A Note on the Decomposition of a Random Sample Size

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Abstract

This note addresses some results of Hess (2000) on the decomposition of a random sample size by using the concept of the multivariate probability generating function.

1 Introduction

Using the concept of the multivariate probability function we prove a nice formula for the vector of thinned random sample sizes. This leads to new proofs of some of the results presented in Hess (2000).

We use the following vector notation: Let $d \in \mathbb{N}$ and denote by $\boldsymbol{e}_i$ the ith unit vector in $\mathbb{R}^d$. Define $\mathbf{1} := \sum_{i=1}^d \boldsymbol{e}_i$. For $\boldsymbol{x} = (x_1 \dots x_d)' \in \mathbb{R}^d$, $\boldsymbol{n} = (n_1 \dots n_d)' \in \mathbb{N}_0^d$ and $n \in \mathbb{N}_0$ such that $n \geq \mathbf{1}'\boldsymbol{n}$ we define (using the definition $0^0 := 1$)

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{n}} &:= \prod_{i=1}^d x_i^{n_i} \\ \boldsymbol{n}! &:= \prod_{i=1}^d n_i! \\ \binom{n}{\boldsymbol{n}} &:= \frac{n!}{\boldsymbol{n}! \cdot (n - \mathbf{1}'\boldsymbol{n})!} \end{aligned}$$

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

2 Probability Generating Function

In this section we consider the multivariate extension of the probability generating function; see also Zocher (2005).

Let $\boldsymbol{N} : \Omega \to \mathbb{N}_0^d$ be a random vector. The function $m_{\boldsymbol{N}} : [0,1]^d \to \mathbb{R}$ with

$$m_{\boldsymbol{N}}(\boldsymbol{t}) \;=\; \mathbf{E}\left[\boldsymbol{t}^{\boldsymbol{N}}\right]$$

is called the *probability generating function* of $\boldsymbol{N}$. In the case $d = 1$ this definition coincides with that in the univariate case.

2.1 Lemma. *The probability generating function $m_{\boldsymbol{N}}$ has the following properties:*
(a) *$m_{\boldsymbol{N}}$ is increasing with respect to the coordinatewise order relation and*

$$0 \;\leq\; m_{\boldsymbol{N}}(\boldsymbol{t}) \;\leq\; m_{\boldsymbol{N}}(\mathbf{1}) \;=\; 1$$

holds for all $\boldsymbol{t} \in [0,1]^d$.
(b) *$m_{\boldsymbol{N}}$ is continuous.*
(c) *$m_{\boldsymbol{N}}$ is infinitely often differentiable on $[0,1)^d$.*
(d) *The identity*

$$\mathbf{P}[\boldsymbol{N} = \boldsymbol{n}] \;=\; \frac{1}{\boldsymbol{n}!} \cdot \frac{\partial^n m_{\boldsymbol{N}}}{\partial t_1^{n_1} \dots \partial t_d^{n_d}}(\mathbf{0})$$

holds for all $\boldsymbol{n} \in \mathbb{N}_0^d$ and $n := \mathbf{1}'\boldsymbol{n}$.
In particular, the distribution of $\boldsymbol{N}$ is uniquely defined by its probability generating function.

The previous lemma was proved by Zocher (2005) using the representation of the probability generating function as a power series in d variables.

2.2 Lemma. *For all $i \in \{1, \dots, d\}$ the probability generating function of the coordinate N_i fulfills*

$$m_{N_i}(t) \;=\; m_{\boldsymbol{N}}(\mathbf{1} - \boldsymbol{e}_i + t\boldsymbol{e}_i)$$

The assertion follows directly from the definition.

2.3 Lemma. *The random vector $\boldsymbol{N}$ has independent coordinates if and only if*

$$m_{\boldsymbol{N}}(\boldsymbol{t}) \;=\; \prod_{i=1}^d m_{N_i}(t_i)$$

holds for all $\boldsymbol{t} \in [0,1]^d$.

