Multivariate Counting Processes

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Abstract

Counting processes are of use in several areas, for example in risk theory. Following the recent development of the transition from univariate models to multivariate models, this paper considers multivariate counting processes. Multivariate versions of the Poisson process and the mixed Poisson process are introduced as well as other classes of multivariate counting processes, like those having the Markov property, the multinomial property and independent increments. These properties are used to characterize the multivariate Poisson process and the multivariate mixed Poisson process.

1 Introduction

This paper extends the theory of the univariate counting process to the multivariate setting. Multivariate versions of the Poisson process and the mixed Poisson process are introduced and will be characterized by simple properties of multivariate counting processes such as independent increments, the multinomial property and the Markov property. Some of the characterizations are similar to the univariate case, but there are also some new results induced by the multivariate setting, as for example the binomial property and independent increments imply independent coordinates.

In Section 2 the multivariate Poisson process and the multivariate mixed Poisson process are introduced. It is shown that the multivariate mixed Poisson process has in general dependent coordinates, in contrast to the multivariate Poisson process.

The multinomial property is introduced in Section 3 and both the multivariate Poisson process and the multivariate mixed Poisson process are characterized in terms of the multinomial property and a particular choice of the one–dimensional distributions. Furthermore, relations between the multinomial property and other properties of multivariate counting processes, like stationary increments, the Markov property and the binomial property are considered.

In Section 4 the previous characterizations of the multivariate Poisson process and the multivariate mixed Poisson process are substantially improved with the help of a multivariate version of the Bernstein–Widder Theorem, which is stated in Appendix A. Thus, a multivariate counting process is a multivariate mixed Poisson process if and only if it has the multinomial property, and it is a multivariate Poisson process if and only if it has independent increments and the binomial property.

Throughout this paper let $(\Omega, \mathcal{F}, \mathrm{P})$ be the underlying probability space. Furthermore, a bold letter represents a vector or a random vector. $\mathbf{1}$ and $\mathbf{0}$ are the vectors having entries all equal to one and zero, respectively, and it will be convenient to use the notations

$$\boldsymbol{\lambda}^{\mathbf{n}} := \prod_{i=1}^{k} \lambda_i^{n^{(i)}} \qquad \text{and} \qquad \mathbf{n}\,! := \prod_{i=1}^{k} n^{(i)}\,!$$

for k–dimensional vectors as well as $\mathbf{n} \leq \mathbf{m}$ if and only if $n^{(i)} \leq m^{(i)}$ for every coordinate i.

A stochastic process $\{N_t\}_{t \in \mathbb{R}_+}$ is said to be a *counting process* if there exists a null set $M \in \mathcal{F}$ (called the exceptional null set) such that the following properties are satisfied for every $\omega \in \Omega \backslash M$:

- (i) $N_0(\omega) = 0$,
- (ii) $N_t(\omega) \in \mathbb{N}_0$ for all $t > 0$,
- (iii) $N_t(\omega) = \inf_{s \in (t,\infty)} N_s(\omega)$ for all $t \in \mathbb{R}_+$,
- (iv) $\sup_{s \in [0,t)} N_s(\omega) \leq N_t(\omega) \leq \sup_{s \in [0,t)} N_s(\omega) + 1$ for all $t \in \mathbb{R}_+$, and
- (v) $\sup_{t \in \mathbb{R}_+} N_t(\omega) = \infty$.

N_t can be interpreted as the number of events occurring in the interval $(0, t]$. The above definition excludes positive probability of infinitely many events occurring in a finite time interval (called explosion) as well as positive probability of only a finite number of events occurring in an infinite time interval.

A multivariate stochastic process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ in k dimensions is said to be a *multivariate counting process* if every coordinate $\{N_t^{(i)}\}_{t \in \mathbb{R}_+}$, $i \in \{1, \dots, k\}$, and the sum $\{N_t\}_{t \in \mathbb{R}_+} := \{\mathbf{1}'\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ of all coordinates is a counting process. Thus, there exists a null set $M \in \mathcal{F}$ (called the exceptional null set of the multivariate counting process) such that for all $\omega \in \Omega \backslash M$ properties (i)–(v) are fulfilled by all coordinates $\{N_t^{(i)}(\omega)\}_{t \in \mathbb{R}_+}$, $i \in \{1, \dots, k\}$, and the sum $\{N_t(\omega)\}_{t \in \mathbb{R}_+}$ of all coordinates. As a consequence, simultaneous jumps of different coordinates are almost surely excluded. From now on k will always be the dimension of the multivariate counting process we are working with and we furthermore assume that M is empty.

