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Abstract

Let $(T_k)_{k\geq 1}$ be a point process on $\mathbb{R}_+$ and $N = (N_t)_{t\geq 0}$ its associated counting process. If N is integrable and adapted to a right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, then a non-negative, $(\mathcal{F}_t)$-adapted cadlag-process $\lambda = (\lambda_t)_{t\geq 0}$ is called the $(\mathcal{F}_t)$-intensity of N, if $(N_t - \int_0^t \lambda_s\, ds)_{t\geq 0}$ is an $(\mathcal{F}_t)$-martingale. The intensity determines essentially the stochastic development of N.

We prove the following main result: Let $(N^{(n)})_{n\geq 1}$ be a sequence of counting processes with $(\mathcal{F}_t^{(n)})$-intensities $\lambda^{(n)} = (\lambda_t^{(n)})_{t\geq 0}$. Suppose that $(N^{(n)})_{n\geq 1}$ is uniformly non-explosive and that the sequence $(\lambda^{(n)})_{n\geq 1}$ converges in probability uniformly on bounded intervals towards a deterministic intensity $\lambda = (\lambda_t)_{t\geq 0}$. Let $N = (N_t)_{t\geq 0}$ be an inhomogeneous Poisson process with intensity λ, and denote for every $T > 0$ by $\mathbf{B}^1(D([0,T]))$ the family of all measurable functions $F : D([0,T]) \to \mathbb{R}$ bounded by 1. Then

$$(*) \quad \lim_{n\to\infty} \sup_{F\in\mathbf{B}^1(D([0,T]))} \left|\mathbf{E}F\big((N_t^{(n)})_{t\leq T}\big) - \mathbf{E}F\big((N_t)_{t\leq T}\big)\right| \;=\; 0\,.$$

This result can be generalized to the case of marked point processes $((T_k, X_k))_{k\geq 1}$, with marks X_k in a polish space $(E, \mathcal{E})$. The marked point process is described equivalently by a counting measure $N = ((N_t(B))_{t\geq 0})_{B\in\mathcal{E}}$, and the notion of an intensity is replaced by an intensity measure $\lambda = ((\lambda_t(B))_{t\geq 0})_{B\in\mathcal{E}}$. Suppose that now $(N^{(n)})_{n\geq 1}$ is a sequence of counting measures with an associated sequence of $(\mathcal{F}_t^{(n)})$-intensity measures $(\lambda^{(n)})_{n\geq 1}$. If $((\lambda_t(B))_{t\geq 0})_{B\in\mathcal{E}}$ is a deterministic intensity measure, then the condition

$$\lim_{n\to\infty} \sup_{t\leq T} \sup_{B\in\mathcal{E}} |\lambda_t^{(n)}(B) - \lambda_t(B)| \;=\; 0$$

in probability for all $T > 0$ implies a convergence result corresponding to $(*)$ for the sequence $(N^{(n)})_{n\geq 1}$.

1 Introduction

A *point process* on the real line is defined as an increasing sequence $(T_n)_{n\geq 1}$ of $\overline{\mathbb{R}}_+$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the additional property that

$$T_n < T_{n+1} \quad , \text{ if } \quad T_n < \infty \; .$$

It is convenient to define $T_0 \equiv 0$.

The *counting process* $N = (N_t)_{t\geq 0}$ associated to $(T_n)_{n\geq 1}$ is given by

$$N_t := \sum_{n\geq 1} 1_{\{T_n \leq t\}} \; ,$$

In this paper we will only consider point processes with an intensity: Suppose that $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ for which $N = (N_t)_{t\geq 0}$ is adapted, and that $\lambda = (\lambda_t)_{t\geq 0}$ is a $\mathbf{F}$-adapted $\mathbb{R}_+$-valued cadlag-process such that for all $t \geq 0$

$$\int_0^t \lambda_r \, dr < \infty \quad \mathbf{P}\text{-a.s.}. \tag{1}$$

Then we say that $N = (N_t)_{t\geq 0}$ (or $(T_n)_{n\geq 1}$) *has the* $\mathbf{F}$*-intensity* λ, if the process

$$\big(N_{t\wedge T_m} - \int_0^{t\wedge T_m} \lambda_r \, dr\big)_{t\geq 0}$$

is an $\mathbf{F}$-martingale for every $m \geq 1$.

If it is known that $T_n \uparrow \infty$ $\mathbf{P}$-a.s. (i.e. if $(T_n)_{n\geq 1}$ is *non-explosive*), then we could say that $N = (N_t)_{t\geq 0}$ has the $\mathbf{F}$-intensity λ, if

$$\big(N_t - \int_0^t \lambda_r \, dr\big)_{t\geq 0}$$

is a local $\mathbf{F}$-martingale (with $(T_n)_{n\geq 1}$ as localizing sequence). If we know that either $(N_t)_{t\geq 0}$ or $(\int_0^t \lambda_r \, dr)_{t\geq 0}$ is integrable, then $(N_t - \int_0^t \lambda_r \, dr)_{t\geq 0}$ is even a martingale.

A point process $(T_n)_{n\geq 1}$ is called *finite*, if $\mathbf{P}\{T_n = \infty\} = 0$ for every $n \geq 1$.

The distribution $\mathbf{P}_{(T_n)_{n\geq 1}}$ of $(T_n)_{n\geq 1}$ is a probability measure on the space

$$S^\infty := \Big\{ (t_k)_{k\geq 0} \in \overline{\mathbf{R}}_+^{\mathbf{Z}_+} \;\Big|\; (i)\ 0 = t_0 \leq t_1 \leq t_2 \leq \cdots, \quad (ii)\ t_k < \infty \;\Rightarrow\; t_k < t_{k+1}, \quad (iii)\ t_k = \infty \;\Rightarrow\; t_{k+1} = \infty \Big\}$$

provided with the σ-algebra

$$\mathcal{G} := \sigma(\{\tau_n \,|\, n \geq 1\}) ,$$

where the $\tau_n : S^\infty \to \overline{\mathbb{R}}_+$ are given by

$$\tau_n((t_k)_{k\geq 0}) := t_n .$$

We call $(\tau_n)_{n\geq 1}$ the *canonical point process on* S^∞. The *associated canonical counting process* $(\nu_t)_{t\geq 0}$ *on* S^∞ then is defined by

$$\nu_t := \sum_{n\geq 1} 1_{\{\tau_n \leq t\}} .$$

We will make the following convention: If $(t_k)_{k\geq 0} \in S^\infty$ has the property that $t_{n+1} = \infty$ for some $n \geq 0$, then we also write $(t_k)_{k\leq n}$ instead of $(t_k)_{k\geq 0}$.

If we denote by $\Phi : \Omega \to S^\infty$ the map defined by

$$\Phi(\omega) := (T_n(\omega))_{n\geq 0} ,$$

then clearly $\Phi(\mathbf{P}) = \mathbf{P}_{(T_n)_{n\geq 1}}$.

For every $t \geq 0$ we set

$$\mathcal{G}_t := \sigma(\{\nu_s \,|\, s \leq t\}) .$$

The filtration $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$ is called the *canonical filtration on* S^∞. If $\mathbf{F}^N = (\mathcal{F}_t^N)_{t\geq 0}$ denotes the canonical filtration of the counting process $(N_t)_{t\geq 0}$ on Ω, then obviously

$$\mathcal{F}_t^N = \Phi^{-1}(\mathcal{G}_t) \tag{2}$$

for every $t \geq 0$.

For any filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ the $\mathbf{F}$-*predictable* σ-algebra $\mathcal{P}(\mathbf{F})$ on $\mathbb{R}_+ \times \Omega$ is the σ-algebra generated by the sets $]s,t] \times F$, where $0 \leq s < t$ and $F \in \mathcal{F}_s$.

We will say that a process $X = (X_t)_{t\geq 0}$ on Ω is $\mathbf{F}$-*predictable*, if X - viewed as a map $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ - is $\mathcal{P}(\mathbf{F})$-measurable.

Now suppose that $(N_t)_{t\geq 0}$ has the $\mathbf{F}$-intensity $(\lambda_t)_{t\geq 0}$. Then the measure $\lambda_t\, dt\otimes d\mathbf{P}$ is absolutely continuous relative to $dt\otimes d\mathbf{P}$ on the measurable space $(\mathbb{R}_+ \times \Omega, \mathcal{P}(\mathbf{F}^N))$. Hence there exists a $\mathcal{P}(\mathbf{F}^N)$-measurable Radon-Nikodym density $\pi(\lambda) = (\pi(\lambda)_t)_{t\geq 0}$. This density is known to have the following two properties:

Proposition 1.1. *If the counting process $N = (N_t)_{t\geq 0}$ is integrable and has the $\mathbf{F}$-intensity λ, then N has the $\mathbf{F}^N$-intensity $\pi(\lambda)$ and there is a $\mathbf{G}$-intensity $\overline{\lambda} = (\overline{\lambda}_t)_{t\geq 0}$ (which is even $\mathbf{G}$-predictable) such that $\pi(\lambda)_t = \overline{\lambda}_t \circ \Phi$ for every $t \geq 0$.*

Proof: We have to prove that for all $0 \leq s < t$

$$\mathbf{E}\Big\{ \int_s^t \lambda_r\, dr - \int_s^t \pi(\lambda)_r\, dr \,\Big|\, \mathcal{F}_s^N \Big\} = 0\ .$$

But this follows, since by definition $]s,t] \times A \in \mathcal{P}(\mathbf{F}^N)$ for every $A \in \mathcal{F}_s^N$, and hence

$$\begin{aligned} \int_A \Big(\int_s^t \lambda_r\, dr \Big)\, d\mathbf{P} \ &= \ \int_{]s,t]\times A} \lambda_r\, dr \otimes\, d\mathbf{P} \\ = \ \int_{]s,t]\times A} \pi(\lambda)_r\, dr \otimes\, d\mathbf{P} \ &= \ \int_A \Big(\int_s^t \pi(\lambda)_r\, dr \Big)\, d\mathbf{P}\ . \end{aligned}$$

If $\tilde{\Phi} : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \times S^\infty$ denotes the map given by

$$\tilde{\Phi}(r, \omega) = (r, \Phi(\omega))\ ,$$

then it follows from (2) that

$$\mathcal{P}(\mathbf{F}^N) = \tilde{\Phi}^{-1}(\mathcal{P}(\mathbf{G}))\ , \tag{3}$$

and this implies (cf. [1]) that for every $\mathcal{P}(\mathbf{F}^N)$-measurable function F there exists a $\mathcal{P}(\mathbf{G})$-measurable function G such that $F = G \circ \tilde{\Phi}$. The second assertion of the proposition is just a special case of this result. □

The $\mathbf{F}^N$-intensity $\pi(\lambda) = (\pi(\lambda)_t)_{t\geq 0}$ will be called the *predictable projection* of λ on Ω, and the intensity $\overline{\lambda} = (\overline{\lambda}_t)_{t\geq 0}$ is called the *predictable projection* of λ on S^∞.

The predictable projection $\overline{\lambda}$ on S^∞ completely determines the distribution $\Phi(\mathbf{P})$ of $(T_n)_{n\geq 1}$. If e.g. $(T_n)_{n\geq 1}$ is finite and $F : \mathbb{R}_+^n \to \mathbb{R}_+$ measurable and bounded, then

$$\begin{aligned}
&\mathbf{E}\, F(T_1, \cdots, T_n) \qquad (4)\\
&\quad = \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{n-1}}^\infty \Big\{ F(t_1, \cdots, t_n) \\
&\qquad \prod_{k=1}^n \Big(\overline{\lambda}_{t_k}((t_i)_{i\leq k-1}) e^{-\int_{t_{k-1}}^{t_k} \overline{\lambda}_r((t_i)_{i\leq k-1})\, dr} \Big) \Big\}\, dt_n \cdots dt_1 \, .
\end{aligned}$$

2 Poisson Approximation

Suppose that on $(\Omega, \mathcal{F}, \mathbf{P})$ we have a sequence of finite point processes $(T_k^{(n)})_{k\geq 1}$, and that the associated counting processes $N^{(n)} = (N_t^{(n)})_{t\geq 0}$ are all integrable. Let further $\mathbf{F}^{(n)} = (\mathcal{F}_t^{(n)})_{t\geq 0}$ be a sequence of filtrations on Ω such that every process $N^{(n)}$ is $\mathbf{F}^{(n)}$-adapted, and suppose in addition that every $N^{(n)}$ has the $\mathbf{F}^{(n)}$-intensity $\lambda^{(n)} = (\lambda_t^{(n)})_{t\geq 0}$.