Proof. If the coordinates are independent, then measurable functions of the coordinates are independent as well and the product formula follows. Conversely, if the product formula holds, then Lemma 2.1 yields for all $\boldsymbol{n} \in \mathbb{N}^d$

$$\begin{aligned}\mathbf{P}[\boldsymbol{N} = \boldsymbol{n}] &= \frac{1}{\boldsymbol{n}!} \frac{\partial^n m_{\boldsymbol{N}}}{\partial t_1^{n_1} \dots \partial t_d^{n_d}}(\mathbf{0}) \\ &= \prod_{i=1}^d \frac{1}{n_i!} \frac{\mathrm{d}^{n_i} m_{N_i}}{\mathrm{d}t^{n_i}}(0) \\ &= \prod_{i=1}^d \mathbf{P}[N_i = n_i]\end{aligned}$$

and hence the coordinates of $\boldsymbol{N}$ are independent. $\square$

2.4 Examples.

(a) **Multinomial distribution:** The random vector $\boldsymbol{N}$ has the *multinomial distribution* $\mathbf{Mult}(n, \boldsymbol{\eta})$ with $n \in \mathbb{N}$ and $\boldsymbol{\eta} \in (0,1)^d$ such that $\mathbf{1}'\boldsymbol{\eta} \leq 1$ if

$$\mathbf{P}[\boldsymbol{N} = \boldsymbol{n}] \;=\; \binom{n}{\boldsymbol{n}} \boldsymbol{\eta}^{\boldsymbol{n}} (1 - \mathbf{1}'\boldsymbol{\eta})^{n - \mathbf{1}'\boldsymbol{n}}$$

holds for all $\boldsymbol{n} \in \mathbb{N}_0^d$ such that $\mathbf{1}'\boldsymbol{n} \leq n$. In this case the probability generating function of $\boldsymbol{N}$ satisfies

$$\begin{aligned} m_{\boldsymbol{N}}(\boldsymbol{t}) &= (\boldsymbol{t}'\boldsymbol{\eta} \;+\; (1 - \mathbf{1}'\boldsymbol{\eta}))^n \\ &= (1 - \mathbf{1}'\boldsymbol{\eta} + \boldsymbol{t}'\boldsymbol{\eta})^n \end{aligned}$$

Since

$$\begin{aligned} m_{N_i}(t) &= m_{\boldsymbol{N}}(\mathbf{1} - (1-t)\boldsymbol{e}_i) \\ &= (1 - \eta_i + t_i \eta_i)^n \end{aligned}$$

the one–dimensional marginal distributions of $\mathbf{Mult}(n, \boldsymbol{\eta})$ are the binomial distributions $\mathbf{Bin}(n, \eta_i)$.

(b) **Multivariate Poisson distribution:** The random vector $\boldsymbol{N}$ has the *multivariate Poisson distribution* $\mathbf{MPoi}(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \in (0, \infty)^d$ if

$$\mathbf{P}[\boldsymbol{N} = \boldsymbol{n}] \;=\; \frac{\boldsymbol{\alpha}^{\boldsymbol{n}}}{\boldsymbol{n}!} \, \mathrm{e}^{-\mathbf{1}'\boldsymbol{\alpha}}$$

holds for all $\boldsymbol{n} \in \mathbb{N}_0^d$. This means that $\boldsymbol{N}$ has independent coordinates which are Poisson distributed. The probability generating function satisfies

$$\begin{aligned} m_{\boldsymbol{N}}(\boldsymbol{t}) &= \prod_{i=1}^d m_{N_i}(t_i) \\ &= \prod_{i=1}^d \mathrm{e}^{-\alpha_i(1-t_i)} \\ &= \mathrm{e}^{-\boldsymbol{\alpha}'(\mathbf{1}-\boldsymbol{t})} \end{aligned}$$

