Multivariate Loss Prediction in the Multivariate Additive Model

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Abstract

In the present paper we propose a multivariate version of the additive model of loss reserving. The multivariate additive model is a linear model with a particular design matrix and a particular variance structure and is suitable for certain portfolios consisting of several correlated subportfolios. Under the assumptions of the multivariate additive model, we derive a formula for the Gauss–Markov predictor for a non–observable incremental claim. We also show that the Gauss–Markov predictors for the reserve of a particular accident year and for the total reserve are obtained by summation over the Gauss–Markov predictors for the corresponding non–observable incremental claims, and that this is also true for the Gauss–Markov predictors for the corresponding quantities of the aggregate portfolio.

1 Introduction

In recent years, major progress in actuarial mathematics has been achieved by the use of multivariate models and methods.

Quite recently, and not yet documented in Radtke and Schmidt [2004], multivariate models and methods have also been introduced in loss reserving. The development started with a particular bivariate extension of the chain–ladder method proposed by Quarg and Mack [2004] which applies to paid and incurred claims of the same portfolio. Subsequently, Braun [2004] proposed another bivariate model in order to construct correlation–dependent estimators for the prediction errors of the univariate chain–ladder predictors, but he did not proceed to correlation–dependent prediction. The model of Braun was then extended by Pröhl and Schmidt [2005] who used a natural optimality criterion to develop a multivariate version of the chain–ladder method which at the same time resolves the problem of additivity; see Ajne [1994]. A related paper is that of Kremer [2005] who proposed another multivariate extension of the chain–ladder method which, however, is less appropriate for applications since it involves the inversion of much larger matrices; also, the problems of optimality and additivity have not been addressed in that paper.

In a personal communication to the first author of the present paper, Braun [2005] pointed out that not only the (multiplicative) chain–ladder method but also the *additive method* could be extended to the multivariate case. This is not really surprising since it has been pointed by Schmidt [2004b] that the predictors of the univariate additive method are nothing else than the Gauss–Markov predictors for the non–observable incremental claims in a suitably chosen linear model which is called the *additive model*; see also Schmidt [2004a, 2004c].

Since a public document justifying the multivariate extension of the additive method does not seem to be available until now, we propose here a rather general multivariate extension of the additive model and determine, under the assumptions of this model, the Gauss–Markov predictors for the non–observable incremental claims. We also show that the Gauss–Markov predictor for any sum of incremental claims coincides with the sums of the Gauss–Markov predictors for the single incremental claims. This implies that the problem of additivity is absent in the case of Gauss–Markov prediction in the multivariate additive model.

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space on which all random variables are defined. We assume that all random variables are square integrable and that all random vectors have square integrable coordinates. Moreover, all equalities involving random variables are understood to hold almost surely with respect to the probability measure P.

In Section 2 of this paper we recall the basic result on Gauss–Markov prediction in the linear model; see also Schmidt [2004a, 2004c]. In Section 3 we introduce the multivariate additive model of loss reserving and determine the Gauss–Markov predictors for the non–observable incremental claims.

2 Prediction in the Linear Model

In the present we consider a random vector $\mathbf{X}$ and we assume that $\mathbf{X}$ satisfies the *linear model*

$$E[\mathbf{X}] = \mathbf{A}\boldsymbol{\beta}$$

where $\mathbf{A}$ is a known *design matrix* and $\boldsymbol{\beta}$ is an unknown *parameter* vector.

We assume further that some of the coordinates of $\mathbf{X}$ are *observable* whereas some other coordinates are *non–observable*. Then the random vector $\mathbf{X}_1$ consisting of the observable coordinates of $\mathbf{X}$ and the random vector $\mathbf{X}_2$ consisting of the non–observable coordinates of $\mathbf{X}$ satisfy

$$\begin{aligned} E[\mathbf{X}_1] &= \mathbf{A}_1\boldsymbol{\beta} \\ E[\mathbf{X}_2] &= \mathbf{A}_2\boldsymbol{\beta} \end{aligned}$$

for some submatrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{A}$.