We will say that the sequence $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ (resp. the sequence $(N^{(n)})_{n\geq 1}$) is *uniformly finite*, if

$$\lim_{t\to\infty} \sup_{n\geq 1} \mathbf{P}\{T_m^{(n)} > t\} = 0 \qquad (5)$$

for every $m \geq 1$, and *uniformly non-explosive*, if

$$\lim_{m\to\infty} \sup_{n\geq 1} \mathbf{P}\{T_m^{(n)} \leq t\} = 0 \qquad (6)$$

for every $t \geq 0$.

Now assume that $\lambda = (\lambda_t)_{t\geq 0}$ is a deterministic, non-negative, right continuous and locally bounded process (i.e. λ is just a function from $\mathbb{R}_+$ into $\mathbb{R}_+$), such that λ has the additional property

$$\inf_{t\geq 0} \lambda_t \geq c \qquad (7)$$

for a certain constant $c > 0$.

We may and will assume that there is a point process $(T_k)_{k\geq 1}$, whose counting process $N = (N_t)_{t\geq 0}$ has the $\mathbf{F}^N$-intensity λ. N is just an inhomogeneous Poisson process, and for our purposes we may and will assume that N and $\bigvee_{n\geq 1} \mathcal{F}_\infty^{(n)}$ are independent.

For any measurable space $(T,\mathcal{T})$ we denote by $\mathbf{B}^1(T)$ the set of all measurable functions $F:T\to\mathbb{R}$ which are bounded by 1.

Now we can formulate the first approximation result:

Theorem 2.1. *Suppose that for every $T>0$*

$$\sup_{t\le T}|\lambda_t^{(n)}-\lambda_t|\to 0 \quad \textit{in probability} . \tag{8}$$

Then for every $m\ge 1$

$$\lim_{n\to\infty}\sup_{F\in\mathbf{B}^1(\mathbb{R}_+^m)}\big|\mathbf{E}\,F(T_1^{(n)},\cdots,T_m^{(n)})-\mathbf{E}\,F(T_1,\cdots,T_m)\big|=0 . \tag{9}$$

For the proof of this theorem we will essentially need the following lemma, which is perhaps of some interest in itself.

Lemma 2.2. *Suppose that $(T_k^{(1)})_{k\ge1}$ and $(T_k^{(2)})_{k\ge1}$ are two point processes with associated counting processes $N^{(1)}$ and $N^{(2)}$. We assume*
(i) that $\mathbf{F}^{(1)}$ is a right continuous filtration, for which $N^{(1)}$ is adapted and has the $\mathbf{F}^{(1)}$-intensity $\lambda^{(1)}$,
(ii) that $N^{(2)}$ has the intensity $\lambda^{(2)}$ relative to the canonical filtration $\mathbf{F}^{N^{(2)}}$ of $N^{(2)}$, and
(iii) that $\mathcal{F}_\infty^{(1)}$ and $N^{(2)}$ are independent.
Let S be a given $\mathbf{F}^{(1)}$-stopping time and define for every $t\ge0$

$$\begin{aligned}\tilde N_t &:= N^{(1)}_{t\wedge S}+(N^{(2)}_{t\vee S}-N^{(2)}_S) ,\\ \tilde{\mathcal{F}}_t &:= \mathcal{F}^{(1)}_{t\wedge S}\vee\sigma\big(\{N^{(2)}_{s\vee S}-N^{(2)}_S\,|\,s\le t\}\big) , \ \textit{and}\\ \tilde\lambda_t &:= \lambda_t^{(1)}1_{\{t<S\}}+\lambda_t^{(2)}1_{\{S\le t\}} .\end{aligned} \tag{10}$$

Then $\tilde N=(\tilde N_t)_{t\ge0}$ is a counting process with the intensity $\tilde\lambda=(\tilde\lambda_t)_{t\ge0}$ relative to the filtration $\tilde{\mathbf{F}}=(\tilde{\mathcal{F}}_t)_{t\ge0}$.

Proof: We have to prove that for every $0\le s<t$

$$\mathbf{E}\{\tilde N_t-\tilde N_s\,|\,\tilde{\mathcal{F}}_s\}=\mathbf{E}\Big\{\int_s^t\tilde\lambda_r\,dr\,\Big|\,\tilde{\mathcal{F}}_s\Big\} . \tag{11}$$

We set

$$\begin{aligned}\mathcal{D}_s := \big\{&\{N^{(2)}_{u_1\vee S}-N^{(2)}_S\in C_1,\cdots,N^{(2)}_{u_m\vee S}-N^{(2)}_S\in C_m\}\big|\\ &\qquad m\in\mathbb{N},\ u_i\le s,\ C_i\in\mathcal{B}(\mathbb{R}_+)\big\} ,\end{aligned}$$

and

$$\mathcal{E}_s := \mathcal{F}_{s\wedge S} \cap \mathcal{D}_s \ .$$

Then $\mathcal{E}_s$ is a $\cap$-stable generator of $\tilde{\mathcal{F}}_s$, and hence it is sufficient to prove

$$\int_{A\cap B} (\tilde{N}_t - \tilde{N}_s)\, d\mathbf{P} = \int_{A\cap B} \Big(\int_s^t \tilde{\lambda}_r \, dr \Big)\, d\mathbf{P} \tag{12}$$

for all $A \in \mathcal{F}^{(1)}_{s\wedge S}$ and all $B \in \mathcal{D}_s$.

We start with the left hand side of (12):

$$\begin{aligned} &\int_{A\cap B} (\tilde{N}_t - \tilde{N}_s)\, d\mathbf{P} \qquad (13)\\ &\quad = \int_{A\cap B} (N^{(1)}_{t\wedge S} - N^{(1)}_{s\wedge S})\, d\mathbf{P} + \int_{A\cap B} (N^{(2)}_{t\vee S} - N^{(2)}_{s\vee S})\, d\mathbf{P} \ . \end{aligned}$$

For the first term on the right hand side we obtain

$$\begin{aligned} &\int_{A\cap B} (N^{(1)}_{t\wedge S} - N^{(1)}_{s\wedge S})\, d\mathbf{P} \qquad (14)\\ &\quad = \int 1_{A\cap\{S>s\}} 1_{B\cap\{T>s\}} (N^{(1)}_{t\wedge S} - N^{(1)}_{s\wedge S})\, d\mathbf{P}\\ &\quad = \int 1_{A\cap\{S>s\}} 1_{B\cap\{T>s\}} \Big(\int_{s\wedge S}^{t\wedge S} \lambda^{(1)}_r \, dr \Big)\, d\mathbf{P}\\ &\quad = \int 1_{A\cap B} \Big(\int_{s\wedge S}^{t\wedge S} \lambda^{(1)}_r \, dr \Big)\, d\mathbf{P} \ , \end{aligned}$$

since $A \cap \{S > s\} \in \mathcal{F}^{(1)}_s$ and since $B \cap \{S > s\}$ is either equal to Ω or empty.

For the second term on the right hand side of (13) we first get

$$\begin{aligned} &\int_{A\cap B} (N^{(2)}_{t\vee S} - N^{(2)}_{s\vee S})\, d\mathbf{P}\\ &\quad = \int 1_A 1_B 1_{\{s\le S<t\}} (N^{(2)}_t - N^{(2)}_S)\, d\mathbf{P} \qquad (15)\\ &\qquad + \int 1_A 1_B 1_{\{S<s\}} (N^{(2)}_t - N^{(2)}_s)\, d\mathbf{P} \ . \qquad (16) \end{aligned}$$

First we consider (15). We set for $n \ge 1$

$$S^{(n)} := \sum_{k\ge 1} \frac{k}{2^n} 1_{\{\frac{k-1}{2^n} \le S < \frac{k}{2^n}\}}$$

Then

$$\int 1_A 1_B 1_{\{s\le S<t\}}(N^{(2)}_t - N^{(2)}_{S^{(n)}})\,d\mathbf{P} \tag{17}$$

$$= \sum_{k\ge 1}\int 1_A 1_B 1_{\{s\le S<t\}} 1_{\{\frac{k-1}{2^n}\le S<\frac{k}{2^n}\}}(N^{(2)}_t - N^{(2)}_{\frac{k}{2^n}})\,d\mathbf{P}$$

$$= \sum_{k\ge 1}\int 1_{A\cap\{s\le S<t\}\cap\{\frac{k-1}{2^n}\le S<\frac{k}{2^n}\}} 1_{B\cap\{s\le S\}}(N^{(2)}_t - N^{(2)}_{\frac{k}{2^n}})\,d\mathbf{P}$$

$$= \sum_{k\ge 1}\int 1_{A\cap\{s\le S<t\}\cap\{\frac{k-1}{2^n}\le S<\frac{k}{2^n}\}} 1_{B\cap\{s\le S\}}\Big(\int_{\frac{k}{2^n}}^t \lambda^{(2)}_r\,dr\Big)\,d\mathbf{P}$$

$$= \int 1_A 1_B 1_{\{s\le S<t\}}\Big(\int_{S^{(n)}}^t \lambda^{(2)}_r\,dr\Big)\,d\mathbf{P}\ ,$$

since by assumption $A\cap\{s\le S<t\}\cap\{\frac{k-1}{2^n}\le S<\frac{k}{2^n}\}$ and $N^{(2)}$ are independent and $B\cap\{s\le S\}$ is either empty or equal to Ω. For $n\to\infty$ we obtain from (17)

$$\int 1_A 1_B 1_{\{s\le S<t\}}(N^{(2)}_t - N^{(2)}_S)\,d\mathbf{P} = \int 1_A 1_B 1_{\{s\le S<t\}}\Big(\int_S^t \lambda^{(2)}_r\,dr\Big)\,d\mathbf{P} \tag{18}$$

Now we consider the term (16), and suppose that B is of the form

$$B = \{N^{(2)}_{u_1\vee S} - N^{(2)}_S \in C_1, \cdots, N^{(2)}_{u_m\vee S} - N^{(2)}_S \in C_m\}$$

with $u_i \le s$, and where we suppose in addition that the Borel sets C_i are all open. Then we get

$$\int 1_A 1_B 1_{\{S<s\}}(N^{(2)}_t - N^{(2)}_s)\,d\mathbf{P} \tag{19}$$

$$= \lim_{n\to\infty}\int 1_{A\cap\{S<s\}} \Big(1_{\{N^{(2)}_{u_1\vee S^{(n)}} - N^{(2)}_{S^{(n)}}\in C_1,\cdots,N^{(2)}_{u_m\vee S^{(n)}} - N^{(2)}_{S^{(n)}}\in C_m\}}(N^{(2)}_t - N^{(2)}_s)\Big)\,d\mathbf{P}$$

$$= \lim_{n\to\infty}\sum_{k\ge 1}\int 1_{A\cap\{S<s\}\cap\{\frac{k-1}{2^n}\le S<\frac{k}{2^n}\}} \Big(1_{\{N^{(2)}_{u_1\vee\frac{k}{2^n}} - N^{(2)}_{\frac{k}{2^n}}\in C_1,\cdots,N^{(2)}_{u_m\vee\frac{k}{2^n}} - N^{(2)}_{\frac{k}{2^n}}\in C_m\}}(N^{(2)}_t - N^{(2)}_s)\Big)\,d\mathbf{P}$$

$$= \lim_{n\to\infty} \sum_{k\geq 1} \int 1_{A\cap\{S<s\}\cap\{\frac{k-1}{2^n}\leq S<\frac{k}{2^n}\}}$$

$$\Big(1_{\{N^{(2)}_{u_1\vee\frac{k}{2^n}}-N^{(2)}_{\frac{k}{2^n}}\in C_1,\cdots,N^{(2)}_{u_m\vee\frac{k}{2^n}}-N^{(2)}_{\frac{k}{2^n}}\in C_m\}}\big(\int_s^t \lambda_r^{(2)}\,dr\big)\Big)\,d\mathbf{P}$$

$$= \int 1_A 1_B 1_{\{S<s\}}\big(\int_s^t \lambda_r^{(2)}\,dr\big)\,d\mathbf{P}$$

again by the independence of $\mathcal{F}_\infty^{(1)}$ and $N^{(2)}$.