(c) **Negativemultinomial distribution:** The random vector $\boldsymbol{N}$ has the *negativemultinomial distribution* $\mathbf{NMult}(\beta, \boldsymbol{\eta})$ with $\beta \in (0, \infty)$ and $\boldsymbol{\eta} \in (0,1)^d$ such that $\mathbf{1}'\boldsymbol{\eta} < 1$ if

$$\mathbf{P}[\boldsymbol{N} = \boldsymbol{n}] \;=\; \frac{\Gamma(\beta + \mathbf{1}'\boldsymbol{n})}{\Gamma(\beta)\, \boldsymbol{n}!} (1 - \mathbf{1}'\boldsymbol{\eta})^\beta \, \boldsymbol{\eta}^{\boldsymbol{n}}$$

holds for all $\boldsymbol{n} \in \mathbb{N}_0^d$. In this case the probability generating function satisfies

$$m_{\boldsymbol{N}}(\boldsymbol{t}) = \left(\frac{1 - \mathbf{1}'\boldsymbol{\eta}}{1 - \boldsymbol{t}'\boldsymbol{\eta}}\right)^\beta$$

Since

$$\begin{aligned} m_{N_i}(t) &= m_{\boldsymbol{N}}(\mathbf{1} - (1-t)\boldsymbol{e}_i) \\ &= \left(\frac{1 - \mathbf{1}'\boldsymbol{\eta}}{1 - \mathbf{1}'\boldsymbol{\eta} + \eta_i - t\eta_i}\right)^\beta \\ &= \left(\frac{1 - \eta_i/(1 - \mathbf{1}'\boldsymbol{\eta} + \eta_i)}{1 - t_i\eta_i/(1 - \mathbf{1}'\boldsymbol{\eta} + \eta_i)}\right)^\beta \end{aligned}$$

the one–dimensional marginal distributions of $\mathbf{NMult}(n, \boldsymbol{\eta})$ are the negative-binomial distributions $\mathbf{NBin}(\beta, \eta_i/(1 - \mathbf{1}'\boldsymbol{\eta} + \eta_i))$.

Since we will consider conditional distributions it is necessary to define conditional probability generating functions. Let $\Theta : \Omega \to \mathbb{R}$ be a random variable and let $K : \mathcal{P}(\mathbb{N}_0^d) \times \Omega \to [0, 1]$ be a Θ–Markov kernel. For each $\omega \in \Omega$, $K(., \omega)$ is a probability measure on $\mathcal{P}(\mathbb{N}_0^d)$ having a probability generating function $m_K(., \omega)$. This defines a function $m_K : [0, 1]^d \times \Omega \to \mathbb{R}$ which is called the *Markov kernel generating function* of K.

For the random vector $\boldsymbol{N} : \Omega \to \mathbb{N}_0^d$ there exists a Θ–conditional distribution $\mathbf{P}_{\boldsymbol{N}|\Theta}$ of $\boldsymbol{N}$. The Markov kernel generating function of $\mathbf{P}_{\boldsymbol{N}|\Theta}$ is called a Θ*–conditional probability generating function* of $\boldsymbol{N}$ and will be denoted by $m_{\boldsymbol{N}|\Theta}$. As usual for conditional expectations we will drop the argument $\omega \in \Omega$.

2.5 Lemma. *The conditional probability generating function $m_{\boldsymbol{N}|\Theta}$ satisfies*

$$m_{\boldsymbol{N}|\Theta}(\boldsymbol{t}) = \mathbf{E}\Big(\boldsymbol{t}^{\boldsymbol{N}}\Big|\,\Theta\Big)$$

and

$$m_{\boldsymbol{N}}(\boldsymbol{t}) = \mathbf{E}\left[m_{\boldsymbol{N}|\Theta}(\boldsymbol{t})\right]$$

2.6 Example. Let $\boldsymbol{N} : \Omega \to \mathbb{N}_0^d$ be a random vector and let $N : \Omega \to \mathbb{N}$ be random variable such that $\mathbf{P}_{\boldsymbol{N}|N} = \mathbf{Mult}(N, \boldsymbol{\eta})$ for some $\boldsymbol{\eta} \in (0, 1)^d$ with $\mathbf{1}'\boldsymbol{\eta} \leq 1$. Then we have

$$m_{\boldsymbol{N}|N}(\boldsymbol{t}) = (1 - \mathbf{1}'\boldsymbol{\eta} + \boldsymbol{t}'\boldsymbol{\eta})^N$$

and

$$m_{\boldsymbol{N}}(\boldsymbol{t}) = \mathbf{E}\left[(1 - \mathbf{1}'\boldsymbol{\eta} + \boldsymbol{t}'\boldsymbol{\eta})^N\right]$$

3 Random Samples and their Decomposition

In this section we prove a formula on the probability generating function of the vector of thinned random sample sizes. This formula leads to new proofs of results of Hess (2000).