2 Multivariate Poisson Processes and Multivariate Mixed Poisson Processes

We start this section with a multivariate version of a most prominent member of univariate counting processes. A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is said to be a *multivariate Poisson process* with parameter $\boldsymbol{\lambda} \in (\mathbf{0}, \boldsymbol{\infty})$ if

$$\mathrm{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] = \prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\lambda}(t_j - t_{j-1})} \frac{(\boldsymbol{\lambda}(t_j - t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$.

A first characterization of the multivariate Poisson process is immediate from the definition.

2.1 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent:*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a multivariate Poisson process.*
(b) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has stationary and independent increments and*

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}$$

holds for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$.

Rewriting the definition as

$$\mathrm{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] = \prod_{i=1}^{k}\prod_{j=1}^{m} e^{-\lambda_i(t_j - t_{j-1})} \frac{(\lambda_i(t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!}$$

we recognize that a multivariate Poisson process has independent coordinates, which are univariate Poisson processes in the usual sense. Since the inverse is also true, we obtain another characterization of the multivariate Poisson process, this time among stochastic processes.

2.2 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a stochastic process. Then the following are equivalent:*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a multivariate Poisson process.*
(b) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has independent coordinates $\{N_t^{(i)}\}_{t \in \mathbb{R}_+}$ which are Poisson processes.*

Proof: Since the coincidence of the finite–dimensional distributions is obvious, it just remains to show that a stochastic process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ fulfilling assumption (b) is a multivariate counting process.
Every coordinate $\{N_t^{(i)}\}_{t \in \mathbb{R}_+}$ is a counting process with jump times distributed according to an Erlang distribution which is continuous w.r.t. the Lebesgue measure. By independence of the coordinates we obtain common jump time distributions which are also continuous w.r.t. the Lebesgue measure. The event of arbitrary jump times of different coordinates being equal has Lebesgue measure zero and therefore

the probability of such an event is also zero. Thus, $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate counting process. □

Hence, a multivariate Poisson process is certainly not a multivariate process of much interest, but serves in this paper as a kind of benchmark for the multivariate mixed Poisson process.

A multivariate counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is said to be a *multivariate mixed Poisson process* with mixing distribution $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ if $U[(\mathbf{0},\boldsymbol{\infty})] = 1$ and if

$$\mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] = \int_{\mathbb{R}^k} \prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\lambda}(t_j-t_{j-1})} \frac{(\boldsymbol{\lambda}(t_j-t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\, dU(\boldsymbol{\lambda})$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$.

Similar to the univariate setting we may obtain a multivariate mixed Poisson process by randomizing the parameter of a Poisson process. Conversely, choosing the mixing distribution U of a multivariate mixed Poisson process to be a Dirac–distribution, we see that every multivariate Poisson process is a multivariate mixed Poisson process. A multivariate mixed Poisson process has stationary increments, but does not have in general independent increments (Corollary 4.3) or independent coordinates (Theorem 2.3). Like the multivariate Poisson process, a multivariate mixed Poisson process is stable in some sense as the i–th coordinate of the process is a (univariate) mixed Poisson process with mixing distribution U_i where U_i represents the corresponding marginal distribution of U.

The following theorem was inspired by Hofmann [1955] and justifies the use of the multivariate mixed Poisson process.

2.3 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then the following are equivalent.*

(a) *The coordinates of $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ are independent.*

(b) *The identity $U = \bigotimes_{i=1}^k U_i$ is valid.*

Proof: The assumption $U = \bigotimes_{i=1}^k U_i$ yields by straightforward calculation the independence of the coordinates of $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$.
Now assume that (a) holds. Denoting by $\mathcal{L}_V$ the Laplace transform of a distribution V on $\mathcal{B}(\mathbb{R}^k)$ we obtain for every $\mathbf{t} \in \mathbb{R}_+^k$

$$\begin{aligned}\mathcal{L}_U(\mathbf{t}) &= \int_{\mathbb{R}^k} e^{-\mathbf{t}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda}) \\ &= \mathrm{P}\left[\bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} = 0\right\}\right]\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{k} \mathrm{P}\left[\left\{N_{t_i}^{(i)} = 0\right\}\right] \\
&= \prod_{i=1}^{k} \int_{\mathbb{R}} e^{-t_i \lambda_i}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} e^{-\mathbf{t}'\boldsymbol{\lambda}}\, d(\textstyle\bigotimes_{i=1}^{k} U_i)\,(\boldsymbol{\lambda}) \\
&= \mathcal{L}_{\bigotimes_{i=1}^{k} U_i}(\mathbf{t})
\end{aligned}
$$

The uniqueness of the Laplace transform (Kallenberg [2002]) implies $U = \bigotimes_{i=1}^{k} U_i$ and the assertion is shown. □

Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U and let $\boldsymbol{\Lambda}$ be a random vector with distribution U. If the corresponding moment of $\boldsymbol{\Lambda}$ exists, we obtain with the help of the Laplace transform

$$
\begin{aligned}
\mathrm{E}[\mathbf{N}_t] &= t\,\mathrm{E}[\boldsymbol{\Lambda}] \quad \text{and} \\
\mathrm{Cov}[\mathbf{N}_s, \mathbf{N}_t] &= \min(s,t)\,\mathrm{Diag}(\mathrm{E}[\boldsymbol{\Lambda}]) + s\,t\,\mathrm{Var}[\boldsymbol{\Lambda}]
\end{aligned}
$$

Since the correlation of the coordinates of the mixing distribution specifies the correlation of the coordinates of the multivariate mixed Poisson process, a wide range of correlations and dependencies can be modelled.

3 Further Classes of Multivariate Counting Processes

The finite–dimensional distributions of the increments of a multivariate Poisson process are completely determined by the one–dimensional distributions (Lemma 2.1). Looking at this relation in a precise manner it is convenient to introduce the following property of multivariate counting processes.

A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the *multinomial property* if

$$
\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \\
&\quad = \left(\prod_{i=1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}} \right) \mathrm{P}\left[\left\{\mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j\right\}\right]
\end{aligned}
$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$. In the univariate setting the multinomial property seems to be introduced first by Schmidt [1996]. Although there are properties which are equivalent to the multinomial property, for example order statistic properties of the jump time distributions (Feigin [1979]), it is convenient to use the multinomial property since it just operates with the process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ itself.

3.1 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent:*
(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate Poisson process.*
(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the multinomial property and*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\,t)^{\mathbf{n}}}{\mathbf{n}!}$$

holds for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$.

The significance of the one–dimensional distributions is valid for the multivariate mixed Poisson process, too.

3.2 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Furthermore, let $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ be a distribution with $U[(\mathbf{0},\boldsymbol{\infty})] = 1$. Then the following are equivalent:*
(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate mixed Poisson process with mixing distribution U.*
(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the multinomial property and*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\,t)^{\mathbf{n}}}{\mathbf{n}!}\,dU(\boldsymbol{\lambda})$$

holds for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$.

The multinomial property can also be written in terms of the conditional distribution of increments up to time t_m w.r.t. the process at time t_m. Furthermore this conditional distribution is a product of multinomial distributions. Thus, given the number of events at present, the distribution of the events into disjoint time intervals in the past corresponds to sampling with replacement. As this sampling is independent for the coordinates of the process, every coordinate could be sampled separately. Hence, it can be statistically tested whether a counting process possesses the multinomial property. By straightforward calculation we obtain the next lemma.

3.3 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process having the multinomial property. Then $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has stationary increments.*

After having considered conditional probabilities of increments in the past w.r.t the process at present, we now change the direction and consider conditional probabilities of increments in the future w.r.t the process at present.

A multivariate counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has the *Markov property* if the identity

$$\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^{m+1}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\right]\,\mathrm{P}\left[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}\right]\\
&\quad=\;\mathrm{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\right]\,\mathrm{P}\left[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}\cap\{\mathbf{N}_{t_{m+1}}-\mathbf{N}_{t_m}=\mathbf{n}_{m+1}\}\right]
\end{aligned}$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_{m+1} \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_{m+1}$ and for all $\mathbf{n}_1, \ldots, \mathbf{n}_{m+1} \in \mathbb{N}_0^k$ with $\mathbf{l}_m := \sum_{j=1}^m \mathbf{n}_j$.

The Markov property, like the multinomial property, is stated here in terms of increments and without using conditional probabilities, since technical calculation becomes more comfortable (see proof of Lemma 3.5). If $\mathrm{P}\left[\cap_{j=1}^m \left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] > 0$, then the previous identities are equivalent to

$$\begin{aligned} &\mathrm{P}\left[\left\{\mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1}\right\} \,\middle|\, \bigcap_{j=1}^m \left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] \\ &\qquad = \mathrm{P}\left[\left\{\mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1}\right\} \,\middle|\, \left\{\mathbf{N}_{t_m} = \mathbf{l}_m\right\}\right] \end{aligned}$$

Roughly speaking, the future increment of a Markov process only depends on the total increment up to the present and not on the partitioning of the increment in the past. Hence it is obvious that a multivariate counting process having independent increments is a Markov process. Another sufficient condition for the Markov property is the multinomial property as stated in the next lemma, which can be obtained by simple calculation.

