int_0^\infty \cdots \int_{t_{n-2}}^\infty \{\prod_{i=1}^{n-1} \lambda^{(i)}(y) e^{-(t_i-t_{i-1})\lambda^{(i)}(y)}\} dt_{n-1}\cdots dt_1 \Big)$$
$$\int_B x g^{(n)}(x;y)\gamma(dx) \prod_{i=1}^{n-1} g^{(i)}(X_i;y) \Big\} \beta(dy)$$

and a similar formula for $\overline{\Psi}_n((X_i)_{1\le i\le n-1})$. Thus (65) follows, since

$$\int_0^\infty \cdots \int_{t_{n-2}}^\infty \{\prod_{i=1}^{n-1} \lambda^{(i)}(y) e^{-(t_i-t_{i-1})\lambda^{(i)}(y)}\} dt_{n-1}\cdots dt_1 \;=\; 1$$

for every $y \in T$. $\square$

**Remark.** At a first glance the formulas (65) and (66) may a little bit surprise, since there is no dependence on the distributions of the claim arrival times $T_1, \cdots, T_{n-1}$. But the reason is simple: The prediction of $X_n 1_{\{X_n\in B\}}$ given $X_1, \cdots, X_{n-1}$ replaces the natural time by the time points $T_1, \cdots, T_{n-1}$ and the independence of $(T_n)_{n\ge 1}$ and $(X_n)_{n\ge 1}$ reduces the prediction to a prediction problem for the discrete time process $(X_n)_{n\ge 1}$. The situation becomes quite different, if we consider the prediction of $X_n 1_{\{X_n\in B\}}$ given $(T_i, X_i)_{i\le n-1}$:

Suppose that $((T_n, X_n))_{n\ge 1}$ is a mixed homogeneous Poisson risk process. Then

$$\begin{aligned}
&\mathbf{E}\{X_n 1_{\{X_n\in B\}}\,|\,(T_i, X_i)_{i\le n-1}\} \\
&= \mathbf{E}\{X_n 1_{\{X_n\in B\}}\,|\,T_{n-1}, X_1, \cdots, X_{n-1}\} \qquad (67) \\
&= \frac{\int_T \left\{ \left( \int_B x g(x;y)\gamma(dx)\right) e^{-T_{n-1}\lambda(y)} \lambda(y)^{n-1} \prod_{i=1}^{n-1} g(X_i;y)\right\} \beta(dy)}{\int_T \left\{ e^{-T_{n-1}\lambda(y)} \lambda(y)^{n-1} \prod_{i=1}^{n-1} g(X_i;y)\right\} \beta(dy)} \ .
\end{aligned}$$

Since an increase of information gives surely more reliable prediction, formula (67) should be better than (65) in case there is the information on $T_{n-1}$.

As a last filtering problem we consider the problem of filtering the intensity measure $\Lambda$ relative to $\mathbf{F}^{\mathcal{N}}$. It will turn out that the filtered intensity measure $\tilde{\Lambda}$ is again regular. We have the following general result, which follows easily from lemma 3.1:

**3.11 Theorem**. *Suppose that the intensities $\lambda_t(B)$ ($t \geq 0$ and $B \in \mathcal{E}$) are integrable. Then the following formula holds:*

$$\mathbf{E}\{\lambda_t(B) \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \sum_{n\geq 1} 1_{[T_{n-1},T_n[}(t)\tilde{\lambda}_t(B;(T_i,X_i)_{i\leq n-1}) \, , \tag{68}$$

*with*

$$\tilde{\lambda}_t(B;(T_i,X_i)_{i\leq n-1}) \;=\; \frac{\phi_t(B;(T_i,X_i)_{i\leq n-1})}{\theta_t((T_i,X_i)_{i\leq n-1})} \, , \tag{69}$$

*where $\phi_t$ and $\theta_t$ are given by (cf. (5))*

$$\begin{aligned}\phi_t(B;(t_i,x_i)_{i\leq n-1}) \;=\; \int_T \Big\{ \int_B & g_t(x;y,(t_i,x_i)_{i\leq n-1})\gamma(dx)\lambda_t(y,(t_i,x_i)_{i\leq n-1}) \\ & e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,x_i)_{i\leq n-1})dr} G^{(n-1)}(y,(t_i,x_i)_{i\leq n-1})\Big\}\beta(dy) \, ,\end{aligned}$$