(14), (17) and (19) together prove that

$$\begin{aligned}
&\int_{A\cap B} (\tilde N_t - \tilde N_s)\,d\mathbf{P} \\
&= \int 1_{A\cap B}\Big\{1_{\{S\geq s\}}\big(\int_{s\wedge S}^{t\wedge S} \lambda_r^{(1)}\,dr\big) \\
&\qquad + 1_{\{s\leq S<t\}}\big(\int_S^t \lambda_r^{(2)}\,dr\big) + 1_{\{S<s\}}\big(\int_s^t \lambda_r^{(2)}\,dr\big)\Big\}\,d\mathbf{P} \\
&= \int 1_{A\cap B}\Big\{1_{\{S\geq s\}}\big(\int_s^t \lambda_r^{(1)}1_{\{S>r\}}\,dr\big) \\
&\qquad + 1_{\{s\leq S<t\}}\big(\int_s^t \lambda_r^{(2)}1_{\{S\leq r\}}\,dr\big) + 1_{\{S<s\}}\big(\int_s^t \lambda_r^{(2)}1_{\{S\leq r\}}\,dr\big)\Big\}\,d\mathbf{P} \\
&= \int 1_{A\cap B}\Big(\int_s^t \big\{\lambda_r^{(1)}1_{\{S>r\}} + \lambda_r^{(2)}1_{\{S\leq r\}}\big\}\,dr\Big)\,d\mathbf{P} \\
&= \int 1_{A\cap B}\big(\int_s^t \tilde\lambda_r\,dr\big)\,d\mathbf{P}\ .
\end{aligned}$$

Thus we have proved (12). □

The counting process $\tilde N$ of lemma 2.2 will be called the *composition of* $N^{(1)}$ *and* $N^{(2)}$ *at* S, and the intensity $\tilde\lambda$ is called the *composition of* $\lambda^{(1)}$ *and* $\lambda^{(2)}$ *at* S.

Proof of theorem 2.1:
(1) For every $n\geq 1$ and every $\varepsilon>0$ we define

$$S^{n,\varepsilon} := \inf\{t>0\ :\ |\lambda_t^{(n)}-\lambda_t|\geq\varepsilon\}\ . \tag{20}$$

Every $S^{n,\varepsilon}$ is an $\mathbf{F}^{(n)}$-stopping time. The assumption (8) implies that for every $\varepsilon>0$, $\delta>0$ and $T>0$ there exists an $n(\varepsilon,\delta,T)\geq 1$ such that

$$\mathbf{P}\{S^{n,\varepsilon}\leq T\}<\delta \quad \text{for all}\ \ n\geq n(\varepsilon,\delta,T)\ . \tag{21}$$

[Suppose that there exist $\varepsilon > 0$, $\delta > 0$ and $T > 0$, such that for every $m \geq 1$ there is an $n \geq 1$, for which

$$\mathbf{P}\{S^{n,\varepsilon} \leq T\} \geq \delta \ .$$

Since

$$\{\sup_{t \leq T} |\lambda_t^{(n)} - \lambda_t| \geq \varepsilon\} \subseteq \{S^{n,\varepsilon} \leq T\} \ ,$$

we get

$$\mathbf{P}\{\sup_{t \leq T} |\lambda_t^{(n)} - \lambda_t| \geq \varepsilon\} \geq \delta$$

in contradiction to assumption (8).]

Hence, if $(\varepsilon_j)_{j\geq 1}$ is a fixed null-sequence of positive numbers, there exists a strictly increasing sequence $(n_j)_{j\geq 1}$ in $\mathbb{N}$, such that

$$\mathbf{P}\{S^{n,\varepsilon_j} \leq j\} < \varepsilon_j \quad \text{for every} \ \ n \geq n_j \ . \tag{22}$$

We set

$$S^n := S^{n,\varepsilon_j} \quad \text{for} \ \ n_j \leq n < n_{j+1} \tag{23}$$

Then

$$\mathbf{P}\{S^n \leq j\} < \varepsilon_j \quad \text{for} \ \ n_j \leq n < n_{j+1} \tag{24}$$

and the sequence $(S^n)_{n\geq 1}$ has especially the property that

$$\lim_{n\to\infty} \mathbf{P}\{S^n \leq T\} = 0 \quad \text{for every} \ \ T > 0 \ . \tag{25}$$

Now let $(\tilde{T}_k^n)_{k\geq 1}$ for every $n \geq 1$ denote the composition of $(T_k^{(n)})_{k\geq 1}$ and $(T_k)_{k\geq 1}$ at S^n, and let $\tilde{\lambda}^n$ denote the corresponding composition of the intensities $\lambda^{(n)}$ and λ at S^n. Then the sequence $(\tilde{\lambda}^n)_{n\geq 1}$ has the property that

$$\sup_{t\geq 0} |\tilde{\lambda}_t^n - \lambda_t| \leq \varepsilon_j \quad \text{for} \ \ n_j \leq n < n_{j+1} \ . \tag{26}$$

This property is inherited by the sequence $(\pi(\tilde{\lambda}^n))_{n\geq 1}$ of the predictable projections: The inequality (26) means that

$$\lambda_t - \varepsilon_j \leq \tilde{\lambda}_t^n(\omega) \leq \lambda_t + \varepsilon_j$$

for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, and hence the uniqueness of the Radon-Nikodym derivative shows that also

$$\lambda - \varepsilon_j \leq \pi(\tilde{\lambda}^n) \leq \lambda + \varepsilon_j \quad (dt \otimes d\mathbf{P}) - \text{a.s.}.$$

Thus we may assume that

$$\sup_{t\geq 0} |\pi(\tilde{\lambda}^n)_t - \lambda_t| \leq \varepsilon_j \tag{27}$$

holds on Ω. The same assertion then holds for the associated sequence $(\overline{\lambda}^n)_{n\geq 1}$ of predictable projections on S^∞. Thus we have for all $n \geq 1$

$$\sup_{t\geq 0} |\overline{\lambda}^n_t - \lambda_t| \leq \varepsilon_j \quad \text{for } \; n_j \leq n < n_{j+1} \; . \tag{28}$$

(2) Now we will prove that the sequence $((\tilde{T}^n_k)_{k\geq 1})_{n\geq 1}$ (and as a consequence also the sequence $((T^{(n)}_k)_{k\geq 1})_{n\geq 1})$ is uniformly finite.

Proof: We choose $j \geq 1$ such that $\varepsilon_j < \frac{c}{2}$, where $c > 0$ is the positive constant, which minorizes λ. Then we have for $n \geq n_j$

$$\begin{aligned}
&\mathbf{P}\{\tilde{T}^n_m > t \geq \tilde{T}^n_{m-1}\} \\
&\quad = \int_0^t \cdots \int_{t_{m-2}}^t \int_t^\infty \Big\{ \prod_{k=1}^m \Big(\overline{\lambda}^n_{t_k}((t_i)_{i\leq k-1}) \\
&\qquad\qquad \cdot e^{-\int_{t_{k-1}}^{t_k} \overline{\lambda}^n_r((t_i)_{i\leq k-1})\,dr} \Big) \Big\} dt_m \cdots dt_1 \\
&= \int_0^t \cdots \int_{t_{m-2}}^t \Big\{ \prod_{k=1}^{m-1} \Big(\overline{\lambda}^n_{t_k}((t_i)_{i\leq k-1}) \\
&\qquad\qquad \cdot e^{-\int_{t_{k-1}}^{t_k} \overline{\lambda}^n_r((t_i)_{i\leq k-1})\,dr} \Big) e^{-\int_{t_{m-1}}^{t} \overline{\lambda}^n_r((t_i)_{i\leq m-1})\,dr} \Big\} dt_{m-1} \cdots dt_1 \\
&\leq e^{2\varepsilon_j t} \int_0^t \cdots \int_{t_{m-2}}^t \Big\{ \prod_{k=1}^{m-1} \Big((\lambda_{t_k} + \varepsilon_j) \\
&\qquad\qquad \cdot e^{-\int_{t_{k-1}}^{t_k} (\lambda_r + \varepsilon_j)\,dr} \Big) e^{-\int_{t_{m-1}}^{t} (\lambda_r + \varepsilon_j)\,dr} \Big\} \, dt_{m-1} \cdots dt_1 \\
&= e^{2\varepsilon_j t} e^{-\int_0^t (\lambda_r + \varepsilon_j)\,dr} \frac{\big(\int_0^t (\lambda_r + \varepsilon_j)\,dr\big)^{m-1}}{(m-1)!} \\
&= e^{-\int_0^t (\lambda_r - \varepsilon_j)\,dr} \frac{\big(\int_0^t (\lambda_r + \varepsilon_j)\,dr\big)^{m-1}}{(m-1)!} \\
&\leq e^{-\frac{c}{2}t} \frac{\big(\int_0^t (\lambda_r + \varepsilon_j)\,dr\big)^{m-1}}{(m-1)!} \; ,
\end{aligned}$$

where we used that for $n \geq n_j$

$$\lambda_t - \varepsilon_j \leq \overline{\lambda}_t^n \leq \lambda_t + \varepsilon_j$$

for all $t \geq 0$. It follows that

$$\begin{aligned} \mathbf{P}\{\tilde{T}_m^n > t\} &= \sum_{k=1}^{m} \mathbf{P}\{\tilde{T}_k^n > t \geq \tilde{T}_{k-1}^n\} \\ &\leq e^{-\frac{c}{2}t} \sum_{k=0}^{m-1} \frac{\left(\int_0^t (\lambda_r + \varepsilon_j)\, dr\right)^k}{k!} , \end{aligned}$$

and hence we have proved

$$\lim_{t\to\infty} \sup_{n \geq n_j} \mathbf{P}\{\tilde{T}_m^n > t\} = 0 . \tag{29}$$

Since the point processes $(\tilde{T}_k^1)_{k\geq 1}, \cdots, (\tilde{T}_k^{n_j-1})_{k\geq 1}$ are all finite, it follows that the sequence $((\tilde{T}_k^n)_{k\geq 1})_{n\geq 1}$ is uniformly finite.

We will prove now that also the original sequence $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ is uniformly finite. Since

$$\{T_m^{(n)} < t\} \cap \{S_n \geq t\} = \{\tilde{T}_m^n < t\} \cap \{S_n \geq t\} ,$$

we have

$$\begin{aligned} &\left|\mathbf{P}\{T_m^{(n)} < t\} - \mathbf{P}\{\tilde{T}_m^n < t\}\right| \\ &\quad \leq \mathbf{P}(\{T_m^{(n)} < t\} \cap \{S_n < t\}) + \mathbf{P}(\{\tilde{T}_m^n < t\} \cap \{S_n < t\}) \leq 2\varepsilon_j , \end{aligned}$$

and hence

$$\mathbf{P}\{T_m^{(n)} \geq t\} \leq \mathbf{P}\{\tilde{T}_m^n \geq t\} + 2\varepsilon_j \tag{30}$$

for $n \geq n_j$. Since for every fixed $j \geq 1$

$$\lim_{t\to\infty} \sup_{1\leq n\leq n_j} \mathbf{P}\{T_m^{(n)} \geq t\} = 0 ,$$

it follows from (30) and the uniform finiteness of $((\tilde{T}_k^n)_{k\geq 1})_{n\geq 1}$ that

$$\limsup_{t\to\infty} \sup_{n\geq 1} \mathbf{P}\{T_m^{(n)} \geq t\} \leq 2\varepsilon_j .$$

Since this holds for every $j \geq 1$, the uniform finiteness of $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ follows.

(3) Now we start with the proof of (9). Let $\varepsilon > 0$ be given. Since the sequence $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ is uniformly finite, we can find a $t = t(\varepsilon) > 0$ such that

$$\mathbf{P}\{T_m^{(n)} > t\} \leq \varepsilon$$

for every $n \geq 1$, and by assumption (7) on the deterministic intensity λ we may at the same time assume that also

$$\mathbf{P}\{T_m > t\} \leq \varepsilon \ .$$

Then

$$\begin{aligned} &\sup_{F\in\mathbf{B}^1(\mathbb{R}_+^m)} \left|\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) - F(T_1, \cdots, T_m)\right| \qquad (31)\\ &\leq \sup_{F\in\mathbf{B}^1(\mathbb{R}_+^m)} \left|\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) 1_{\{T_m^{(n)}\leq t\}} - F(T_1, \cdots, T_m) 1_{\{T_m\leq t\}}\right| + 2\varepsilon \ , \end{aligned}$$

and it is sufficient to prove

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1(\mathbb{R}_+^m)} \left|\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) 1_{\{T_m^{(n)}\leq t\}} - F(T_1, \cdots, T_m) 1_{\{T_m\leq t\}}\right| = 0 \ . \qquad (32)$$

Now we compare the expectation $\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) 1_{\{T_m^{(n)}\leq t\}}$ for a fixed $F \in \mathbf{B}^1(\mathbb{R}_+^m)$ with the corresponding expectation $\mathbf{E}\, F(\tilde{T}_1^n, \cdots, \tilde{T}_m^n) 1_{\{\tilde{T}_m^n\leq t\}}$ of the composed process $(\tilde{T}_k^n)_{k\geq 1}$. Since $T_k^{(n)} = \tilde{T}_k^n$ on $\{S^n > t\} \cap \{T_m^{(n)} \leq t\}$ and on $\{S^n > t\} \cap \{\tilde{T}_m^n \leq t\}$ resp. for $k \leq m$, we have

$$\begin{aligned} &\left|\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) 1_{\{T_m^{(n)}\leq t\}} - \mathbf{E}\, F(\tilde{T}_1^n, \cdots, \tilde{T}_m^n) 1_{\{\tilde{T}_m^n\leq t\}}\right| \\ &= \left|\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) 1_{\{T_m^{(n)}\leq t\}\cap\{S^n\leq t\}}\right| + \left|\mathbf{E}\, F(\tilde{T}_1^n, \cdots, \tilde{T}_m^n) 1_{\{\tilde{T}_m^n\leq t\}\cap\{S^n\leq t\}}\right| \\ &\leq 2\varepsilon_j \quad \text{for} \quad n \geq n_j \ . \end{aligned}$$

and thus

$$\sup_{F\in\mathbf{B}^1(\mathbb{R}_+^m)} \left|\mathbf{E}\, F(T_1^{(n)}, \cdots, T_m^{(n)}) 1_{\{T_m^{(n)}\leq t\}} - \mathbf{E}\, F(\tilde{T}_1^n, \cdots, \tilde{T}_m^n) 1_{\{\tilde{T}_m^n\leq t\}}\right| \leq 2\varepsilon_j \qquad (33)$$

for every $n \geq n_j$.