3.4 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process having the multinomial property. Then $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a Markov process.*

On the other hand the Markov property is not sufficient for the multinomial property without adding another property, which we obtain by restricting the multinomial property to the case $m = 2$. A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the *binomial property* if

$$\begin{aligned} &\mathrm{P}\left[\{\mathbf{N}_s = \mathbf{l}\} \cap \{\mathbf{N}_t - \mathbf{N}_s = \mathbf{n}\}\right] \\ &\quad = \left(\prod_{i=1}^k \binom{n^{(i)} + l^{(i)}}{l^{(i)}} \left(\frac{s}{t}\right)^{l^{(i)}} \left(1 - \frac{s}{t}\right)^{n^{(i)}}\right) \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n} + \mathbf{l}\}\right] \end{aligned}$$

holds for all $s, t \in \mathbb{R}_+$ with $0 < s < t$ and all $\mathbf{l}, \mathbf{n} \in \mathbb{N}_0^k$. In contrast to the multinomial property the binomial property has been widely used (see e.g Lundberg [1964]).

3.5 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent*
(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has the multinomial property.*
(b) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has the binomial property and the Markov property.*

Before proving Lemma 3.5, we state a consequence of the binomial property which is derived from the properties of the paths of a multivariate counting process.

3.6 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the binomial property, then*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;>\; 0$$

holds for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: First, we assume there exists some $\mathbf{m} \in \mathbb{N}_0^k$ such that

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \;=\; 0$$

holds for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{n} \geq \mathbf{m}$. Then we have $\mathrm{P}[\{\mathbf{N}_t \geq \mathbf{m}\}] = 0$ for all $t > 0$, which is a contradiction to $\lim_{t\to\infty} \mathrm{P}[\{\mathbf{N}_t \geq \mathbf{n}\}] = 1$ for all $\mathbf{n} \in \mathbb{N}_0^k$. The last identities are valid since all paths of every coordinate of a multivariate counting process increase and have no upper limit.
Now, consider $\mathbf{m} \in \mathbb{N}_0^k$. By the first part of the proof there exists some $t > 0$ and some $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{n} \geq \mathbf{m}$ such that

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \;>\; 0$$

The binomial property leads to

$$\begin{aligned}\mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\}] &\geq \mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\} \cap \{\mathbf{N}_t - \mathbf{N}_s = \mathbf{n} - \mathbf{l}\}] \\ &= \left(\prod_{i=1}^{k} \binom{n^{(i)}}{l^{(i)}} \left(\frac{s}{t}\right)^{l^{(i)}} \left(1 - \frac{s}{t}\right)^{n^{(i)}-l^{(i)}}\right) \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]\end{aligned}$$

and hence

$$\mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\}] \;>\; 0$$

for all $s \in (0,t)$ and all $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{l} \leq \mathbf{n}$. Moreover, for all $u \in (t,\infty)$ the identity $\sum_{\mathbf{p}\geq\mathbf{n}} \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\} \,|\, \{\mathbf{N}_t = \mathbf{n}\}] = 1$ yields the existence of some $\mathbf{p} \in \mathbb{N}_0^k$ with $\mathbf{p} \geq \mathbf{n}$ such that

$$\begin{aligned}\mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\}] &\geq \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\} \cap \{\mathbf{N}_t = \mathbf{n}\}] \\ &= \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\} \,|\, \{\mathbf{N}_t = \mathbf{n}\}]\; \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \\ &> 0\end{aligned}$$

Replacing t and $\mathbf{n}$ by u and $\mathbf{p}$ in the preceding argument, we get

$$\mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\}] \;>\; 0$$

for all $s > 0$ and all $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{l} \leq \mathbf{n}$.
Since $\mathbf{m} \in \mathbb{N}_0^k$ was arbitrary, the assertion is shown. □

Now, we can prove Lemma 3.5 in a simplified manner.
Proof: Since (a) obviously implies (b), we only have to show that the Markov property and the binomial property imply the multinomial property. We proceed by induction over the number m of time periods in the equation

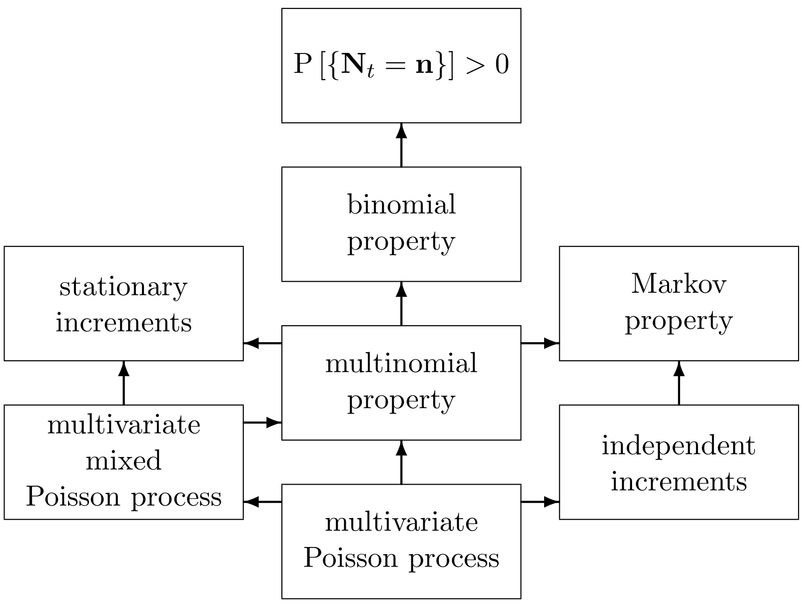
$$\begin{aligned}&\mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \\ &\quad= \left(\prod_{i=1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \mathrm{P}\left[\left\{\mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j\right\}\right] \qquad (*)\end{aligned}$$