*and*

$$\theta_t((t_i,x_i)_{i\leq n-1}) \;=\; \int_T \Big\{ e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,x_i)_{i\leq n-1})dr} G^{(n-1)}(y,(t_i,x_i)_{i\leq n-1})\Big\}\beta(dy) \, ,$$

*for $0 < t_1 < \cdots < t_{n-1}$ and $x_1, \cdots, x_{n-1} \in E$.*

*The $\mathbf{F}^{\mathcal{N}}$-intensity measure*

$$\tilde{\Lambda} \;=\; \big((\tilde{\lambda}_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}} \tag{70}$$

*given by (69) has a similar structure as the $\mathbf{F}^{\Theta}$-intensity measure $\Lambda$. On the set $\{T_{n-1} \leq t < T_n\}$ we have*

$$\tilde{\lambda}_t(dx;(T_i,X_i)_{i\leq n-1}) \;=\; \tilde{\gamma}_t(dx;(T_i,X_i)_{i\leq n-1}) \cdot \tilde{\lambda}_t((T_i,X_i)_{i\leq n-1}) \, , \tag{71}$$

*where*

$$\tilde{\lambda}_t((T_i,X_i)_{i\leq n-1}) \;=\; \frac{\phi_t(E;(T_i,X_i)_{i\leq n-1})}{\theta_t((T_i,X_i)_{i\leq n-1})} \, , \tag{72}$$

*and where the probability measure $\tilde{\gamma}_t(dx;(T_i,X_i)_{i\leq n-1})$ has a density*

$$\tilde{g}_t(x;(T_i,X_i)_{i\leq n-1})$$

*relative to $\gamma(dx)$, which is given as follows:*

*We set for $x \in E$*

$$\tilde{\phi}_t(x;(t_i,x_i)_{i\le n-1}) \;=\; \int_T \Big\{ g_t(x;y,(t_i,x_i)_{i\le n-1})\lambda_t(y,(t_i,x_i)_{i\le n-1}) \tag{73}$$
$$e^{-\int_{t_{n-1}}^t \lambda_r(y,(t_i,x_i)_{i\le n-1})dr} G^{(n-1)}(y,(t_i,x_i)_{i\le n-1}) \Big\} \beta(dy)\;,$$

*and choose a fixed, measurable $g^* : E \to \mathbb{R}_+$ such that $g^* \cdot \gamma$ is a probability measure. Then*

$$\tilde{g}_t(x;(T_i,X_i)_{i\le n-1}) \;=\; \frac{\tilde{\phi}_t(x;(T_i,X_i)_{i\le n-1})}{\phi_t(E;(T_i,X_i)_{i\le n-1})} 1_{\{\phi_t(E;(T_i,X_i)_{i\le n-1})>0\}} \tag{74}$$
$$+\, g^*(x) 1_{\{\phi_t(E;(T_i,X_i)_{i\le n-1})=0\}}\;.$$

**3.12 Corollary**. *Let $((T_n,X_n))_{n\ge 1}$ be a mixed homogeneous Poisson risk process with the regular $\mathbf{F}^\Theta$-intensity measure $\Lambda$ given by (10) and (11). Then*

$$\tilde{\lambda}_t(dx;(t_n,x_n)_{n\ge 0}) \;=\; \sum_{n\ge 1} 1_{[t_{n-1},t_n[}(t)\tilde{\lambda}_t(dx;(t_i,x_i)_{i\le n-1})$$

*with*

$$\tilde{\lambda}_t(dx;(t_i,x_i)_{i\le n-1}) \;=\; \tilde{g}_t(x;(t_i,x_i)_{i\le n-1})\gamma(dx)\tilde{\lambda}_t((t_i,x_i)_{i\le n-1})\;, \tag{75}$$