(33) means that (32) follows, if we prove that

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1(\mathbb{R}_+^m)} \left|\mathbf{E}\, F(\tilde{T}_1^n, \cdots, \tilde{T}_m^n) 1_{\{\tilde{T}_m^n\leq t\}} - F(T_1, \cdots, T_m) 1_{\{T_m\leq t\}}\right| = 0 \ . \qquad (34)$$

For every $F \in \mathbf{B}^1(\mathbb{R}_+^m)$ we know (cf. (4)) that

$$\begin{aligned}
&\left|\mathbf{E}\, F(\tilde{T}_1^n, \cdots, \tilde{T}_m^n) 1_{\{\tilde{T}_m^n \le t\}} - F(T_1, \cdots, T_m) 1_{\{T_m \le t\}}\right| \\
&= \Bigg| \int_0^t \cdots \int_{t_{m-1}}^t F(t_1, \cdots, t_m) \\
&\qquad\qquad \prod_{k=1}^m \left(\overline{\lambda}_{t_k}^n((t_i)_{i \le k-1}) \cdot e^{-\int_{t_{k-1}}^{t_k} \overline{\lambda}_r^n((t_i)_{i\le k-1})\, dr} \right) dt_m \cdots dt_1 \\
&\qquad - \int_0^t \cdots \int_{t_{m-1}}^t F(t_1, \cdots, t_m) \prod_{k=1}^m \left(\lambda_{t_k} \cdot e^{-\int_{t_{k-1}}^{t_k} \lambda_r\, dr} \right) dt_m \cdots dt_1 \Bigg| \\
&\le \int_0^t \cdots \int_{t_{m-1}}^t \Bigg| \prod_{k=1}^m \left(\overline{\lambda}_{t_k}^n((t_i)_{i \le k-1}) \cdot e^{-\int_{t_{k-1}}^{t_k} \overline{\lambda}_r^n((t_i)_{i\le k-1})\, dr} \right) \\
&\qquad\qquad - \prod_{k=1}^m \left(\lambda_{t_k} \cdot e^{-\int_{t_{k-1}}^{t_k} \lambda_r\, dr} \right) \Bigg| \, dt_m \cdots dt_1 \,.
\end{aligned} \tag{35}$$

Since

$$|\overline{\lambda}_{t_k}^n((t_i)_{i \le k-1}) - \lambda_{t_k}| \le \varepsilon_j$$

for all $1 \le k \le m$ and all $n \ge n_j$, it follows easily from Lebesgue's convergence theorem that (34) holds, and the theorem is proved. □.

Remark 2.3. The dominated convergence argument at the end of the proof of theorem 2.1 is just a qualitative argument and it may be useful to have a better quantitative estimate. Such an estimate is given - in the more general context of marked point processes - at the end of the proof of theorem 3.1 below.

In case that the sequence $((T_k^{(n)})_{k\ge1})_{n\ge1}$ is also uniformly non-explosive, theorem 2.1 can be generalized.

Theorem 2.4. *Suppose that - in addition to the hypotheses of theorem 2.1 - the sequence $((T_k^{(n)})_{k\ge1})_{n\ge1}$ is uniformly non-explosive. Then for every $T > 0$*

$$\lim_{n\to\infty} \sup_{F \in \mathbf{B}^1(S^\infty, \mathcal{G}_T)} \left|\mathbf{E}\, F((T_k^{(n)})_{k\ge1}) - \mathbf{E}\, F((T_k)_{k\ge1})\right| = 0 \,. \tag{36}$$

Before we start with the proof of the theorem, we give a different interpretation of the limit relation (36). Let $D([0,T])$ denote the space of all right continuous functions with left hand limits on the interval $[0,T]$ provided with the σ-algebra $\mathcal{D}_T$ generated by the maps $f \mapsto f(t)$ $(t \in [0,T])$. Then the map

$$\psi_T : (S^\infty, \mathcal{G}_T) \to (D([0,T]), \mathcal{D}_T) ,$$

defined by

$$\psi_T((t_k)_{k\geq 1})(t) := \sum_{k\geq 1} 1_{[0,t]}(t_k) ,$$

is $\mathcal{G}_T$-$\mathcal{D}_T$-measurable. Now let

$$N^{(n),T} = (N_t^{(n)})_{0\leq t\leq T}$$

denote the counting process associated to $(T_k^{(n)})_{k\geq 1}$ and restricted to $[0,T]$. Then $N^{(n),T}$ can be viewed as a $D([0,T])$-valued random vector. Moreover,

$$N^{(n),T} = \psi_T((T_k^{(n)})_{k\geq 1}) .$$

If $N^T = (N_t)_{0\leq t\leq T}$ denotes the counting process associated to $(T_k)_{k\geq 1}$, then (36) is equivalent to the assertion

$$\lim_{n\to\infty} \sup_{F\in \mathbf{B}^1(D([0,T]),\mathcal{D}_T)} \left|\mathbf{E}\, F(N^{(n),T}) - \mathbf{E}\, F(N^T)\right| = 0 \tag{37}$$

for every $T > 0$.

Proof of theorem 2.4: Since

$$\mathcal{G}_T = \sigma(\{\tau_k \wedge T \,|\, k \geq 1\}) ,$$

a function $F : S^\infty \to \mathbb{R}$ is $\mathcal{G}_T$-measurable, if and only if F is of the form

$$F((t_k)_{k\geq 1}) = \tilde{F}((t_k \wedge T)_{k\geq 1})$$

for every $(t_k)_{k\geq 1} \in S^\infty$. Since $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ is uniformly non-explosive, for every $\varepsilon > 0$ we can find an $m = m(\varepsilon) \geq 1$ such that

$$\mathbf{P}\{T_m^{(n)} \leq T\} < \varepsilon \quad \text{for all } n \geq 1 \text{ and } \quad \mathbf{P}\{T_m \leq T\} < \varepsilon . \tag{38}$$

Thus we have for every $F \in \mathbf{B}^1(S^\infty, \mathcal{G}_T)$

$$\begin{aligned} &\left|\mathbf{E}\, F((T_k^{(n)})_{k\geq 1}) - \mathbf{E}\, F((T_k)_{k\geq 1})\right| \\ &\quad\leq\ \mathbf{E}\left|\tilde{F}((T_k^{(n)} \wedge T)_{k\geq 1}) 1_{\{T_m^{(n)} > T\}} - \tilde{F}((T_k \wedge T)_{k\geq 1}) 1_{\{T_m > T\}}\right| + 2\varepsilon , \end{aligned}$$

and for the assertion (36) it is sufficient to prove

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1(S^\infty,\mathcal{G}_T)} \mathbf{E}\Big|\tilde{F}((T_k^{(n)}\wedge T)_{k\geq 1})1_{\{T_m^{(n)}>T\}} - \tilde{F}((T_k\wedge T)_{k\geq 1})1_{\{T_m>T\}}\Big| = 0 \ . \tag{39}$$

Let us define the transformation

$$\Psi_T : \mathbf{B}^1(S^\infty, \mathcal{G}_T) \to \mathbf{B}^1(\mathbb{R}_+^m)$$

by

$$\Psi_T(F)(t_1, \cdots, t_m) := F(t_1 \wedge T, \cdots, t_m \wedge T, T, T, \cdots)1_{\{t_m>T\}} \ .$$

Then (40) means that we have to prove

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1(S^\infty,\mathcal{G}_T)} \mathbf{E}\Big|\Psi_T(F)(T_1^{(n)}, \cdots, T_m^{(n)}) - \Psi_T(F)(T_1, \cdots, T_m)\Big| = 0 \ , \tag{40}$$

but this assertion follows immediately from theorem 2.1. □

We now present a general example, which shows that the general assumptions on the sequence $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ of point processes are very natural.

We start with a single point process $(\overline{T}_n)_{n\geq 1}$ and assume that for a right continuous filtration $\overline{\mathbf{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$ the associated counting process $\overline{N} = (\overline{N}_t)_{t\geq 0}$ is integrable, finite and has the $\overline{\mathbf{F}}$-intensity $\overline{\lambda} = (\overline{\lambda}_t)_{t\geq 0}$.

Let $(N^k, \mathbf{F}^k, \lambda^k)_{k\geq 1}$ be a sequence of independent copies of $(\overline{N}, \overline{\mathbf{F}}, \overline{\lambda})$ and define for every $n \geq 1$ the counting process $N^{(n)} = (N_t^{(n)})_{t\geq 0}$ by

$$N_t^{(n)} := \sum_{k=1}^{n} N_{\frac{t}{n}}^k \ . \tag{41}$$

Corresponding to $N^{(n)}$ we define the filtration $\mathbf{F}^{(n)}$ by

$$\mathcal{F}_t^{(n)} := \bigvee_{k=1}^{n} \mathcal{F}_{\frac{t}{n}}^k \ . \tag{42}$$

Then it is easy to show that every $N^{(n)}$ has the $\mathbf{F}^{(n)}$-intensity $\lambda^{(n)}$, given by

$$\lambda_t^{(n)} = \frac{1}{n}\sum_{k=1}^{n} \lambda_{\frac{t}{n}}^k \ . \tag{43}$$

Theorem 2.5. *Suppose that* $\mathbf{E}\sup_{0\le t\le T}\overline{\lambda}_t < \infty$ *for every* $T > 0$ *and that* $\alpha := \mathbf{E}\overline{\lambda}_0 > 0$. *Let* $N = (N_t)_{t\ge 0}$ *be a Poisson process with parameter* α. *Then (with the notations introduced after theorem 2.4)*

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1(D([0,T]),\mathcal{D}_T)} \left|\mathbf{E}\,F(N^{(n),T}) - \mathbf{E}\,F(N^T)\right| = 0 \tag{44}$$

for every $T > 0$.

Proof: We prove that for the sequence $(N^{(n)})$ the assumptions of the theorems 2.1 and 2.4 are fulfilled.

Let $D([0,T])$ denote the space of all functions $f : [0,T] \to \mathbb{R}$, which are right continuous and have left hand limits. For $f \in D([0,T])$ we set $\|f\| := \sup_{0\le t\le T}|f(t)|$. Then $D([0,T])$ is a Banach space relative to this norm. If we define

$$X_k^T := (\lambda_t^k)_{0\le t\le T}\ ,$$

then $(X_k^T)_{k\ge 1}$ is an independent, identically distributed sequence of $D([0,T])$-valued random vectors. Since by assumption

$$\mathbf{E}\,\|X_1^T\| = \mathbf{E}\sup_{0\le t\le T}\overline{\lambda}_t < \infty\ ,$$

we know from the strong law of large numbers for Banach space valued random vectors (see e.g. [4]) that

$$\frac{1}{n}\sum_{k=1}^n X_k^T = \mathbf{E}\,X_1^T \quad \mathbf{P}\text{-a.s.}$$

in $D([0,T])$, which means that $\mathbf{P}$-a-s.

$$\sup_{0\le t\le T}\Big|\frac{1}{n}\sum_{k=1}^n \lambda_t^k - \mathbf{E}\overline{\lambda}_t\Big| \to 0\ .$$

Thus we have that for $\mathbf{P}$-almost every $\omega\in\Omega$ and every given $\varepsilon > 0$ there exists an $n(\varepsilon,\omega)\ge 1$ such that

$$\sup_{0\le t\le T}\Big|\frac{1}{n}\sum_{k=1}^n \lambda_t^k(\omega) - \mathbf{E}\overline{\lambda}_t\Big| < \frac{\varepsilon}{2}$$

for every $n \geq n(\varepsilon, \omega)$. Since $t \mapsto \mathbf{E}\overline{\lambda}_t$ is right continuous, there exists also some $t(\varepsilon) > 0$ such that

$$|\mathbf{E}\overline{\lambda}_t - \mathbf{E}\overline{\lambda}_0| < \frac{\varepsilon}{2}$$

for every $t \leq t(\varepsilon)$. If we assume that $n(\varepsilon, \omega)$ is large enough such that $T \leq n(\varepsilon, \omega)t(\varepsilon)$, then

$$\sup_{0 \leq t \leq T} |\lambda_t^{(n)}(\omega) - \alpha| < \varepsilon$$

for every $n \geq n(\varepsilon, \omega)$, and we have proved condition (8) of theorem 2.1.