We also assume that the matrix $\mathbf{A}_1$ has full column rank, that the matrices

$$\begin{aligned}\mathbf{\Sigma}_{11} &:= \mathrm{Var}[\mathbf{X}_1] \\ \mathbf{\Sigma}_{21} &:= \mathrm{Cov}[\mathbf{X}_2, \mathbf{X}_1]\end{aligned}$$

are known, and that $\mathbf{\Sigma}_{11}$ is invertible. Since the random vector $\mathbf{X}_2$ is non–observable, only the random vector $\mathbf{X}_1$ can be used for the estimation of the parameter $\boldsymbol{\beta}$. The estimator

$$\boldsymbol{\beta}^* \;:=\; (\mathbf{A}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}\mathbf{A}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1$$

is the *Gauss–Markov estimator* of $\boldsymbol{\beta}$ on the basis of $\mathbf{X}_1$.

Let us now consider the prediction problem for the non–observable random vector $\mathbf{D}\mathbf{X}_2$, where $\mathbf{D}$ is any matrix of suitable dimension.

A random variable $\widehat{\mathbf{Y}}$ is said to be a *predictor* of $\mathbf{D}\mathbf{X}_2$ if it is a measurable transformation of the observable random vector $\mathbf{X}_1$. For a predictor $\widehat{\mathbf{Y}}$ of $\mathbf{D}\mathbf{X}_2$, the real number

$$E\Big[\Big(\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\Big)'\Big(\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\Big)\Big]$$

is said to be the *expected squared prediction error* of $\widehat{\mathbf{Y}}$. Since

$$\begin{aligned}&E\Big[\Big(\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\Big)'\Big(\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\Big)\Big] \\ &\quad= \mathrm{trace}\Big(\mathrm{Var}\big[\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\big]\Big) + E\big[\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\big]'E\big[\widehat{\mathbf{Y}} - \mathbf{D}\mathbf{X}_2\big]\end{aligned}$$

the expected squared prediction error is determined by the variance and the expectation of the prediction error.

An observable random vector $\widehat{\mathbf{Y}}$ is said to be
- a *linear predictor* of $\mathbf{D}\mathbf{X}_2$ if there exists a matrix $\mathbf{Q}$ such that $\widehat{\mathbf{Y}} = \mathbf{Q}\mathbf{X}_1$.
- an *unbiased predictor* of $\mathbf{D}\mathbf{X}_2$ if $E[\widehat{\mathbf{Y}}] = E[\mathbf{D}\mathbf{X}_2]$.
- a *Gauss–Markov predictor* of $\mathbf{D}\mathbf{X}_2$ if it is an unbiased linear predictor of $\mathbf{D}\mathbf{X}_2$ and if it minimizes the expected squared prediction error over all unbiased linear predictors of $\mathbf{D}\mathbf{X}_2$.

We have the following result:

2.1 Proposition (Gauss–Markov Theorem). *There exists a unique Gauss–Markov predictor* $\mathbf{Y}^*(\mathbf{D}\mathbf{X}_2)$ *of* $\mathbf{D}\mathbf{X}_2$ *and it satisfies*

$$\mathbf{Y}^*(\mathbf{D}\mathbf{X}_2) \;=\; \mathbf{D}\Big(\mathbf{A}_2\boldsymbol{\beta}^* + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\big(\mathbf{X}_1 - \mathbf{A}_1\boldsymbol{\beta}^*\big)\Big)$$

In particular, $\mathbf{Y}^*(\mathbf{D}\mathbf{X}_2) = \mathbf{D}\mathbf{Y}^*(\mathbf{X}_2)$.

Proposition 2.1 shows that the Gauss–Markov predictor depends not only on the Gauss–Markov estimator of the unknown parameter but also on the covariance between the non–observable random vector and the observable one. Moreover, the final assertion of Proposition 2.1 shows that the coordinates of the Gauss–Markov predictor of the non–observable random vector coincide with the Gauss–Markov predictors of its coordinates.

3 Application to Loss Reserving

In the present section we consider $m \in \mathbf{N}$ portfolios all having the same number of development years. The m portfolios may be interpreted as subportfolios of an aggregate portfolio.

For portfolio $p \in \{1, \ldots, m\}$, we denote by

$$Z_{i,k}^{(p)}$$

the incremental claim size of accident year $i \in \{0, 1, \ldots, n\}$ and development year $k \in \{0, 1, \ldots, n\}$.