*where*

$$\tilde{\lambda}_t((t_i,x_i)_{i\le n-1}) \;=\; \frac{\int_T \prod_{i=1}^{n-1} g(x_i;y)\,\lambda(y)^n e^{-t\lambda(y)}\,\beta(dy)}{\int_T \prod_{i=1}^{n-1} g(x_i;y)\,\lambda(y)^{n-1} e^{-t\lambda(y)}\,\beta(dy)}$$

*and*

$$\tilde{g}_t(x;(t_i,x_i)_{i\le n-1}) \;=\; \frac{\int_T g(x;y)\prod_{i=1}^{n-1} g(x_i;y)\,\lambda(y)^n e^{-t\lambda(y)}\,\beta(dy)}{\int_T \prod_{i=1}^{n-1} g(x_i;y)\,\lambda(y)^{n} e^{-t\lambda(y)}\,\beta(dy)}\;.$$

**Remark.** It is well known (cf. e.g. Schmidt [1996]) that the filtering of a mixed Poisson process gives a Markov process. Corollary 3.12 shows that the filtering of a mixed Poisson *risk* process gives no longer a Markovian risk process.

**3.14 Corollary**. *Let $((T_n,X_n))_{n\ge 1}$ be a mixed inhomogeneous Poisson risk process with the regular $\mathbf{F}^\Theta$-intensity measure $\Lambda$ given by (12) and (13). Then*

$$\tilde{\lambda}_t(dx;(t_n,x_n)_{n\ge 0}) \;=\; \sum_{n\ge 1} 1_{[t_{n-1},t_n[}(t)\tilde{\lambda}_t(dx;(t_i,x_i)_{i\le n-1})$$

*with*

$$\tilde{\lambda}_t(dx;(t_i,x_i)_{i\le n-1}) \;=\; \tilde{g}_t(x;(t_i,x_i)_{i\le n-1})\gamma(dx)\tilde{\lambda}_t((t_i,x_i)_{i\le n-1})\;, \tag{76}$$

*where*

$$\tilde{\lambda}_t((t_i,x_i)_{i\le n-1}) \;=\; \frac{\int_T \prod_{i=1}^{n-1}\big(g_{t_i}(x_i;y)\,\lambda_{t_i}(y)\big)e^{-\int_0^t \lambda_s(y)\,ds}\,\beta(dy)}{\int_T \prod_{i=1}^{n-1}\big(g_{t_i}(x_i;y)\,\lambda_{t_i}(y)\big)e^{-\int_0^t \lambda_s(y)\,ds}\,\beta(dy)}$$

*and*

$$\tilde{g}_t(x;(t_i,x_i)_{i\le n-1}) \;=\; \frac{\int_T g_t(x;y)\lambda_t(y)\prod_{i=1}^{n-1} g_{t_i}(x_i;y)\,\lambda_{t_i}(y)e^{-\int_0^t \lambda_s(y)\,ds}\,\beta(dy)}{\int_T \prod_{i=1}^{n-1}\big(g_{t_i}(x_i;y)\,\lambda_{t_i}(y)\big)e^{-\int_0^t \lambda_s(y)\,ds}\,\beta(dy)}\;.$$

We omit the corresponding filtering result for the mixed Markovian risk process.

Theorem 3.11 has the following interpretation:

**3.15 Proposition**. *Let $((T_n,X_n))_{n\ge1}$ be an integrable, mixed risk process with regular $\mathbf{F}^\Theta$-intensity measure*

$$\Lambda \;=\; \big((\lambda_t(B))_{t\ge0}\big)_{B\in\mathcal{E}}\;,$$

*and let*

$$\tilde{\Lambda} \;=\; \big((\tilde{\lambda}_t(B))_{t\ge0}\big)_{B\in\mathcal{E}}\;,$$

*be the intensity measure defined in theorem 3.11. Then $((T_n,X_n))_{n\ge1}$ has the intensity measure $\tilde{\Lambda}$ for the filtration $\mathbf{F}^{\mathcal{N}}$.*