For an application of theorem 2.3 it remains to prove that the sequence $(N^{(n)})_{n \geq 1}$ is uniformly non-explosive. We have

$$\begin{aligned}
\mathbf{P}\{N_t^{(n)} \geq m\} \ &\leq \ \frac{1}{m}\mathbf{E}\, N_t^{(n)} \\
&= \ \frac{1}{m}\sum_{k=1}^{n} \mathbf{E} N_{\frac{t}{n}}^{k} \ = \ \frac{1}{m} n \mathbf{E}\, \overline{N}_{\frac{t}{n}} \\
&= \ \frac{1}{m} n \mathbf{E} \int_0^{\frac{t}{n}} \overline{\lambda}_s \, ds \ \leq \ \frac{1}{m} t \mathbf{E} \sup_{s \leq \frac{t}{n}} \overline{\lambda}_s \\
&\leq \ \frac{1}{m} t \mathbf{E} \sup_{s \leq t} \overline{\lambda}_s \ ,
\end{aligned}$$

and hence $(N^{(n)})_{n \geq 1}$ is uniformly non-explosive. □

3 The Case of Marked Point Processes

Let $(E, \mathcal{E})$ be a measurable space and let Δ denote an artificial element outside of E. We set $E_\Delta := E \cup \{\Delta\}$ and provide E_Δ with the σ-algebra $\mathcal{E}_\Delta := \sigma(\mathcal{E} \cup \{\{\Delta\}\})$. Now suppose that $(T_n)_{n \geq 1}$ is a point process and that $(X_n)_{n \geq 1}$ is a sequence of E_Δ-valued random variables such that the following condition holds:

$$T_n = \infty \iff X_n = \Delta \ . \tag{45}$$

Then the double sequence $((T_n, X_n))_{n \geq 1}$ is called a *marked point process*. For $n = 0$ we make the convention $X_0 := \epsilon$ with a fixed $\epsilon \in E$.

In the following we will always assume that E is a polish space and that $\mathcal{E}$ is the Borel field of E.

The marked point process $((T_n, X_n))_{n\geq 1}$ can be equivalently described by the family

$$N = \big((N_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$$

of counting processes, where

$$N_t(B) = \sum_{n\geq 1} 1_{\{T_n\leq t\}} 1_{\{X_n\in B\}} \ .$$

For $B = E$ we just write $(N_t)_{t\geq 0}$ instead of $(N_t(E))_{t\geq 0}$.

N can be viewed as a random measure on $\mathbb{R}_+ \times E$ and will be called the *counting measure* of $((T_n, X_n))_{n\geq 1}$.

The rôle of the intensity is now replaced by the notion of an intensity measure: Let $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ be a right continuous filtration on $(\Omega, \mathcal{F}, \mathbf{P})$, such that N is $\mathbf{F}$-adapted in the sense that every process $(N_t(B))_{t\geq 0}$ is $\mathbf{F}$-adapted. The smallest filtration, for which N is adapted in this sense, is the canonical filtration $\mathbf{F}^N = (\mathcal{F}_t^N)_{t\geq 0}$, defined by

$$\mathcal{F}_t^N := \sigma(\{N_s(B) \,|\, s \leq t\,,\ B \in \mathcal{E}\}) \ .$$

A family

$$\lambda = \big((\lambda_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}}$$

of $\mathbf{F}$-adapted, $\mathbb{R}_+$-valued cadlag-processes $(\lambda_t(B))_{t\geq 0}$ is called an *intensity measure*, if for every $t \geq 0$ the map

$$\mathcal{E} \ni B \mapsto \lambda_t(B)$$

is a finite measure on E, and if for every $t \geq 0$

$$\int_0^t \lambda_s(E)\, ds < \infty \quad \mathbf{P}\text{-a.s.}\ .$$

Again we just write $(\lambda_t)_{t\geq 0}$ instead of $(\lambda_t(E))_{t\geq 0}$.

Then we say that N (or $((T_n, X_n))_{n\geq 1}$) has the $\mathbf{F}$-intensity measure λ, if for every $B \in \mathcal{E}$ and every $m \geq 1$ the process

$$\Big(N_{t\wedge T_m}(B) - \int_0^{t\wedge T_m} \lambda_s(B)\, ds\Big)_{t\geq 0} \tag{46}$$

is a $\mathbf{F}$-martingale. If N is non-explosive (i.e. if $(T_n)_{n\geq 1}$ is non-explosive), then (46) just means that every process

$$\big(N_t(B) - \int_0^t \lambda_s(B)\,ds\big)_{t\geq 0} \tag{47}$$

is a local $\mathbf{F}$-martingale (with $(T_n)_{n\geq 1}$ as localizing sequence), and if $(N_t)_{t\geq 0}$ or $(\int_0^t \lambda_s\,ds)_{t\geq 0}$ are integrable processes, then the processes in (47) are all $\mathbf{F}$-martingales.

The canonical space for the concrete construction of marked point processes is here the space

$$S^\infty(E) \;:=\; \Big\{ ((t_k, x_k))_{k\geq 0} \in (\overline{\mathbb{R}}_+ \times E_\Delta)^{\mathbb{Z}_+} \;\Big|\; (i)\;\; (t_k)_{k\geq 0} \in S^\infty\;,\; x_0 = \epsilon\,, \\ (ii)\;\; t_k = \infty \;\Leftrightarrow\; x_k = \triangle \Big\}\;.$$

Again we write $((t_k, x_k))_{k\leq n}$ for $((t_k, x_k))_{k\geq 0} \in S^\infty(E)$, if $t_{n+1} = \infty$.

Now we define $\tau_n : S^\infty(E) \to \overline{\mathbb{R}}_+$ and $\xi_n : S^\infty(E) \to \overline{E}_\Delta$ for $n \geq 0$ by

$$\tau_n(((t_k, x_k))_{k\geq 0}) = t_n \quad \text{and} \quad \xi_n(((t_k, x_k))_{k\geq 0}) = x_n\;.$$

Then we set for every $B \in \mathcal{E}$

$$\nu_t(B) := \sum_{n\geq 1} 1_{\{\tau_n\leq t\}} 1_{\{\xi_n \in B\}}\;,$$

and define on $S^\infty(E)$ the filtration $\mathbf{H} = (\mathcal{H}_t)_{t\geq 0}$ by

$$\mathcal{H}_t := \sigma(\{\nu_s(B)\,|\, s\leq t\,,\; B\in\mathcal{E}\})\;.$$

Now let $\Psi : \Omega \to S^\infty(E)$ be defined by

$$\Psi(\omega) := ((T_n(\omega), X_n(\omega)))_{n\geq 0}\;.$$

It follows that

$$\Psi^{-1}(\mathcal{H}_t) = \mathcal{F}_t^N \tag{48}$$

for every $t \geq 0$, and

$$\Psi(\mathbf{P}) = \mathbf{P}_{((T_n,X_n))_{n\geq 1}}\;.$$

From now on we make the general assumption that every marked point process $((T_n, X_n))_{n\geq 1}$ considered in this paper is integrable and finite in the sense that $(N_t)_{t\geq 0}$ is integrable and finite. This is equivalent to the properties that $(\lambda_t)_{t\geq 0}$ is integrable and that $\int_0^\infty \lambda_s \, ds = \infty$.

Similarly to the simple point process case, the given intensity measure λ determines an associated predictable intensity measure: For every fixed $B \in \mathcal{E}$ the measure $\lambda_t(B)\, dt \otimes d\mathbf{P}$ is absolutely continuous relative to $dt \otimes d\mathbf{P}$ on $(\mathbb{R}_+ \times \Omega, \mathcal{P}(\mathbf{F}^N))$, and hence there exists a $\mathcal{P}(\mathbf{F}^N)$-measurable Radon-Nikodym density $\pi(\lambda)(B) = (\pi(\lambda)_t(B))_{t\geq 0}$. By the same arguments as for the proof of the existence of conditional distributions one can see that one can choose a version of $\pi(\lambda)(B)$, such that again

$$B \mapsto \pi(\lambda)_t(B)$$

is a finite measure on $(E, \mathcal{E})$. It follows from (48) that there is also an $\mathbf{H}$-intensity measure

$$\overline{\lambda} = \big((\overline{\lambda}_t(B))_{t\geq 0}\big)_{B\in\mathcal{E}} \tag{49}$$

such that

$$\pi(\lambda)_t(B) = \overline{\lambda}_t(B) \circ \Psi \tag{50}$$

for every $t \geq 0$ and every $B \in \mathcal{E}$. Again we will call $\pi(\lambda)$ the *predictable projection* of λ on Ω and $\overline{\lambda}$ the *predictable projection* of λ on $S^\infty(E)$.

The predictable projection $\overline{\lambda}$ determines completely the distribution of $((T_n, X_n))_{n\geq 1}$: If $F : (\mathbb{R}_+ \times E)^m \to \mathbb{R}$ is bounded and measurable, then

$$\begin{aligned}
&\mathbf{E}\, F(((T_k, X_k))_{k\leq m}) \\
&= \underbrace{\int_0^\infty \int_E \cdots \int_{t_{m-1}}^\infty \int_E}_{2m \text{ integrals}} F(((t_k, x_k))_{k\leq m}) \\
&\qquad \overline{\lambda}_{t_m}(dx_m; ((t_k, x_k))_{k\leq m-1}) e^{-\int_{t_{m-1}}^{t_m} \overline{\lambda}_r(((t_k,x_k))_{k\leq m-1})\, dr}\, dt_m \\
&\qquad \cdots \overline{\lambda}_{t_1}(dx_1) e^{-\int_0^{t_1} \overline{\lambda}_r \, dr}\, dt_1
\end{aligned} \tag{51}$$

(where in the last line $\overline{\lambda}_t(dx_1) = \overline{\lambda}_t(dx_1, ((t_k, x_k))_{k\leq 0})$).

The stochastic kernel

$$\overline{\lambda}_t(dx; ((t_i, x_i))_{i\leq k-1}) e^{-\int_{t_{k-1}}^t \overline{\lambda}_r(((t_i,x_i))_{i\leq k-1})\, dr}\, dt$$

has the probabilistic interpretation that

$$\begin{aligned}
&\overline{\lambda}_t(dx;((T_i,X_i))_{i\le k-1})e^{-\int_{t_{k-1}}^{t}\overline{\lambda}_r(((T_i,X_i))_{i\le k-1})\,dr}\,dt \\
&\quad = \ \mathbf{P}\big\{(T_k,X_k)\in dt\times dx\,\big|\,((T_i,X_i))_{i\le k-1}\big\}\ . \qquad (52)
\end{aligned}$$

It is sometimes convenient to write formula (51) in a little bit different form. We define

$$\overline{\gamma}_t(dx;((t_k,x_k))_{k\le m}) \ = \ \begin{cases} \frac{\overline{\lambda}_t(dx;((t_k,x_k))_{k\le m})}{\overline{\lambda}_t(((t_k,x_k))_{k\le m})} & \text{if} \ \ \overline{\lambda}_t(((t_k,x_k))_{k\le m})>0 \\ \delta_\epsilon & \text{if} \ \ \lambda_t(((t_k,x_k))_{k\le m})=0 \end{cases} .$$

Then $\overline{\gamma}_t(dx;((t_k,x_k))_{k\le m})$ is a stochastic kernel,

$$\overline{\lambda}_t(dx;((t_k,x_k))_{k\le m})=\overline{\gamma}_t(dx;((t_k,x_k))_{k\le m})\cdot\overline{\lambda}_t(((t_k,x_k))_{k\le m}) \qquad (53)$$

for every $t\ge 0$, and formula (51) is now of the form

$$\begin{aligned}
&\mathbf{E}\,F(((T_k,X_k))_{k\le m}) \qquad (54)\\
&= \ \underbrace{\int_0^\infty\cdots\int_{t_{m-1}}^\infty}_{m-times}\underbrace{\int_E\cdots\int_E}_{m-times}\Big\{F(((t_k,x_k))_{k\le m}) \\
&\qquad\qquad \prod_{k=1}^{m}\overline{\lambda}_{t_k}(((t_i,x_i))_{i\le k-1})e^{-\int_{t_{k-1}}^{t_k}\overline{\lambda}_r(((t_i,x_i))_{i\le k-1})\,dr}\Big\} \\
&\qquad\qquad \overline{\gamma}_{t_m}(dx_m;((t_k,x_k))_{k\le m-1})\cdots\overline{\gamma}_{t_1}(dx_1;((t_k,x_k))_{k\le 0})\,dt_m\cdots dt_1\ .
\end{aligned}$$

The kernel $\overline{\gamma}$ has the probabilistic interpretation:

$$\overline{\gamma}_{T_m}(dx;((T_k,X_k))_{k\le m-1})=\mathbf{P}\{X_m\in dx\,|\,T_m\ ,\ ((T_k,X_k))_{k\le m-1}\}\ .$$

For the approximation results we need the following data: We suppose that we are given a sequence

$$\big(((T_k^{(n)},X_k^{(n)}))_{k\ge 1},\mathbf{F}^{(n)},\lambda^{(n)}\big)_{n\ge 1}\ ,$$

where
(i) every $((T_k^{(n)},X_k^{(n)})_{k\ge 1}$ is an integrable and finite marked point process with associated family $N^{(n)}=((N_t^{(n)}(B))_{t\ge 0})_{B\in\mathcal{E}}$ of counting measures,
(ii) every $\mathbf{F}^{(n)}=(\mathcal{F}_t^{(n)})_{t\ge 0}$ is a right continuous filtration, and
(iii) every $N^{(n)}$ has the $\mathbf{F}^{(n)}$-intensity measure $\lambda^{(n)}=((\lambda_t^{(n)}(B))_{t\ge 0})_{B\in\mathcal{E}}$.