For $i, k \in \{0, 1, \ldots, n\}$, we thus obtain the m–dimensional random vector of *incremental claims*

$$\mathbf{Z}_{i,k} \;:=\; \left(Z_{i,k}^{(p)}\right)_{p \in \{1,\ldots,m\}}$$

The observable incremental claims are represented by the following run–off triangle:

Accident Year	**Development Year** 0	1	…	k	…	$n-i$	…	$n-1$	n
0	$\mathbf{Z}_{0,0}$	$\mathbf{Z}_{0,1}$	…	$\mathbf{Z}_{0,k}$	…	$\mathbf{Z}_{0,n-i}$	…	$\mathbf{Z}_{0,n-1}$	$\mathbf{Z}_{0,n}$
1	$\mathbf{Z}_{1,0}$	$\mathbf{Z}_{1,1}$	…	$\mathbf{Z}_{1,k}$	…	$\mathbf{Z}_{1,n-i}$	…	$\mathbf{Z}_{1,n-1}$	
⋮	⋮	⋮		⋮		⋮			
i	$\mathbf{Z}_{i,0}$	$\mathbf{Z}_{i,1}$	…	$\mathbf{Z}_{i,k}$	…	$\mathbf{Z}_{i,n-i}$			
⋮	⋮	⋮		⋮					
$n-k$	$\mathbf{Z}_{n-k,0}$	$\mathbf{Z}_{n-k,1}$	…	$\mathbf{Z}_{n-k,k}$					
⋮	⋮	⋮							
$n-1$	$\mathbf{Z}_{n-1,0}$	$\mathbf{Z}_{n-1,1}$							
n	$\mathbf{Z}_{n,0}$								

We can now formulate the *multivariate additive model*:

> **The Multivariate Additive Model:** There exist positive definite symmetric matrices $\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_n$ and $\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_n$ and unknown vectors $\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n$ such that
>
> $$E[\mathbf{Z}_{i,k}] \;=\; \mathbf{V}_i\, \boldsymbol{\zeta}_k$$
>
> and
>
> $$\mathrm{Cov}[\mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}] \;=\; \begin{cases} \mathbf{V}_i^{1/2} \boldsymbol{\Sigma}_k \mathbf{V}_i^{1/2} & \text{if } i = j \text{ and } k = l \\ \mathbf{O} & \text{else} \end{cases}$$
>
> holds for all $i, j, k, l \in \{0, 1, \ldots, n\}$.

The multivariate additive model is a general but straightforward extension of the univariate additive model documented by Schmidt [2004b]. In particular, each of the matrices $\mathbf{V}_i$ may be chosen to be diagonal as to represent volume measures of accident year i.

We assume henceforth that the assumptions of the multivariate additive model are fulfilled.

Because of the assumption on the expectations of the incremental claims, the multivariate additive model is a linear model. This can be seen as follows: Define

$$\boldsymbol{\beta} \;:=\; \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_{k-1} \\ \zeta_k \\ \zeta_{k+1} \\ \vdots \\ \zeta_n \end{pmatrix}$$

and, for all $i, k \in \{0, 1, \dots, n\}$, define

$$\mathbf{A}_{i,k} \;:=\; \begin{pmatrix} \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{V}_i & \mathbf{O} & \dots & \mathbf{O} \end{pmatrix}$$

where the matrix $\mathbf{V}_i$ occurs in position $k+1$. Then we have

$$E[\mathbf{Z}_{i,k}] \;=\; \mathbf{A}_{i,k}\boldsymbol{\beta}$$

for all $i, k \in \{0, 1, \dots, n\}$. Let $\mathbf{Z}_1$ and $\mathbf{A}_1$ denote a block vector and a block matrix consisting of the vectors $\mathbf{Z}_{i,k}$ and the matrices $\mathbf{A}_{i,k}$ with $i + k \le n$ (arranged in the same order) and let $\mathbf{Z}_2$ and $\mathbf{A}_2$ denote a block vector and a block matrix consisting of the vectors $\mathbf{Z}_{i,k}$ and the matrices $\mathbf{A}_{i,k}$ with $i + k > n$. Then we have

$$\begin{aligned} E[\mathbf{Z}_1] &= \mathbf{A}_1\boldsymbol{\beta} \\ E[\mathbf{Z}_2] &= \mathbf{A}_2\boldsymbol{\beta} \end{aligned}$$

Therefore, the multivariate additive model is indeed a linear model.