**Proof.** We have to prove that for every $B\in\mathcal{E}$ the process

$$\Big(N_t(B)-\int_0^t \tilde{\lambda}_s\,ds\Big)_{t\ge0}$$

is an $\mathbf{F}^{\mathcal{N}}$-martingale. This follows, since for $0\le s<t$

$$\begin{aligned}
\mathbf{E}\{N_t(B)-N_s(B)\,|\,\mathcal{F}_s^{\mathcal{N}}\} &= \mathbf{E}\big\{\mathbf{E}\{N_t(B)-N_s(B)\,|\,\mathcal{F}_s^{\Theta}\}\,\big|\,\mathcal{F}_s^{\mathcal{N}}\big\}\\
&= \mathbf{E}\big\{\mathbf{E}\{\int_s^t \lambda_r\,dr\,|\,\mathcal{F}_s^{\Theta}\}\,\big|\,\mathcal{F}_s^{\mathcal{N}}\big\}\\
&= \mathbf{E}\{\int_s^t \lambda_r\,dr\,|\,\mathcal{F}_s^{\mathcal{N}}\}\\
&= \mathbf{E}\{\int_s^t \tilde{\lambda}_r\,dr\,|\,\mathcal{F}_s^{\mathcal{N}}\}\;.
\end{aligned}$$

$\square$

**Remark.** Proposition 3.15 could be used to compute directly the prediction formulas relative to $\mathcal{F}_t^{\mathcal{N}}$ proved in this section. But this looks much more complicated than the method of first predicting relative to $\mathcal{F}_t^{\Theta}$ and then filtering relative to $\mathcal{F}_t^{\mathcal{N}}$.

# 4 Prediction of the Claim Amount Distribution

The prediction problems of the foregoing sections are only a first step to get some insight into the future behaviour of mixed risk processes. One should have in mind that e.g. the prediction of the first claim amount

$$X_{t,u} \;:=\; S_{T_t \wedge u} - S_t \quad (S_t = S_t^E)$$

in the planning period $[t,u]$ is just a mean value on the basis of the informations given by $\mathcal{F}_t^{\mathcal{N}}$. For more reliable information one should not only consider the prediction of this mean value, but also (at least theoretically) the prediction of the distribution of $X_{t,u}$. Suppose that this distribution is given by

$$P_{t,u}(dz; \mathcal{F}_t^{\mathcal{N}}) \,. \tag{77}$$

Then one can easily predict in addition the "distance" $d(X_{(t)}, \varphi_t)$ to some given $\mathcal{F}_t^{\mathcal{N}}$-measurable function $\varphi_t$ [one could imagine that $\varphi_t$ is (connected with) the individual premium predicted by the last section]. This prediction would simply be given by the formula

$$\mathbf{E}\{d(X_{t,u}, \varphi_t) \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \int_E d(z, \varphi_t) P_{t,u}(dz; \mathcal{F}_t^{\mathcal{N}}) \,. \tag{78}$$

Such a distance $d(\cdot,\cdot)$ could be the euclidian distance

$$d_2(z,v) \;:=\; (z-v)^2 \,. \tag{79}$$

Thus for $\varphi_t = \mathbf{E}\{X_{t,u} \,|\, \mathcal{F}_t^{\mathcal{N}}\}$ the prediction of $d_2(X_{t,u}, \varphi_t)$ is just the prediction of the conditional variance of $X_{t,u}$ given $\mathcal{F}_t^{\mathcal{N}}$. But there are also other "distances" (which are even more important): We set for a given $\varepsilon > 0$

$$d_\varepsilon(z,v) \;:=\; 1_{\{z \geq v+\varepsilon\}} \,, \tag{80}$$

and

$$d_+(z,v) := (z-v)_+ \,. \tag{81}$$

Then the prediction of $d_\varepsilon(X_{t,u}, \varphi_t)$ e.g. is just the conditional probability of $\{X_{t,u} \geq \varphi_t + \varepsilon\}$ given $\mathcal{F}_t^{\mathcal{N}}$.