Furthermore, we suppose that we are given a deterministic kernel $\lambda = ((\lambda_t(B))_{t\geq 0})_{B\in\mathcal{E}}$ such that

$$\inf_{t\geq 0} \lambda_t \geq c \tag{55}$$

for some positive constant c. Now let $((T_k, X_k))_{k\geq 1}$ be a marked point process, whose counting measure $N = ((N_t(B))_{t\geq 0})_{B\in\mathcal{E}}$ has the intensity measure λ. We call such a marked point process an *inhomogeneous marked Poisson process*. W.l.o.g. we assume that $((T_k, X_k))_{k\geq 1}$ is independent of $\bigvee_{n\geq 1} \mathcal{F}_\infty^{(n)}$. Again we have that

$$\lambda_t(dx) = \gamma_t(dx)\lambda_t \ ,$$

where $\gamma_t(dx)$ is just a probability measure on E.

Theorem 3.1. *Assume that the sequence $(\lambda^{(n)})_{n\geq 1}$ has the property*

$$\sup_{t\leq T} \sup_{B\in\mathcal{E}} |\lambda_t^{(n)}(B) - \lambda_t(B)| \to 0 \quad \textit{in probability} \tag{56}$$

for every $T > 0$. Then for every $m \geq 1$

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1((\mathbb{R}_+\times E)^m)} \left|\mathbf{E}F\big(((T_k^{(n)}, X_k^{(n)})_{k\leq m}\big) - \mathbf{E}F\big(((T_k, X_k)_{k\leq m}\big)\right| = 0. \tag{57}$$

Remark 3.2. Condition (56) is mainly a condition on the convergence behaviour of certain conditional distributions of the $X_k^{(n)}$. Suppose that $\tilde{\lambda}^{(n)}$ denotes the predictable projection of $\lambda^{(n)}$ on $S^\infty(E)$, and that according to (53)

$$\tilde{\lambda}_t^{(n)}(dx; ((t_k, x_k))_{k\leq m-1}) = \tilde{\gamma}_t^{(n)}(dx; ((t_k, x_k))_{k\leq m-1}) \cdot \tilde{\lambda}_t^{(n)}(((t_k, x_k))_{k\leq m-1}) \ ,$$

with the interpretation

$$\tilde{\gamma}_{T_m}^{(n)}(dx; ((T_k^{(n)}, X_k^{(n)}))_{k\leq m-1}) = \mathbf{P}\big\{X_m^{(n)} \in dx \,\big|\, T_m^{(n)}\ ,\ ((T_k^{(n)}, X_k^{(n)}))_{k\leq m-1}\big\} \ .$$

If also λ is given in the form

$$\lambda_t(B) = \gamma_t(B)\lambda_t \ ,$$

then (56) implies that

$$\sup_{t\leq T} \sup_{B\in\mathcal{E}} |\tilde{\gamma}_t^{(n)}(B) - \gamma_t(B)| \to 0 \quad \text{in probability}$$

for every $T > 0$. This will follow from a remark at the end of step (1) of the proof of theorem 3.1 below.

For the proof of theorem 3.1 we will need the following lemma, which is well known, but which we prove for the lack of an exact reference.

Lemma 3.3. *Let $(X, \mathcal{X})$ be a measurable space and suppose that μ and ν are two finite measures on X. Then*

$$\sup_{F \in \mathbf{B}^1(X)} \Big| \int F\, d\mu - \int F\, d\nu \Big| \leq 2 \sup_{A \in \mathcal{X}} |\mu(A) - \nu(A)| \; .$$

Proof: (cf. [5], sec. 29) We set $\lambda = \mu - \nu$. Then λ is a finite signed measure on X, and there exists a decomposition of X in two sets $B_+, B_- \in \mathcal{X}$ such that for every $A \in \mathcal{X}$

$$\lambda(A \cap B_+) \geq 0 \quad \text{and} \quad \lambda(A \cap B_-) \leq 0$$

(this is the so called Hahn decomposition of X rel. to λ). The non-negative finite measures λ^+ and λ^-, defined by

$$\lambda^+(A) = \lambda(A \cap B_+) \quad \text{and} \quad \lambda^-(A) = -\lambda(A \cap B_-)$$

for all $A \in \mathcal{X}$, are called the upper and the lower variation of λ resp.. One has $\lambda = \lambda^+ - \lambda^-$ (this is the so called Jordan decomposition of λ), and $|\lambda| := \lambda^+ + \lambda^-$ is called the total variation of λ. The Hahn decomposition shows that

$$\max(\lambda^+(X), \lambda^-(X)) \leq \sup_{A \in \mathcal{X}} |\lambda(A)| \; .$$

On the other side

$$\sup_{A \in \mathcal{X}} |\lambda(A)| \leq \sup_{A \in \mathcal{X}} \max(\lambda^+(A), \lambda^-(A)) \leq \max(\lambda^+(X), \lambda^-(X)) \; .$$

Thus we have

$$\sup_{A \in \mathcal{X}} |\lambda(A)| = \max(\lambda^+(X), \lambda^-(X)) \; .$$

Since

$$|\lambda|(X) = \lambda^+(X) + \lambda^-(X) \leq 2 \max(\lambda^+(X), \lambda^-(X)) \; ,$$

we get for $F \in \mathbf{B}^1(X)$

$$\begin{aligned} \Big| \int F\, \lambda(dx) \Big| \leq |\lambda|(X) \;\; &\leq \;\; 2 \max(\lambda^+(X), \lambda^-(X)) \\ &= \;\; 2 \sup_{A \in \mathcal{X}} |\lambda(A) \; , \end{aligned}$$

and the lemma is proved. □

the next lemma generalizes lemma 2.2 to the case of marked point processes.

Lemma 3.4. *Suppose that* $((T_k^{(1)}, X_k^{(1)}))_{k\geq 1}$ *and* $((T_k^{(2)}, X_k^{(2)}))_{k\geq 1}$ *are two marked point processes with associated counting measures* $N^{(1)}$ *and* $N^{(2)}$. *We assume that*
(i) $\mathbf{F}^{(1)}$ *is a right continuous filtration, for which* $N^{(1)}$ *is adapted and has the* $\mathbf{F}^{(1)}$*-intensity measure* $\lambda^{(1)}$, *that*
(ii) $N^{(2)}$ *has the intensity measure* $\lambda^{(2)}$ *relative to the canonical filtration* $\mathbf{F}^{N^{(2)}}$ *of* $N^{(2)}$, *and that*
(iii) $\mathcal{F}_\infty^{(1)}$ *and* $N^{(2)}$ *are independent.*
Let S *be a given* $\mathbf{F}^{(1)}$*-stopping time and define for every* $t \geq 0$ *and every* $B \in \mathcal{E}$

$$\begin{aligned} \tilde{N}_t(B) &:= N_{t\wedge S}^{(1)}(B) + (N_{t\vee S}^{(2)}(B) - N_S^{(2)}(B)) \,, \qquad (58)\\ \tilde{\mathcal{F}}_t &:= \mathcal{F}_{t\wedge S}^{(1)} \vee \sigma\big(\{N_{s\vee S}^{(2)}(B) - N_S^{(2)}(B) \,|\, s \leq t \,,\, B \in \mathcal{E}\}\big) \,, \ \text{and} \\ \tilde{\lambda}_t(B) &:= \lambda_t^{(1)}(B) 1_{\{t<S\}} + \lambda_t^{(2)}(B) 1_{\{S\leq t\}} \,. \end{aligned}$$

Then $\tilde{N} = ((\tilde{N}_t(B))_{t\geq 0})_{B\in\mathcal{E}}$ *is a counting measure with the intensity measure* $\tilde{\lambda} = ((\tilde{\lambda}_t(B))_{t\geq 0})_{B\in\mathcal{E}}$ *relative to the filtration* $\tilde{\mathbf{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$. *The corresponding marked point process* $((\tilde{T}_k, \tilde{X}_k))_{k\geq 1}$ *is given by*

$$\begin{aligned} (\tilde{T}_k, \tilde{X}_k) &= (T_k^{(1)}, X_k^{(1)}) 1_{\{T_k^{(1)} \leq S\}} \qquad (59)\\ &\quad + 1_{\{S<T_k^{(1)}\}} \sum_{j=1}^{k} (T_{k-(j-1)}^{(2)}, X_{k-(j-1)}^{(2)}) 1_{\{T_{j-1}^{(1)} \leq S < T_j^{(1)}\}} \,. \end{aligned}$$

We omit the proof, which would be very similar to the proof of lemma 2.2. The marked point process $((\tilde{T}_k, \tilde{X}_k))_{k\geq 1}$ will be called the *composition of* $((T_k^{(1)}, X_k^{(1)}))_{k\geq 1}$ *and* $((T_k^{(2)}, X_k^{(2)}))_{k\geq 1}$ *at* S. Similarly, the intensity measure $\tilde{\lambda}$ is called the *composition of* $\lambda^{(1)}$ *and* $\lambda^{(2)}$ *at* S.

Proof of theorem 3.1:
(1) For every $n \geq 1$ and every $\varepsilon > 0$ we define

$$S^{n,\varepsilon} := \inf\{t > 0 \,:\, \sup_{B\in\mathcal{E}} |\lambda_t^{(n)}(B) - \lambda_t(B)| > \varepsilon\} \,. \qquad (60)$$

We claim that $S^{n,\varepsilon}$ is an $\mathbf{F}^{(n)}$-stopping time: Since E is a polish space, there exists (cf. [1]) an at most countable algebra $\mathcal{G} \subseteq \mathcal{E}$ such that $\mathcal{E} = \sigma(\mathcal{G})$. Thus suppose that $\mathcal{G} = \{B_j \,|\, j \geq 1\}$. Then

$$\sup_{B\in\mathcal{E}} |\lambda_t^{(n)}(B) - \lambda_t(B)| = \sup_{j\geq 1} |\lambda_t^{(n)}(B_j) - \lambda_t(B_j)| \ ,$$

and therefore

$$\{S^{n,\varepsilon} < t\} = \bigcup_{\substack{s<t \\ s\in\mathbb{Q}_+}} \bigcup_{j\geq 1} \big\{|\lambda_s^{(n)}(B_j) - \lambda_s(B_j)| > \varepsilon\big\} \ . \tag{61}$$

Since $\mathbf{F}^{(n)}$ is right continuous, (61) shows that $S^{n,\varepsilon}$ is a stopping time.