with arbitrary $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$, $0 = t_0 < t_1 < \ldots < t_m$ and arbitrary $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$.
For $m = 1$ $(*)$ is evidently satisfied.
Now, assume that $(*)$ holds for $m \in \mathbb{N}$. Consider $t_0, t_1, \ldots, t_m, t_{m+1} \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m < t_{m+1}$ and $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m+1\}$. Setting $\mathbf{l}_j := \sum_{h=1}^{j} \mathbf{n}_h$ for $j \in \{1, \ldots, m+1\}$ and using the Markov property and the binomial property we get

$$
\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^{m+1} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}\right] \\
&= \mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\} \cap \{\mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1}\}\right] \\
&= \left(\prod_{i=1}^{k} \frac{l_m^{(i)}!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}\right] \\
&\quad \cdot \left(\prod_{i=1}^{k} \binom{l_{m+1}^{(i)}}{l_m^{(i)}} \left(\frac{t_m}{t_{m+1}}\right)^{l_m^{(i)}} \left(\frac{t_{m+1} - t_m}{t_{m+1}}\right)^{n_{m+1}^{(i)}}\right) \mathrm{P}\left[\{\mathbf{N}_{t_{m+1}} = \mathbf{l}_{m+1}\}\right] \\
&= \left(\prod_{i=1}^{k} \frac{l_{m+1}^{(i)}!}{\prod_{j=1}^{m+1} n_j^{(i)}!} \prod_{j=1}^{m+1} \left(\frac{t_j - t_{j-1}}{t_{m+1}}\right)^{n_j^{(i)}}\right) \mathrm{P}\left[\{\mathbf{N}_{t_{m+1}} = \mathbf{l}_{m+1}\}\right] \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}\right]
\end{aligned}
$$

Since we obtain from the binomial property $\mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}] > 0$ (Lemma 3.6), the above identity yields that $(*)$ is valid for $m+1$ time periods. Hence, the binomial property and the Markov property imply the multinomial property. □

The following picture recapitulates the implications between the properties of multivariate counting processes introduced so far.



As the picture shows the multinomial property provides several links between the properties considered here. This statement will be strengthened in the next section.

4 Characterizations

Lemma 3.2 can be substantially improved as follows:

4.1 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate mixed Poisson process.*
(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the multinomial property.*
(c) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the binomial property and the Markov property.*

Proof: Because of Lemma 3.2 and 3.5 it remains to show that (b) implies (a).
We set

$$\Pi_{\mathbf{n}}(\mathbf{t}) \;:=\; \mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)} = n^{(i)}\right\}\right]$$

for all $\mathbf{t} \in \mathbb{R}_+^k$ and $\mathbf{n} \in \mathbb{N}_0^k$.
Consider $\mathbf{t} \in \mathbb{R}_+^k$, $t \in \mathbb{R}_+$ with $\mathbf{t} \in (\mathbf{0}, t\mathbf{1})$ and $\mathbf{n} \in \mathbb{N}_0^k$. The multinomial property implies

$$\begin{aligned}
&\mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)} = n^{(i)}\right\} \cap \left\{N_t^{(i)} - N_{t_i}^{(i)} = l^{(i)} - n^{(i)}\right\}\right] \\
&\quad = \left(\prod_{i=1}^{k}\binom{l^{(i)}}{n^{(i)}}\left(\frac{t_i}{t}\right)^{n^{(i)}}\left(1-\frac{t_i}{t}\right)^{l^{(i)}-n^{(i)}}\right)\Pi_{\mathbf{l}}(t\mathbf{1})
\end{aligned}$$