The formulas for the prediction of the claim amount distribution are derived similarly as the prediction formulas for $X_{t,u}$. Analogously to proposition 3.6 we get

**4.1 Theorem**. *Let $((T_n, X_n))_{n\geq 1}$ be a integrable, mixed risk process with regular $\mathbf{F}^\Theta$-intensity measure*

$$\Lambda \;=\; \big((\lambda_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}} \,,$$

*and define for $0 \leq t < u$, $n \geq 1$, $y \in T$, $0 < t_1 < \cdots < t_{n-1}$, $x_1, \cdots, x_{n-1} \in E$ and $A \in \mathcal{E}$*

$$J_n^{t,u}(A; y, (t_i, x_i)_{i\le n-1}) \;=\; \int_t^u \Big( \int_A g_{t_n}(x; y, (t_i, x_i)_{i\le n-1}) \gamma(dx) \Big) \lambda_{t_n}(y, (t_i, x_i)_{i\le n-1}) e^{-\int_t^{t_n} \lambda_s(y,(t_i,x_i)_{i\le n-1})\, ds}\, dt_n + 1_A(0) e^{-\int_t^u \lambda_s(y,(t_i,x_i)_{i\le n-1}) ds} . \tag{82}$$

*Then*

$$\mathbf{P}\{X_{t,u} \in A \,|\, \mathcal{F}_t^{\mathcal{N}}\} \;=\; \sum_{n\ge 1} 1_{[T_{n-1},T_n[}(t) \frac{\Phi_n^{t,u}(A; (T_i, X_i)_{i\le n-1})}{\Psi_n^t((T_i, X_i)_{i\le n-1})} , \tag{83}$$

*where* $\Phi_n^{t,u}(A)$ *and* $\Psi_n^t$ *are given by*

$$\Phi_n^{t,u}(A; (T_i, X_i)_{i\le n-1}) \;=\; \int_T \Big\{ J_n^{t,u}(A; y, (T_i, X_i)_{i\le n-1}) e^{-\int_{t_{n-1}}^t \lambda_r(y,(T_i,X_i)_{i\le n-1}) dr} G^{(n-1)}(y, (T_i, X_i)_{i\le n-1}) \Big\} \beta(dy) ,$$

*and*

$$\Psi_n^t((T_i, X_i)_{i\le n-1}) = \int_T \Big\{ e^{-\int_{t_{n-1}}^t \lambda_r(y,(T_i,X_i)_{i\le n-1}) dr} G^{(n-1)}(y, (T_i, X_i)_{i\le n-1}) \Big\} \beta(dy) .$$

*Thus we have the formula*

$$P_{t,u}(dz; \mathcal{F}_t^{\mathcal{N}}) \;=\; \sum_{n\ge 1} 1_{[T_{n-1},T_n[}(t) \frac{\Phi_n^{t,u}(dz; (T_i, X_i)_{i\le n-1})}{\Psi_n^t((T_i, X_i)_{i\le n-1})} . \tag{84}$$

**Proof.** Since

$$X_{t,u} \;=\; \sum_{n\ge 1} 1_{\{T_{n-1}\le t<T_n\}} X_n 1_{\{T_n\le u\}} ,$$

we have

$$\{X_{t,u} \in A\} \cap \{T_{n-1} \le t < T_n\} \;=\; \{X_n 1_{\{T_n\le u\}} \in A\} \cap \{T_{n-1} \le t < T_n\} .$$

If $0 \notin A$, then

$$\{X_n 1_{\{T_n\le u\}} \in A\} \;=\; \{X_n \in A\} \cap \{T_n \le u\} ,$$

and if $0 \in A$, then

$$\{X_n 1_{\{T_n\le u\}} \in A\} \;=\; (\{X_n \in A\} \cap \{T_n \le u\}) \cup \{u < T_n\}$$

where the union on the right hand side clearly is disjoint. Altogether we have

$$1_{\{X_{t,u}\in A\}} \;=\; \sum_{n\ge 1} 1_{\{T_{n-1}\le t<T_n\}} \big( 1_{\{X_n\in A\}\cap\{T_n\le u\}} + 1_A(0) 1_{\{u<T_n\}} \big) .$$