Let $(M, \mathcal{M})$ be a measurable space. Given a random variable $N : \Omega \to \mathbb{N}_0$ and a sequence $\{Y_j\}_{j\in\mathbb{N}}$ of random variables $Y_j : \Omega \to M$, the pair

$$\left\langle N, \{Y_j\}_{j\in\mathbb{N}} \right\rangle$$

is called a *random sample* if the sequence $\{Y_j\}_{j\in\mathbb{N}}$ is i.i.d. and independent of N. In this case N, is called the *random sample size* and Y_j is called a *sample variable.*

Let $\{C_1, \dots, C_d\}$ be a measurable partition of M such that $\mathbf{P}[Y_1 \in C_i] > 0$ holds for all $i \in \{1, \dots, d\}$. Furthermore define $\eta_i := \mathbf{P}[Y_1 \in C_i]$ for all $i \in \{1, \dots, d\}$ and $\boldsymbol{\eta} := (\eta_1 \dots \eta_d)'$.

For each $i \in \{1, \dots, d\}$, the *thinned random sample size* of group i is defined as

$$N_i \;:=\; \sum_{j=1}^{N} \chi_{\{Y_j \in C_i\}}$$

and $\boldsymbol{N} := (N_1, \dots, N_d)'$ is called *vector of the thinned random sample sizes.*

The next theorem gives a nice formula for the probability generating function of the vector of the thinned random sample sizes:

3.1 Theorem. *It holds*

$$\begin{aligned} m_{\boldsymbol{N}|N}(\boldsymbol{t}) &= (\boldsymbol{t}'\boldsymbol{\eta})^N \\ m_{\boldsymbol{N}}(\boldsymbol{t}) &= m_N(\boldsymbol{t}'\boldsymbol{\eta}) \\ m_{N_i}(t) &= m_N(1 - \eta_i + t\eta_i) \end{aligned}$$

for all $\boldsymbol{t} \in [0,1]^d, t \in [0,1]$ *and* $i \in \{1, \dots, d\}$.

In particular, the conditional distribution of $\boldsymbol{N}$ *given* N *is the multinomial distribution* $\mathbf{M}(N, \boldsymbol{\eta})$.

Proof. Since the random sample size and the sample variables are independent, we get

$$\begin{aligned} m_{\boldsymbol{N}|N}(\boldsymbol{t}) &= \mathbf{E}\left(\prod_{i=1}^{d} t_i^{N_i} \middle| N \right) \\ &= \mathbf{E}\left(\prod_{i=1}^{d} t_i^{\sum_{j=1}^{N} \chi_{\{Y_j \in C_i\}}} \middle| N \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \mathbf{E}\left[\prod_{i=1}^{d} t_i^{\sum_{j=1}^{n} \chi_{\{Y_j \in C_i\}}} \cdot \chi_{\{N=n\}}\right] \cdot \Big(\mathbf{P}[N=n]\Big)^{-1} \cdot \chi_{\{N=n\}} \\
&= \sum_{n=0}^{\infty} \mathbf{E}\left[\prod_{i=1}^{d}\prod_{j=1}^{n} t_i^{\chi_{\{Y_j \in C_i\}}}\right] \cdot \chi_{\{N=n\}} \\
&= \sum_{n=0}^{\infty} \mathbf{E}\left[\prod_{j=1}^{n}\prod_{i=1}^{d} t_i^{\chi_{\{Y_j \in C_i\}}}\right] \cdot \chi_{\{N=n\}} \\
&= \sum_{n=0}^{\infty} \prod_{j=1}^{n} \mathbf{E}\left[\prod_{i=1}^{d} t_i^{\chi_{\{Y_j \in C_i\}}}\right] \cdot \chi_{\{N=n\}}
\end{aligned}
$$