for all $\mathbf{l} \geq \mathbf{n}$ and thus

$$\begin{aligned}
\Pi_{\mathbf{n}}(\mathbf{t}) &= \sum_{\mathbf{l}\in[\mathbf{n},\boldsymbol{\infty})} \mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)} = n^{(i)}\right\} \cap \left\{N_t^{(i)} - N_{t_i}^{(i)} = l^{(i)} - n^{(i)}\right\}\right] \\
&= \sum_{\mathbf{l}\in[\mathbf{n},\boldsymbol{\infty})} \left(\prod_{i=1}^{k}\binom{l^{(i)}}{n^{(i)}}\left(\frac{t_i}{t}\right)^{n^{(i)}}\left(1-\frac{t_i}{t}\right)^{l^{(i)}-n^{(i)}}\right)\Pi_{\mathbf{l}}(t\mathbf{1})
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\Pi_{\mathbf{0}}(\mathbf{t}) &= \sum_{\mathbf{l}\in\mathbb{N}_0^k} \left(\prod_{i=1}^{k}\left(1-\frac{t_i}{t}\right)^{l^{(i)}}\right)\Pi_{\mathbf{l}}(t\mathbf{1}) \\
&= \sum_{\mathbf{l}\in\mathbb{N}_0^k} \left(\prod_{i=1}^{k}(t_i - t)^{l^{(i)}}\right)\frac{\Pi_{\mathbf{l}}(t\mathbf{1})}{t^{\mathbf{1}'\mathbf{l}}}
\end{aligned}$$

The power series $\Pi_{\mathbf{0}}(\mathbf{t})$ in k coordinates is absolutely bounded for $\mathbf{t} \in (\mathbf{0}, 2t\mathbf{1})$ by $\sum_{\mathbf{l}\,\in\,\mathbb{N}_0^k} \Pi_{\mathbf{l}}(t\mathbf{1}) = 1$ and therefore absolutely convergent. Thus, $\Pi_{\mathbf{0}}$ is continuous on $(\mathbf{0}, 2t\mathbf{1})$ and the power series can infinitely often be differentiated in this open set (Dieudonné [1960]).

$$\begin{aligned} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t}) &= \sum_{\mathbf{l}\,\in\,[\mathbf{n},\boldsymbol{\infty})} \left(\prod_{i=1}^{k} \frac{l^{(i)}!}{(l^{(i)}-n^{(i)})!} \left(1-\frac{t_i}{t}\right)^{l^{(i)}-n^{(i)}} \left(\frac{-1}{t}\right)^{n^{(i)}} \right) \Pi_{\mathbf{l}}(t\mathbf{1}) \\ &= \frac{\mathbf{n}!}{(-\mathbf{t})^{\mathbf{n}}} \sum_{\mathbf{l}\,\in\,[\mathbf{n},\boldsymbol{\infty})} \left(\prod_{i=1}^{k} \binom{l^{(i)}}{n^{(i)}} \left(\frac{t_i}{t}\right)^{n^{(i)}} \left(1-\frac{t_i}{t}\right)^{l^{(i)}-n^{(i)}} \right) \Pi_{\mathbf{l}}(t\mathbf{1}) \end{aligned}$$

and hence

$$\Pi_{\mathbf{n}}(\mathbf{t}) = \frac{(-\mathbf{t})^{\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t})$$

for all $\mathbf{t} \in (\mathbf{0}, 2t\mathbf{1})$. Since t was arbitrary, the inequality

$$(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t}) \geq 0$$

holds for all $\mathbf{t} \in (\mathbf{0}, \boldsymbol{\infty})$ and $\Pi_{\mathbf{0}}$ is continuous on $(\mathbf{0}, \boldsymbol{\infty})$. The right continuity of the path of the coordinates of a multivariate counting process yields that $\Pi_{\mathbf{0}}$ is right continuous on $\mathbb{R}_+^k$ and so $\Pi_{\mathbf{0}}$ is continuous on $\mathbb{R}_+^k$.
Last but not least, we have $\Pi_{\mathbf{0}}(\mathbf{0}) = 1$ since $\{\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ is a multivariate counting process. Thus, $\Pi_{\mathbf{0}}$ fulfils all conditions of the Multivariate Bernstein–Widder Theorem (Theorem A.1) which yields the existence of a distribution $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ with $U\left[\mathbb{R}_+^k\right] = 1$ such that

$$\Pi_{\mathbf{0}}(\mathbf{t}) = \int_{\mathbb{R}^k} e^{-\mathbf{t}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda})$$

holds for all $\mathbf{t} \in \mathbb{R}_+^k$. So we obtain $\Pi_{\mathbf{0}}(\mathbf{t}) = \mathcal{L}_U(\mathbf{t})$. Since $\mathcal{L}_U$ is finite on $[\mathbf{0}, \boldsymbol{\infty})$ we can interchange differentiation and integration on $(\mathbf{0}, \boldsymbol{\infty})$. Thus, differentiating $\Pi_{\mathbf{0}}$ on $(\mathbf{0}, \boldsymbol{\infty})$ we get

$$\begin{aligned} \Pi_{\mathbf{n}}(\mathbf{t}) &= \frac{(-\mathbf{t})^{\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t}) \\ &= \frac{(-\mathbf{t})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} (-\boldsymbol{\lambda})^{\mathbf{n}}\, e^{-\mathbf{t}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{R}^k} e^{-\mathbf{t}'\boldsymbol{\lambda}}\, \frac{\mathbf{t}^{\mathbf{n}}\,\boldsymbol{\lambda}^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \end{aligned}$$

Using $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = \Pi_{\mathbf{n}}(t\mathbf{1})$, we immediately obtain

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})$$

for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. The last identities are also valid for $t = 0$ as all paths start at $\mathbf{0}$.