Thus the prediction of $1_{\{X_{t,u}\in A\}}$ given $\mathcal{F}_t^{\Theta}$ is given by (cf. section 2, esp. (27))

$$\mathbf{E}\{1_{\{X_{t,u}\in A\}}\,|\,\mathcal{F}_t^{\Theta}\} \;=\; \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}} J_n^{t,u}(A;\Theta,(t_i,x_i)_{i\leq n-1}) \;,$$

where $J_n^{t,u}$ is defined by (82). An application of lemma 3.1 then proves (83). $\square$

Thus we get for example, if $d$ denotes one of the "distances" introduced above, the following prediction formula:

**4.2 Corollary**. *Let $\varphi_t$ be $\mathcal{F}_t^{\mathcal{N}}$-measurable and suppose that $d(X_{t,u},\varphi_t)$ is integrable. Then*

$$\mathbf{E}\{d(X_{t,u},\varphi_t)\,|\,\mathcal{F}_t^{\mathcal{N}}\} \;=\; 1_{\{T_{n-1}\leq t<T_n\}} \frac{\Phi_n^{t,u}(G;(T_i,X_i)_{i\leq n-1})}{\Psi_n^{t}((T_i,X_i)_{i\leq n-1})} \;,$$

*where*

$$\begin{aligned} &G_n^{t,u}(y,(t_i,x_i)_{i\leq n-1}) \\ &\quad= \int_t^u \Big(\int_E d(x,\varphi_t((t_i,x_i)_{i\leq n-1}))g_{t_n}(x;y,(t_i,x_i)_{i\leq n-1})\gamma(dx)\Big) \\ &\qquad\qquad \lambda_{t_n}(y,(t_i,x_i)_{i\leq n-1})e^{-\int_t^{t_n}\lambda_s(y,(t_i,x_i)_{i\leq n-1})\,ds}\,dt_n \\ &\qquad +d(0,\varphi_t((t_i,x_i)_{i\leq n-1}))e^{-\int_t^u \lambda_s(y,(t_i,x_i)_{i\leq n-1})ds} \;, \end{aligned}$$

*and where $\Phi_n^{t,u}(G)$ is defined analogously to $\Phi_n^{t,u}(A)$ (with $J_n^{t,u}(A)$ replaced by $G_n^{t,u}$).*

Of course, there are corresponding results for

$$P_{t,u}(dz;\mathcal{G}_t) \;:=\; \mathbf{P}\{X_{t,u}\in dz\,|\,\mathcal{G}_t\} \;, \tag{85}$$

where $\mathcal{G}_t$ was defined in (53). Corresponding results also hold for

$$P_n(dz;(T_i,X_i)_{i\leq n-1}) \;:=\; \mathbf{P}\{X_n\in dz\,|\,\mathcal{F}_{T_{n-1}}\} \tag{86}$$

or

$$P_n(dz;(X_i)_{i\leq n-1}) \;:=\; \mathbf{P}\{X_n\in dz\,|\,X_1,\cdots,X_{n-1}\} \;.$$

(cf. theorem 3.8 and proposition 3.9).

For the mixed Poisson risk process we have the following corollary from theorem 4.1:

**4.3 Corollary**. *Let $((T_n,X_n))_{n\geq 1}$ be a mixed homogeneous Poisson risk process with densities $g_t$ and intensities $\lambda_t$ given by (10) and (11), and define for $A\in\mathcal{E}$, and $y\in T$*

$$H(A,y) \;:=\; \int_A g(x;y)\gamma(dx) \;.$$

*Then the following prediction formula holds:*

$$\mathbf{P}\{X_{t,u} \in A \,|\, \mathcal{F}_t^{\mathcal{N}}\} = \sum_{n \geq 1} 1_{\{T_{n-1} \leq t < T_n\}} \frac{\Phi_n^{t,u}(A; (X_i)_{1 \leq i \leq n-1})}{\Psi_n^t((X_i)_{1 \leq i \leq n-1})} ,$$

*with*

$$\begin{aligned} &\Phi_n^{t,u}(A; (x_i)_{1 \leq i \leq n-1}) \\ &= \int_T \Big\{ H(A, y)(1 - e^{-(u-t)\lambda(y)}) e^{-t\lambda(y)} \lambda(y)^{n-1} \prod_{i=1}^{n-1} g(x_i; y) \Big\} \, \beta(dy) \\ &\quad + 1_A(0) e^{-(u-t)\lambda(y)} , \end{aligned}$$