The assumption (56) implies that for every $\varepsilon > 0$, $\delta > 0$ and $T > 0$ there exists an $n(\varepsilon, \delta, T) \geq 1$ such that

$$\mathbf{P}\{S^{n,\varepsilon} \leq T\} < \delta \quad \text{for all} \;\; n \geq n(\varepsilon, \delta, T) \ . \tag{62}$$

Thus we may argue as in the proof of theorem 2.1: If $(\varepsilon_j)_{j\geq 1}$ is a fixed null-sequence of positive numbers, there exists a strictly increasing sequence $(n_j)_{j\geq 1}$ in $\mathbb{N}$, such that

$$\mathbf{P}\{S^{n,\varepsilon_j} \leq j\} < \varepsilon_j \quad \text{for every} \;\; n \geq n_j \ , \tag{63}$$

and we set

$$S^n := S^{n,\varepsilon_j} \quad \text{for} \;\; n_j \leq n < n_{j+1} \tag{64}$$

Then

$$\mathbf{P}\{S^n \leq j\} < \varepsilon_j \quad \text{for} \;\; n_j \leq n < n_{j+1} \ . \tag{65}$$

Let $((\tilde{T}_k^n, \tilde{X}_k^n))_{k\geq 1}$ $(n \geq 1)$ denote the composition of $((T_k^{(n)}, X_k^{(n)}))_{k\geq 1}$ and $((T_k, X_k))_{k\geq 1}$ at S^n, and let $\tilde{\lambda}^n$ denote the corresponding composition of the intensity measures $\lambda^{(n)}$ and λ at S^n. Then the sequence $(\tilde{\lambda}^n)_{n\geq 1}$ has the property that

$$\sup_{t\geq 0} \sup_{B\in\mathcal{E}} |\tilde{\lambda}_t^n(B) - \lambda_t(B)| \leq \varepsilon_j \quad \text{for} \;\; n_j \leq n < n_{j+1} \ . \tag{66}$$

By the same argument as in the proof of theorem 2.1 this property carries over to the predictable projections $\overline{\lambda}^n$ of $\tilde{\lambda}^n$ on $S^\infty(E)$, i.e. we have also

$$\sup_{t\geq 0} \sup_{B\in\mathcal{E}} |\overline{\lambda}_t^n(B) - \lambda_t(B)| \leq \varepsilon_j \quad \text{for} \;\; n_j \leq n < n_{j+1} \ . \tag{67}$$

Remark: Assume (cf. 53) that $\overline{\lambda}^n$ is given in the form

$$\overline{\lambda}_t^n(dx;((t_k,x_k))_{k\le m}) = \overline{\gamma}_t^n(dx;((t_k,x_k))_{k\le m}) \cdot \overline{\lambda}_t^n(((t_k,x_k))_{k\le m}) \tag{68}$$

for every $m \ge 0$ and $((t_k,x_k))_{k\le m} \in S^\infty(E)$. For $\overline{\lambda}_t^n(((t_k,x_k))_{k\le m}) > 0$ and $B \in \mathcal{E}$ we know that

$$\overline{\gamma}_t^n(B;((t_k,x_k))_{k\le m}) = \frac{\overline{\lambda}_t^n(B;((t_k,x_k))_{k\le m})}{\overline{\lambda}_t^n(((t_k,x_k))_{k\le m})} .$$

Thus we have

$$\begin{aligned}&\left|\overline{\gamma}_t^n(B;((t_k,x_k))_{k\le m}) - \gamma_t(B)\right| \\ &\quad\le \frac{\left|\overline{\lambda}_t^n(B;((t_k,x_k))_{k\le m}) - \lambda_t(B)\right|}{\overline{\lambda}_t^n(((t_k,x_k))_{k\le m})} + \frac{\left|\overline{\lambda}_t^n(((t_k,x_k))_{k\le m}) - \lambda_t\right|}{\overline{\lambda}_t^n(((t_k,x_k))_{k\le m})} .\end{aligned}$$

For large enough $j \ge 1$ we have $\varepsilon_j \le \frac{c}{2}$, which implies

$$\overline{\lambda}_t^n(((t_k,x_k))_{k\le m}) \ge \lambda_t - \varepsilon_j \ge \lambda_t - \frac{c}{2} \ge \frac{c}{2}$$

for $n \ge n_j$ and we have proved that

$$\sup_{B\in\mathcal{E}} \left|\overline{\gamma}_t^n(B;((t_k,x_k))_{k\le m}) - \gamma_t(B)\right| \le \frac{4}{c}\varepsilon_j \tag{69}$$

for all $m \ge 0$, $((t_k,x_k))_{k\le m}$ and all $n \ge n_j$, if j is large enough. This limit relation can now be used to prove the remark 3.2.

(2) Now we start with proof of (57). As in part (2) of the proof of theorem 2.1 it follows that the sequences $((T_k^{(n)})_{k\ge1})_{n\ge1}$ and $((\tilde{T}_k^n)_{k\ge1})_{n\ge1}$ are uniformly finite. Thus for a fixed given $\varepsilon > 0$ there exists a $t = t(\varepsilon) > 0$ such that

$$\mathbf{P}\{T_m^{(n)} > t\} \le \varepsilon \quad \text{and} \quad \mathbf{P}\{\tilde{T}_m^n > t\} \le \varepsilon$$

for every $n \ge 1$, and by assumption (55) on the deterministic intensity measure λ we may at the same time assume that also

$$\mathbf{P}\{T_m > t\} \le \varepsilon .$$

As a consequence, for the proof of (57) it is sufficient to prove

$$\begin{aligned}\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1((\mathbb{R}_+\times E)^m)} &\Big|\mathbf{E}F\big(((T_k^{(n)},X_k^{(n)})_{k\le m}\big)1_{\{T_m^{(n)}\le t\}} \\ &\qquad - \mathbf{E}F\big(((T_k,X_k)_{k\le m}\big)1_{\{T_m\le t\}}\Big| = 0 ,\end{aligned} \tag{70}$$

and again by the same arguments as in the proof of theorem 2.1 this follows if we prove that

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1((\mathbb{R}_+\times E)^m)} \Big|\mathbf{E}F\big(((\tilde{T}^n_k,\tilde{X}^n_k)_{k\le m}\big)1_{\{\tilde{T}^n_m\le t\}} - \mathbf{E}F\big(((T_k,X_k)_{k\le m}\big)1_{\{T_m\le t\}}\Big| = 0\ . \tag{71}$$

We have (cf. 51)

$$\begin{aligned}
&\mathbf{E}\,F(((\tilde{T}^n_k,\tilde{X}^n_k))_{k\le m})1_{\{\tilde{T}^n_m\le t\}} \\
&= \underbrace{\int_0^\infty\int_E\cdots\int_{t_{m-1}}^\infty\int_E}_{2m\ integrals} F(((t_k,x_k))_{k\le m})1_{[0,t]}(t_m) \\
&\qquad \overline{\lambda}^n_{t_m}(dx_m;((t_k,x_k))_{k\le m-1})e^{-\int_{t_{m-1}}^{t_m}\overline{\lambda}^n_r(((t_k,x_k))_{k\le m-1})\,dr}\,dt_m \\
&\qquad \cdots\overline{\lambda}^n_{t_1}(dx_1)e^{-\int_0^{t_1}\overline{\lambda}^n_r\,dr}\,dt_1\ .
\end{aligned}\tag{72}$$

If we set

$$\begin{aligned}
&\psi^n_{m,t}(F)\big(((t_k,x_k))_{k\le m-1}\big) \\
&= \int_{t_{m-1}}^\infty\int_E F(((t_k,x_k))_{k\le m})1_{[0,t]}(t_m) \\
&\qquad \overline{\lambda}^n_{t_m}(dx_m;((t_k,x_k))_{k\le m-1})e^{-\int_{t_{m-1}}^{t_m}\overline{\lambda}^n_r(((t_i,x_i))_{i\le m-1})\,dr}\,dt_m\ ,
\end{aligned}$$

then

$$\psi^n_{m,t}\big(\mathbf{B}^1((\mathbb{R}_+\times E)^m)\big)\subseteq\mathbf{B}^1((\mathbb{R}_+\times E)^{m-1})\ , \tag{73}$$

and

$$\mathbf{E}\,F\big(((\tilde{T}^n_k,\tilde{X}^n_k)_{k\le m}\big)1_{\{\tilde{T}^n_m\le t\}} = \mathbf{E}\,\psi^n_{m,t}(F)\big(((\tilde{T}^n_k,\tilde{X}^n_k))_{k\le m-1}\big)1_{\{\tilde{T}^n_{m-1}\le t\}}\ . \tag{74}$$

Similarly, for the transformation $\psi_{m,t}$, defined by

$$\begin{aligned}
&\psi_{m,t}(F)\big(((t_k,x_k))_{k\le m-1}\big) \\
&= \int_{t_{m-1}}^\infty\int_E F(((t_k,x_k))_{k\le m})1_{[0,t]}(t_m) \\
&\qquad \lambda_{t_m}(dx_m)e^{-\int_{t_{m-1}}^{t_m}\lambda_r\,dr}\,dt_m\ ,
\end{aligned}$$

we have

$$\mathbf{E}\,F\big(((T_k,X_k)_{k\le m}\big)1_{\{T_m\le t\}} = \mathbf{E}\,\psi_{m,t}(F)\big(((T_k,X_k))_{k\le m-1}\big)1_{\{T_{m-1}\le t\}}\ . \tag{75}$$

From (74) and (75) we obtain now for (71)

$$
\begin{aligned}
&\Big|\mathbf{E}F\big(((\tilde{T}_k^n,\tilde{X}_k^n))_{k\le m}\big)1_{\{\tilde{T}_m^n\le t\}} - \mathbf{E}F\big(((T_k,X_k))_{k\le m}\big)1_{\{T_m\le t\}}\Big| \\
\le\ & \Big|\mathbf{E}\,\psi_{m,t}^n(F)\big(((\tilde{T}_k^n,\tilde{X}_k^n))_{k\le m-1}\big)1_{\{\tilde{T}_{m-1}^n\le t\}} \qquad (76)\\
&\qquad - \mathbf{E}\,\psi_{m,t}^n(F)\big(((T_k,X_k))_{k\le m-1}\big)1_{\{T_{m-1}\le t\}}\Big| \\
&+\Big|\mathbf{E}\,\psi_{m,t}^n(F)\big(((T_k,X_k))_{k\le m-1}\big)1_{\{T_{m-1}\le t\}} \qquad (77)\\
&\qquad - \mathbf{E}\,\psi_{m,t}(F)\big(((T_k,X_k))_{k\le m-1}\big)1_{\{T_{m-1}\le t\}}\Big|\,.
\end{aligned}
$$

From (74) we get for the term(76)

$$
\begin{aligned}
&\sup_{F\in\mathbf{B}^1((\mathbb{R}_+\times E)^m)}\Big|\mathbf{E}\,\psi_{m,t}^n(F)\big(((\tilde{T}_k^n,\tilde{X}_k^n))_{k\le m-1}\big)1_{\{\tilde{T}_{m-1}^n\le t\}} \\
&\qquad - \mathbf{E}\,\psi_{m,t}^n(F)\big(((T_k,X_k))_{k\le m-1}\big)1_{\{T_{m-1}\le t\}}\Big| \\
\le\ & \sup_{F\in\mathbf{B}^1((\mathbb{R}_+\times E)^{m-1})}\Big|\mathbf{E}\,F\big(((\tilde{T}_k^n,\tilde{X}_k^n))_{k\le m-1}\big)1_{\{\tilde{T}_{m-1}^n\le t\}} \qquad (78)\\
&\qquad - \mathbf{E}\,F\big(((T_k,X_k))_{k\le m-1}\big)1_{\{T_{m-1}\le t\}}\Big|\,.
\end{aligned}
$$

Now consider the term (77). We have

$$
\begin{aligned}
&\Big|\psi_{m,t}^n(F)\big(((t_k,x_k))_{k\le m-1}\big) - \psi_{m,t}(F)\big(((t_k,x_k))_{k\le m-1}\big)\Big| \qquad (79)\\
\le\ & \int_{t_{m-1}}^{\infty}\Big|\int_E F(((t_k,x_k))_{k\le m})\overline{\lambda}_{t_m}^n(dx_m;((t_k,x_k))_{k\le m-1}) \\
&\qquad - \int_E F(((t_k,x_k))_{k\le m})\lambda_{t_m}(dx_m)\Big|1_{[0,t]}(t_m)\,dt_m \\
&+\int_{t_{m-1}}^{\infty}\Big|e^{-\int_{t_{m-1}}^{t_m}\overline{\lambda}_r^n(((t_i,x_i))_{i\le m-1})\,dr} - e^{-\int_{t_{m-1}}^{t_m}\lambda_r\,dr}\Big|\lambda_{t_m}1_{[0,t]}(t_m)\,dt_m\,.
\end{aligned}
$$