Because of

$$
t_i^{\chi_{\{Y_j \in C_i\}}(\omega)} = \begin{cases} t_i & \text{if} \quad Y_j(\omega) \in C_i \\ 1 & \text{if} \quad Y_j(\omega) \notin C_i \end{cases}
$$

and since $\{C_1, \ldots, C_d\}$ is a partition of M, the previous identity yields

$$
\begin{aligned}
m_{\boldsymbol{N}|N}(\boldsymbol{t}) &= \sum_{n=0}^{\infty} \prod_{j=1}^{n} \mathbf{E}\left[\sum_{i=1}^{d} t_i \chi_{\{Y_j \in C_i\}}\right] \cdot \chi_{\{N=n\}} \\
&= \sum_{n=0}^{\infty} \prod_{j=1}^{n} \sum_{i=1}^{d} t_i \, \mathbf{P}[Y_j \in C_i] \cdot \chi_{\{N=n\}} \\
&= \sum_{n=0}^{\infty} \prod_{j=1}^{n} \sum_{i=1}^{d} t_i \, \eta_i \cdot \chi_{\{N=n\}} \\
&= \sum_{n=0}^{\infty} \left(\sum_{h=1}^{d} t_i \, \eta_i\right)^n \cdot \chi_{\{N=n\}} \\
&= (\boldsymbol{t}'\boldsymbol{\eta})^N
\end{aligned}
$$

This proves the first equation. Using this equation we get from Lemma 2.5

$$
\begin{aligned}
m_{\boldsymbol{N}}(\boldsymbol{t}) &= \mathbf{E}\left[m_{\boldsymbol{N}|N}(\boldsymbol{t})\right] \\
&= \mathbf{E}\left[(\boldsymbol{t}'\boldsymbol{\eta})^N\right] \\
&= m_N(\boldsymbol{t}'\boldsymbol{\eta})
\end{aligned}
$$

Lemma 2.2 yields

$$
\begin{aligned}
m_{N_i}(t) &= m_{\boldsymbol{N}}(\mathbf{1} - \boldsymbol{e}_i + t\boldsymbol{e}_i) \\
&= m_N\Big((\mathbf{1} - \boldsymbol{e}_i + t\boldsymbol{e}_i)'\,\boldsymbol{\eta}\Big) \\
&= m_N(1 - \eta_i + t\eta_i)
\end{aligned}
$$

Since $\mathbf{1}'\boldsymbol{\eta} = 1$, the final assertion follows from Example 2.6. □

Theorem 3.1 can be used to prove the following corollary via probability generating functions, see Example 2.4:

3.2 Corollary.

(a) *If N has the binomial distribution* $\mathbf{Bin}(n, \vartheta)$ *with* $n \in \mathbb{N}$ *and* $\vartheta \in (0,1)$*, then* $\boldsymbol{N}$ *has the multinomial distribution* $\mathbf{Mult}(n, \vartheta\boldsymbol{\eta})$ *and each* N_i *has the binomial distribution* $\mathbf{Bin}(n, \vartheta\eta_i)$*.*

(b) *If N has the Poisson distribution* $\mathbf{Poi}(\alpha)$ *with* $\alpha \in (0,\infty)$*, then* $\boldsymbol{N}$ *has the multivariate Poisson distribution* $\mathbf{MPoi}(\alpha\boldsymbol{\eta})$ *and each* N_i *has the Poisson distribution* $\mathbf{Poi}(\alpha\eta_i)$*. In this case,* $\boldsymbol{N}$ *has independent coordinates.*

(c) *If N has the negativebinomial distribution* $\mathbf{NBin}(\beta, \vartheta)$ *with* $\beta \in (0,\infty)$ *and* $\vartheta \in (0,1)$*, then* $\boldsymbol{N}$ *has the negativemultinomial distribution* $\mathbf{NMult}(\beta, \vartheta\boldsymbol{\eta})$ *and each* N_i *has the negativebinomial distribution* $\mathbf{NBin}(\beta, \vartheta\eta_i/(1-\vartheta+\vartheta\eta_i))$*.*

Next we want to show that the Poisson case is the only case in which the coordinates are independent. A proof of this result was given by Hess and Schmidt (2002), but here we want to give an analytic one.