The following result provides a formula for the Gauss–Markov predictors of the non–observable incremental claims:

3.1 Theorem. *For all $i, k \in \{0, 1, \dots, n\}$ such that $i + k > n$, the Gauss–Markov predictor $\mathbf{Y}^*(\mathbf{Z}_{i,k})$ of $\mathbf{Z}_{i,k}$ satisfies*

$$\mathbf{Y}^*(\mathbf{Z}_{i,k}) \;=\; \mathbf{V}_i \left(\sum_{j=0}^{n-k} \mathbf{V}_j^{1/2} \boldsymbol{\Sigma}_k^{-1} \mathbf{V}_j^{1/2} \right)^{-1} \sum_{j=0}^{n-k} \left(\mathbf{V}_j^{1/2} \boldsymbol{\Sigma}_k^{-1} \mathbf{V}_j^{1/2} \right) \mathbf{V}_j^{-1} \mathbf{Z}_{j,k}$$

Proof. Because of to the diagonal block structure of $\mathbf{\Sigma}_{11} = \mathrm{Var}[\mathbf{Z}_1]$ and the block structure of $\mathbf{A}_1$ we obtain

$$\mathbf{A}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{A}_1 \;=\; \mathrm{diag}\left(\sum_{j=0}^{n-k}\mathbf{V}_j^{1/2}\mathbf{\Sigma}_k^{-1}\mathbf{V}_j^{1/2}\right)_{k\in\{0,1,\dots,n\}}$$

and

$$\mathbf{A}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{Z}_1 \;=\; \left(\sum_{j=0}^{n-k}\left(\mathbf{V}_j^{1/2}\mathbf{\Sigma}_k^{-1}\mathbf{V}_j^{1/2}\right)\mathbf{V}_j^{-1}\mathbf{Z}_{j,k}\right)_{k\in\{0,1,\dots,n\}}$$

and hence

$$\boldsymbol{\beta}^* \;=\; \left(\left(\sum_{j=0}^{n-k}\mathbf{V}_j^{1/2}\mathbf{\Sigma}_k^{-1}\mathbf{V}_j^{1/2}\right)^{-1}\sum_{j=0}^{n-k}\left(\mathbf{V}_j^{1/2}\mathbf{\Sigma}_k^{-1}\mathbf{V}_j^{1/2}\right)\mathbf{V}_j^{-1}\mathbf{Z}_{j,k}\right)_{k\in\{0,1,\dots,n\}}$$

Since $\mathbf{\Sigma}_{21} = \mathrm{Cov}[\mathbf{Z}_2, \mathbf{Z}_1] = \mathbf{O}$, we obtain

$$\begin{aligned}\mathbf{Y}^*(\mathbf{Z}_{i,k}) &= \mathbf{A}_{i,k}\boldsymbol{\beta}^* \\ &= \mathbf{V}_i\left(\sum_{j=0}^{n-k}\mathbf{V}_j^{1/2}\mathbf{\Sigma}_k^{-1}\mathbf{V}_j^{1/2}\right)^{-1}\sum_{j=0}^{n-k}\left(\mathbf{V}_j^{1/2}\mathbf{\Sigma}_k^{-1}\mathbf{V}_j^{1/2}\right)\mathbf{V}_j^{-1}\mathbf{Z}_{j,k}\end{aligned}$$

as was to be shown. □

The Gauss–Markov Theorem implies that the Gauss–Markov predictors for the sum of the non–observable incremental claims of a given accident year and for the sum of all non–observable incremental claims are obtained by summation from the Gauss–Markov predictors for single non–observable claims.

The Gauss–Markov Theorem also implies that, for a given accident year and a given development year, the Gauss–Markov predictor for the non–observable incremental claim $\mathbf{1}'\mathbf{Z}_{i,k}$ of the aggregate portfolio is equal to the sum $\mathbf{1}'\mathbf{Y}^*(\mathbf{Z}_{i,k})$ of the Gauss–Markov predictors of the non–observable incremental claims of the subportfolios; here $\mathbf{1}$ denotes the m–dimensional vector with all coordinates being equal to 1. Of course, the preceding remarks on the Gauss–Markov predictors for sums of non–observable incremental claims apply to the aggregate portfolio as well.

4 Remark

The multivariate additive model involves a particular assumption on the structure of the variances of the incremental claims. This assumption is not really essential: If it is modified or dropped, then an obvious modification or extension of Theorem 3.1 is easily obtained from Proposition 2.1.

Acknowledgement

The authors would like to thank Christian Braun who inspired this paper.

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November 13, 2005