Assume now $U[\mathbb{R}_+^k \backslash (\mathbf{0}, \boldsymbol{\infty})] > 0$. Then there would exist a coordinate such that the mixing distribution fulfils $U_i[\{0\}] > 0$. This yields $\lim_{t\to\infty} \mathrm{P}[\{N_t^{(i)} = 0\}] > 0$ which is a contradiction to $\{N_t^{(i)}\}_{t \in \mathbb{R}_+}$ being a counting process and having paths going to infinity. Thus we obtain $U[\mathbb{R}_+^k \backslash (\mathbf{0}, \boldsymbol{\infty})] = 0$ and therefore $U[(\mathbf{0}, \boldsymbol{\infty})] = 1$ which completes the proof. □

In the univariate setting related results have been obtained by Nawrotzki [1955], Lundberg [1964], Feigin [1979], Albrecht [1981] as well as Schmidt and Zocher [2003]. Having characterized the multivariate mixed Poisson process we can now characterize the multivariate Poisson process such that we are able to answer the question under which condition a multivariate mixed Poisson process possesses independent increments.

4.2 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent:*

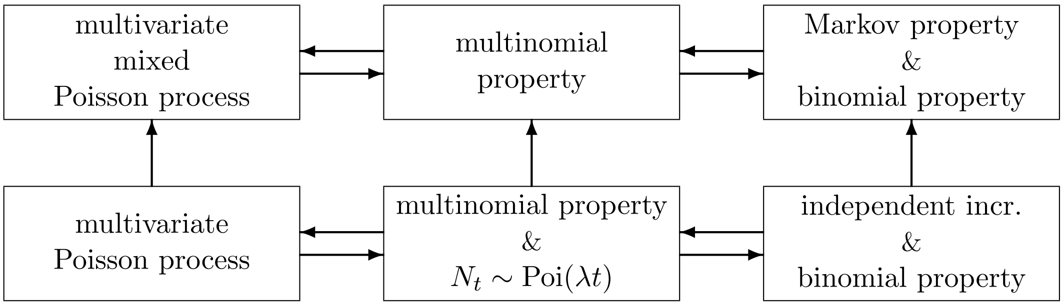
(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a multivariate Poisson process.*
(b) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has the binomial property and independent increments.*

Proof: By Section 3 we already know that multivariate Poisson process possesses the binomial property and independent increments.
Now assume that $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the binomial property and independent increments. Independent increments imply the Markov property and therefore (compare Theorem 4.1) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a multivariate mixed Poisson process with mixing distribution U. The coordinate i of $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a mixed Poisson process with mixing distribution U_i.
Since the multivariate counting process has the binomial property and independent increments, every coordinate has the same properties. Thus, by Schmidt and Zocher [2003] every coordinate is a Poisson process and hence a mixed Poisson process with mixing distribution being the Dirac distribution in $x_i \in (0, \infty)$, in formula $U_i = \delta_{x_i}$. The only distribution U fulfilling $U_i = \delta_{x_i}$ for all $i \in \{1, \ldots, k\}$ is $U = \delta_{\mathbf{x}}$. Hence, $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a multivariate Poisson process. □

4.3 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent:*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a multivariate mixed Poisson process with independent increments.*
(b) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a multivariate Poisson process.*

As a consequence of the previous corollary a multivariate mixed Poisson process with independent increments has always independent coordinates. With the characterizations stated above an illustration of the properties of multivariate counting processes may look like the following:



Appendix

A Multivariate Bernstein–Widder Theorem

For the main result in Section 4 we need a multivariate extension of the famous Bernstein–Widder theorem, which states that a completely monotone function has a representation as Laplace transform of a distribution. The Bernstein–Widder theorem possesses a lot of different proofs from various fields of mathematics. However, the proof of the multivariate extension is often taken for granted and therefore not carried out (compare Bochner [1955, Theorem 4.2.1] and Berg et al. [1984, Exercise 4.6.27]). So in this section we state the multivariate Bernstein–Widder theorem in a fashion fitting our purpose and give a proof, which is inspired by Berg et al. [1984].