3.3 Theorem. *The thinned random sample sizes are independent if and only if the original sample size has a Poisson distribution.*

Proof. In Corollary 3.2 we have established that the Poisson distribution of the original sample size implies independence of the thinned random sample sizes. Now let the thinned random sample sizes be independent. Using $\mathbf{1}'\boldsymbol{\eta} = 1$, Theorem 3.1, and Lemma 2.3, we get

$$\begin{aligned} m_N\left(1 - \sum_{i=1}^d (\eta_i - t_i\eta_i)\right) &= m_N(\boldsymbol{t}'\boldsymbol{\eta}) \\ &= m_{\boldsymbol{N}}(\boldsymbol{t}) \\ &= \prod_{i=1}^d m_{N_i}(t_i) \\ &= \prod_{i=1}^d m_N(1 - \eta_i + t_i\eta_i) \\ &= \prod_{i=1}^d m_N(1 - (\eta_i - t_i\eta_i)) \end{aligned}$$

for all $\boldsymbol{t} = (t_1, \ldots, t_d)' \in [0,1]^d$. Lemma 2.1 yields that m_N is continuous, increasing and fulfills $m_N(1) = 1$. Therefore m_N is an increasing solution of the functional equation

$$f\left(1 - \sum_{i=1}^d u_i\right) = \prod_{i=1}^d f(1 - u_i)$$

for all $u_1, \ldots, u_d \in [0,1]$ fulfilling $\sum_{i=1}^d u_i \leq 1$ in the class of all continuous functions $f : [0,1] \to \mathbb{R}$ fulfilling the initial condition $f(1) = 1$. Define $\varphi : [0,1] \to [0,1]$ with

$\varphi(u) = 1 - u$. Then φ is a bijection. This leads to the functional equation

$$(f \circ \varphi)\left(\sum_{i=1}^d u_i\right) = \prod_{i=1}^d (f \circ \varphi)(u_i)$$

for all $u_1, \ldots, u_d \in [0,1]$ fulfilling $\sum_{i=1}^d u_i \leq 1$ in the class of all continuous functions $f : [0,1] \to \mathbb{R}$ fulfilling the initial condition $f(1) = 1$, and hence $(f \circ \varphi)(0) = 1$. Since this characterized the exponential function we get $(f \circ \varphi)(t) = \mathrm{e}^{-\alpha t}$ for some $\alpha \in \mathbb{R}$ and therefore $f(t) = \mathrm{e}^{-\alpha(1-t)}$. As mentioned before, m_N is increasing. Therefore wet get $\alpha \in (0, \infty)$. This is the probability generating function of the Poisson distribution $\mathbf{Poi}(\alpha)$. □

We complete this note with a result on mixed Poisson distributions.

3.4 Theorem. *If N has the mixed Poisson distribution with parameter Θ, then $\boldsymbol{N}$ has the mixed multivariate Poisson distribution with parameter $\Theta\boldsymbol{\eta}$.*

Proof. Let $\boldsymbol{n} \in \mathbb{N}_0^d$. Using Theorem 3.1 we get

$$\begin{aligned}
\mathbf{P}[\boldsymbol{N} = \boldsymbol{n}] &= \mathbf{P}\left(\boldsymbol{N} = \boldsymbol{n} \,\middle|\, N = \sum_{i=1}^d n_i\right) \cdot \mathbf{P}\left[N = \sum_{i=1}^d n_i\right] \\
&= \frac{\left(\sum_{i=1}^d n_i\right)!}{\prod_{i=1}^d n_i!} \cdot \prod_{i=1}^d \eta_i^{n_i} \cdot \int_\Omega \mathrm{e}^{-\Theta} \frac{\Theta^{\sum_{i=1}^d n_i}}{\left(\sum_{i=1}^d n_i\right)!} \, d\mathbf{P} \\
&= \int_\Omega \prod_{i=1}^d \mathrm{e}^{-\Theta\eta_i} \frac{(\Theta\eta_i)^{n_i}}{n_i!} \, d\mathbf{P}
\end{aligned}$$

as was to be shown. □

For related results on the decomposition of sample variables see Hess (2000) who generalized results by Franke and Macht (1995) and Hess, Schmidt and Macht (1995). For applications to counting processes see Schmidt (1996) and Hess (2003).

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