For the first term on the right hand side of (79) we obtain from lemma 3.3

$$
\begin{aligned}
&\int_{t_{m-1}}^{\infty}\Big|\int_E F(((t_k,x_k))_{k\le m})\overline{\lambda}_{t_m}^n(dx_m;((t_k,x_k))_{k\le m-1}) \\
&\qquad - \int_E F(((t_k,x_k))_{k\le m})\lambda_{t_m}(dx_m)\Big|1_{[0,t]}(t_m)\,dt_m \\
\le\ & 2t\varepsilon_j \qquad (80)
\end{aligned}
$$

for $n \geq n_j$, and for the second term on the right hand side of (79) we get

$$\begin{aligned}
&\int_{t_{m-1}}^{\infty} \left| e^{-\int_{t_{m-1}}^{t_m} \overline{\lambda}_r^n(((t_i,x_i))_{i\leq m-1})\,dr} - e^{-\int_{t_{m-1}}^{t_m} \lambda_r\,dr} \right| \lambda_{t_m} 1_{[0,t]}(t_m)\,dt_m \\
&\leq \int_{t_{m-1}}^{\infty} 1_{[0,t]}(t_m)\lambda_{t_m} e^{-\int_{t_{m-1}}^{t_m} \lambda_r\,dr} \\
&\qquad\qquad \left| e^{-\int_{t_{m-1}}^{t_m} \left(\overline{\lambda}_r^n(((t_i,x_i))_{i\leq m-1})-\lambda_r\right)\,dr} - 1 \right| dt_m \\
&\leq t\varepsilon_j \int_{t_{m-1}}^{\infty} 1_{[0,t]}(t_m)\lambda_{t_m} e^{-\int_{t_{m-1}}^{t_m} \lambda_r\,dr}\,dt_m
\end{aligned} \tag{81}$$

With the aid of the inequalities (80) and (81) it is now easy to see that we get for the term (77)

$$\begin{aligned}
&\Big| \mathbf{E}\,\psi_{m,t}^n(F)\big(((T_k,X_k))_{k\leq m-1}\big)1_{\{T_{m-1}\leq t\}} \\
&\qquad\qquad - \mathbf{E}\,\psi_{m,t}(F)\big(((T_k,X_k))_{k\leq m-1}\big)1_{\{T_{m-1}\leq t\}} \Big| \\
&\leq 3t\varepsilon_j\,.
\end{aligned}$$

Together with (78) it follows finally that

$$\begin{aligned}
\sup_{F\in\mathbf{B}^1((\mathbb{R}_+\times E)^m)} &\Big| \mathbf{E}F\big(((\tilde{T}_k^n,\tilde{X}_k^n))_{k\leq m}\big)1_{\{\tilde{T}_m^n\leq t\}} \\
&\qquad - \mathbf{E}F\big(((T_k,X_k))_{k\leq m}\big)1_{\{T_m\leq t\}} \Big| \leq 3mt\varepsilon_j\,,
\end{aligned}$$

and the theorem is proved. □

As in the simple point process case, theorem 3.1 can be generalized in case that the sequence $\big(((T_k^{(n)},X_k^{(n)}))_{k\geq 1}\big)_{n\geq 1}$ is uniformly non-explosive.

Theorem 3.5. *Suppose that $\big(((T_k^{(n)},X_k^{(n)}))_{k\geq 1}\big)_{n\geq 1}$ is uniformly non-explosive. Then under the assumptions of theorem 3.1*

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1((S^\infty(E),\mathcal{H}_t))} \Big| \mathbf{E}\,F\big(((T_k^{(n)},X_k^{(n)}))_{k\geq 1}\big) - \mathbf{E}\,F\big(((T_k,X_k))_{k\geq 1}\big) \Big| = 0 \tag{82}$$

for every $t > 0$.

Proof: We set for $t \geq 0$ and every $k \geq 1$

$$\xi_k^t := \xi_k 1_{\{\tau_k\leq t\}} + \epsilon 1_{\{\tau_k>t\}}\,.$$

Then it is not difficult to prove that

$$\mathcal{H}_t = \sigma(\{(\tau_k \wedge t)_{k\geq 1}\,,\ (\xi_k^t)_{k\geq 1}\})\ .$$

Thus a function $F : S^\infty(E) \to \mathbb{R}$ is $\mathcal{H}_t$-measurable, if and only if F is of the form

$$F\big(((t_k, x_k))_{k\geq 1}\big) = \tilde{F}\big(((t_k \wedge t, \xi_k^t))_{k\geq 1}\big)$$

for every $((t_k, x_k))_{k\geq 1} \in S^\infty(E)$. Since our sequence of marked point processes is uniformly non-explosive, for every given $\varepsilon > 0$ we can find an $m = m(\varepsilon) \geq 1$ such that for every $n \geq 1$

$$\mathbf{P}\{T_m^{(n)} \leq t\} < \varepsilon \quad \text{and} \quad \mathbf{P}\{T_m \leq t\} < \varepsilon\ .$$

Thus for every $F \in \mathbf{B}^1((S^\infty(E), \mathcal{H}_t))$

$$\begin{aligned}
&\big|\mathbf{E}\, F\big(((T_k^{(n)}, X_k^{(n)}))_{k\geq 1}\big) - \mathbf{E}\, F\big(((T_k, X_k))_{k\geq 1}\big)\big| \\
&\quad \leq \ \mathbf{E}\big|\tilde{F}\big(((T^{(n)} \wedge t, X_k^{(n),t}))_{k\geq 1}\big) 1_{\{T_m^{(n)} > t\}} \\
&\qquad\qquad\qquad - \tilde{F}\big(((T_k \wedge t, X_k^t))_{k\geq 1}\big) 1_{\{T_m > t\}}\big| + 2\varepsilon\ ,
\end{aligned}$$

where of course

$$X_k^{(n),t} := X_k^{(n)} 1_{\{T_k^{(n)} \leq t\}} + \epsilon 1_{\{T_k^{(n)} > t\}}\ .$$

Now the proof of the theorem is continued similarly to the proof of theorem 2.4. $\square$

The next result is in a certain sense connected with the problem of approximating a compound Poisson distribution as studied in [6].

Corollary 3.6. *Suppose that $\big(((T_k^{(n)}, X_k^{(n)}))_{k\geq 1}\big)_{n\geq 1}$ is uniformly non-explosive and that the assumptions of theorem 3.1 hold. For every $n \geq 1$ and every $t \geq 0$ we set*

$$S_t^{(n)} = \sum_{k\geq 1} X_k^{(n)} 1_{\{T_k^{(n)} \leq t\}} \quad \textit{and} \quad S_t = \sum_{k\geq 1} X_k 1_{\{T_k \leq t\}}\ .$$

Then

$$\lim_{n\to\infty} \sup_{F \in \mathbf{B}^1(D([0,T]))} \big|\mathbf{E}\, F\big((S_t^{(n)})_{0\leq t\leq T}\big) - \mathbf{E}\, F\big((S_t)_{0\leq t\leq T}\big)\big| = 0 \tag{83}$$

for every $T > 0$.

Proof: Let $T > 0$ be fixed and define the map $\varphi_T : S^\infty \to \mathbf{B}^1(D([0,T]))$ by

$$\varphi_T\big(((t_k, x_k))_{k\geq 1}\big) = \Big(\sum_{k\geq 1} x_k 1_{[0,t]}(t_k)\Big)_{0\leq t\leq T}$$

for $((t_k, x_k))_{k\geq 1} \in S^\infty(E)$. Then $F \circ \varphi_T \in \mathbf{B}^1((S^\infty(E), \mathcal{H}_T))$ and

$$\sup_{F\in\mathbf{B}^1(D([0,T]))} \left|\mathbf{E}\,F\big((S_t^{(n)})_{0\leq t\leq T}\big) - \mathbf{E}\,F\big((S_t)_{0\leq t\leq T}\big)\right| =$$
$$\sup_{F\in\mathbf{B}^1(D([0,T]))} \left|\mathbf{E}\,(F\circ\varphi_T)\big(((T_k^{(n)}, X_k^{(n)}))_{k\geq 1}\big) - \mathbf{E}\,(F\circ\varphi_T)\big(((T_k, X_k))_{k\geq 1}\big)\right| ,$$

and (83) follows from theorem 3.5. □

As a general example we prove an analogue to theorem 2.5 in the marked point process case. We start with a single marked point process $((\overline{T}_k, \overline{X}_k))_{k\geq 1}$ and assume that the associated counting measure $\overline{N}$ is integrable, finite, and has the intensity measure $\overline{\lambda}$ relative to a right continuous filtration $\overline{\mathbf{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$. We suppose in addition that $\overline{\lambda}$ has the properties that

$$\mathbf{E} \sup_{0\leq t\leq T} \overline{\lambda}_t < \infty \quad \text{for every} \quad T > 0\ , \tag{84}$$

that

$$\alpha := \mathbf{E}\overline{\lambda}_0 > 0\ , \tag{85}$$

and that

$$\lim_{t\downarrow 0} \sup_{B\in\mathcal{E}} \left|\mathbf{E}\overline{\lambda}_t(B) - \mathbf{E}\overline{\lambda}_0(B)\right| = 0\ . \tag{86}$$

Let $(N^k, \mathbf{F}^k, \lambda^k)_{k\geq 1}$ be a sequence of independent copies of $(\overline{N}, \overline{\mathbf{F}}, \overline{\lambda})$ and define for every $n \geq 1$ the counting measure $N^{(n)} = ((N_t^{(n)}(B))_{t\geq 0})_{B\in\mathcal{E}}$ by

$$N_t^{(n)}(B) := \sum_{k=1}^n N^k_{\frac{t}{n}}(B)\ . \tag{87}$$

Corresponding to $N^{(n)}$ we define the filtration $\mathbf{F}^{(n)}$ by

$$\mathcal{F}_t^{(n)} := \bigvee_{k=1}^n \mathcal{F}^k_{\frac{t}{n}}\ . \tag{88}$$

Then it is easy to show that every $N^{(n)}$ has the $\mathbf{F}^{(n)}$-intensity measure $\lambda^{(n)}$, given by

$$\lambda_t^{(n)}(B) = \frac{1}{n}\sum_{k=1}^{n} \lambda_{\frac{t}{n}}^{k}(B) \tag{89}$$

for $t \geq 0$ and $B \in \mathcal{E}$.

Theorem 3.7. *Let $((T_k, X_k))_{k\geq 1}$ be a marked point process, whose counting measure has the intensity measure λ given by $\lambda_t(B) = \mathbf{E}\overline{\lambda}_0(B)$. Then*

$$\lim_{n\to\infty} \sup_{F\in\mathbf{B}^1((S^\infty(E),\mathcal{H}_T))} \left|\mathbf{E}F\big(((T_k^{(n)}, X_k^{(n)}))_{k\geq 1}\big) - \mathbf{E}\,F\big(((T_k, X_k))_{k\geq 1}\big)\right| = 0 \tag{90}$$

for every $T > 0$.

Proof: For a fixed $T > 0$ we define

$$\mathbf{D}_T := \left\{\mathbf{x} = (f_B)_{B\in\mathcal{E}} \in D([0,T])^{\mathcal{E}} \;\middle|\; \|\mathbf{x}\|_T := \sup_{t\leq T}\sup_{B\in\mathcal{E}} |f_B(t)| < \infty\right\}.$$

Then $(\mathbf{D}_T, \|\cdot\|_T)$ is a Banach space. If we define

$$X_k := ((\lambda_t^k(B))_{t\leq T})_{B\in\mathcal{E}} ,$$

then $(X_k)_{k\geq 1}$ is an independent, identically distributed sequence of $\mathbf{D}_T$-valued random vectors. Since by assumption (84)

$$\mathbf{E}\,\|X_1\|_T = \mathbf{E}\sup_{t\leq T}\sup_{B\in\mathcal{E}} \overline{\lambda}_t(B) < \infty ,$$

we know from the strong law of large numbers for Banach space valued random vectors (see e.g. [4]) that

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \mathbf{E}\,X_1 \quad \mathbf{P}\text{-a.s..}$$

Thus $\mathbf{P}$-a.s.

$$\sup_{t\leq T}\sup_{B\in\mathcal{E}} \left|\frac{1}{n}\sum_{k=1}^{n} \lambda_t^k(B) - \mathbf{E}\overline{\lambda}_t(B)\right| \to 0 \quad \text{for } n\to\infty .$$

Together with assumption (86) the condition (56) of theorem 3.1 follows. Since the sequence $((T_k^{(n)})_{k\geq 1})_{n\geq 1}$ is also uniformly non-explosive (see the end of the proof of theorem 2.4), the assertion of the theorem follows from theorem 3.5. □

Remark 3.8. The deterministic limit intensity λ in theorem 3.7 is of the form $\gamma(dx)\alpha$, where $\alpha > 0$ and where $\gamma(dx)$ is a probability measure on E. This means that $((T_k, X_k))_{k\geq 1}$ is a (homogeneous) marked Poisson process: $(T_k)_{k\geq 1}$ is a classical Poisson process with parameter $\alpha > 0$, $(X_k)_{k\geq 1}$ is an i.i.d. sequence independent of $(T_k)_{k\geq 1}$, and $\gamma(dx) = \mathbf{P}_{X_1}$. Thus the distribution of every S_t is compound Poisson and corollary 3.6 gives a result for convergence towards a compound Poisson process.

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