A.1 Theorem (Multivariate Bernstein–Widder). *Let $f : \mathbb{R}_+^k \to \mathbb{R}$ be a continuous function with $f(\mathbf{0}) = 1$ and*

$$(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} f(\mathbf{t}) \;\; \geq \;\; 0$$

for all $\mathbf{n} \in \mathbb{N}_0^k$. Then there exists a distribution U on $\mathcal{B}(\mathbb{R}^k)$ with $U\left[\mathbb{R}_+^k\right] = 1$ such that

$$f(\mathbf{t}) \;\; = \;\; \int_{\mathbb{R}^k} e^{-\mathbf{t}'\mathbf{x}} \, dU(\mathbf{x})$$

holds for all $\mathbf{t} \in \mathbb{R}_+^k$.

Proof: Every numeration used in this proof refers to Berg et al. [1984].
First, we show that f is completely monotone in the sense of Definition 4.6.1, which states that a function has to be nonnegative and fulfils for all finite sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{R}_+^k$ and all $\mathbf{s} \in \mathbb{R}_+^k$ the inequality $\nabla_{\mathbf{a}_1} \cdots \nabla_{\mathbf{a}_n} f(\mathbf{s}) \geq 0$ in order to be completely monotone, where $\nabla_{\mathbf{a}}$ is defined by $\nabla_{\mathbf{a}} f(\mathbf{s}) := f(\mathbf{s}) - f(\mathbf{s} + \mathbf{a})$. Thus, we generalize a part of the proof of Theorem 4.6.13. Consider $\mathbf{a} \in \mathbb{R}_+^k$, then the function $\nabla_{\mathbf{a}} f$ is continuous on $\mathbb{R}_+^k$. Furthermore, we have for all $\mathbf{n} \in \mathbb{N}_0^k$ and $\mathbf{t} > \mathbf{0}$ with the mean value theorem (Browder [1996])

$$\begin{aligned} (-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} (\nabla_{\mathbf{a}} f)(\mathbf{t}) \;\; &= \;\; (-1)^{\mathbf{1}'\mathbf{n}} \nabla_{\mathbf{a}} D^{\mathbf{n}} f(\mathbf{t}) \\ &= \;\; (-1)^{\mathbf{1}'\mathbf{n}} \big(D^{\mathbf{n}} f(\mathbf{t}) - D^{\mathbf{n}} f(\mathbf{t} + \mathbf{a}) \big) \end{aligned}$$

$$= (-1)^{\mathbf{1}'\mathbf{n}+1} \sum_{i=1}^{k} a_i \, D^{\mathbf{n}+\mathbf{e}_i} f(\boldsymbol{\xi})$$

with $\boldsymbol{\xi} \in [\mathbf{t}, \mathbf{t}+\mathbf{a}]$. And so we have $(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}}(\nabla_{\mathbf{a}} f)(\mathbf{t}) \geq 0$. By iteration we get for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}_+^k$, $n \in \mathbb{N}$ that the function $\nabla_{\mathbf{a}_1} \ldots \nabla_{\mathbf{a}_n} f$ is continuous on $\mathbb{R}_+^k$ and fulfils $(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}}(\nabla_{\mathbf{a}_1} \ldots \nabla_{\mathbf{a}_n} f)(\mathbf{t}) \geq 0$ for all $\mathbf{n} \in \mathbb{N}_0^k$ and $\mathbf{t} > \mathbf{0}$. In particular, $\nabla_{\mathbf{a}_1} \ldots \nabla_{\mathbf{a}_n} f(\mathbf{t}) \geq 0$ for all $\mathbf{t} > \mathbf{0}$ and by continuity $\nabla_{\mathbf{a}_1} \ldots \nabla_{\mathbf{a}_n} f(\mathbf{t}) \geq 0$ for all $\mathbf{t} \geq \mathbf{0}$. As f is by assumption nonnegative it is completely monotone.
It follows from Theorem 4.6.5 that f is positive definite and bounded (in notation of Berg et al. [1984] $f \in \mathscr{P}^b(\mathbb{R}_+^k)$). Thus, the continuity of f in connection with Proposition 4.4.7. yields the existence of a finite, nonnegative measure U on $\mathcal{B}(\mathbb{R}_+^k)$ with

$$f(\mathbf{t}) = \int_{\mathbb{R}_+^k} e^{-\mathbf{t}'\mathbf{x}} \, dU(\mathbf{x})$$

for all $\mathbf{t} \in \mathbb{R}_+^k$. Finally

$$\begin{aligned} U\left[\mathbb{R}_+^k\right] &= \int_{\mathbb{R}_+^k} dU(\mathbf{x}) \\ &= f(\mathbf{0}) \\ &= 1 \end{aligned}$$

and the assertion is shown. □

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