*and*

$$\Psi_t^n((x_i)_{1 \leq i \leq n-1}) = \int_T \Big\{ e^{-t\lambda(y)} \lambda(y)^{n-1} \prod_{i=1}^{n-1} g(x_i; y) \Big\} \, \beta(dy) .$$

*Thus we have*

$$P_{t,u}(dz; \mathcal{F}_t^{\mathcal{N}}) = \sum_{n \geq 1} 1_{\{T_{n-1} \leq t < T_n\}} \frac{\Phi_n^{t,u}(dz; (X_i)_{i \leq n-1})}{\Psi_n^t((X_i)_{i \leq n-1})} .$$

Let us suppose that $\varphi_t$ only depends on $X_1, \cdots, X_{n-1}$ on the set $\{T_{n-1} \leq t < T_n\}$, which is the case, if

$$\varphi_t = \mathbf{E}\{X_{t,u} \,|\, \mathcal{F}_t^{\mathcal{N}}\}$$

(cf. (61). Then we obtain from the corollary the following two examples:

(1) For the distance $d_+$ we have

$$\begin{aligned} &\mathbf{E}\{d_+(X_{t,u}, \varphi_t) \,|\, \mathcal{F}_t^{\mathcal{N}}\} \\ &= \sum_{n \geq 1} 1_{\{T_{n-1} \leq t < T_n\}} \frac{\int_E d_+(z, \varphi_t((X_i)_{i \leq n-1})) \Phi_n^t(dz; (X_i)_{i \leq n-1})}{\Psi_n^t((X_i)_{i \leq n-1})} , \end{aligned}$$

where

$$\begin{aligned} &\int_E d_+(z, \varphi_t((X_i)_{i \leq n-1})) \Phi_n^t(dz; (X_i)_{i \leq n-1}) \\ &= \int_T \Big\{ \Big( \int_E d_+(x, \varphi_t((X_i)_{i \leq n-1})) g(x; y) \gamma(dx) \Big) \\ &\qquad (1 - e^{-(u-t)\lambda(y)}) e^{-t\lambda(y)} \lambda(y)^{n-1} \prod_{i=1}^{n-1} g(x_i; y) \Big\} \, \beta(dy) . \end{aligned}$$

(2) For the distance $d_\varepsilon$ we have

$$
\begin{aligned}
&\mathbf{P}\{X_{t,u} \geq \varphi_t + \varepsilon \,|\, \mathcal{F}_t^{\mathcal{N}}\} \\
&\quad = \sum_{n\geq 1} 1_{\{T_{n-1}\leq t<T_n\}} \frac{\int_E d_\varepsilon(z,\varphi_t((X_i)_{i\leq n-1}))\Phi_n^t(dz;(X_i)_{i\leq n-1})}{\Psi_n^t((X_i)_{i\leq n-1})} \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_E d_\varepsilon(z,\varphi_t((X_i)_{i\leq n-1}))\Phi_n^t(dz;(X_i)_{i\leq n-1}) \\
&\quad = \int_T \Big\{ \Big( \int_E 1_{[\varphi_t((X_i)_{i\leq n-1})+\varepsilon[}(x) g(x;y)\gamma(dx) \Big) \\
&\qquad\qquad (1-e^{-(u-t)\lambda(y)})e^{-t\lambda(y)}\lambda(y)^{n-1} \prod_{i=1}^{n-1} g(x_i;y) \Big\} \, \beta(dy) \ .
\end{aligned}
$$

# References


**Brémaud, P. [1981]:** *Point Processes and Queues - Martingale Dynamics.* Berlin – Heidelberg – New York: Springer.

**Dettweiler, E. [2004]:** *Risk Processes.* Leipzig: Edition am Gutenbergplatz.

**Schmidt, K.D. [1996]:** *Lectures on Risk Theory.* Stuttgart: Teubner.



Egbert Dettweiler
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D–72076 Tübingen

E–mail: e.dettweiler@web.de


30th October 2005