Linear Models in Loss Reserving

Abschlussbericht zu einem Forschungsprojekt, das im Auftrag der

Hannover Rück

unter der Leitung von

Prof. Dr. Klaus D. Schmidt

durchgeführt und am 30. April 2009 abgeschlossen wurde.

Alexander Ludwig, Christiane Schmeißer und Katrin Thänert

Lehrstuhl für Versicherungsmathematik
Technische Universität Dresden

Danksagung

An dieser Stelle möchten wir uns bei Herrn Prof. Dr. Klaus D. Schmidt für seine Betreuung, Unterstützung und konstruktive Kritik bedanken. Des weiteren gilt unser Dank Herrn Andreas Ringel für viele hilfreiche Anregungen hinsichtlich der Formulierung sowie Programmierung von Ergebnissen.

Besonderen Dank schulden wir der Hannover Rück, insbesondere Herrn Eberhard Müller (CRO, Chief Actuary), für die finanzielle Unterstützung dieses Forschungsprojektes sowie die Bereitstellung von Datenmaterial.

Unseren Familien und Partnern sei ebenso für ihre vielseitige Unterstützung und ihr Interesse an unserer Projektarbeit herzlich gedankt.

Contents

Introduction

The linear model is one of the most frequently applied models in statistics. It assumes the expected value of a vector of observable random variables as a linear function of a parameter vector. This parameter vector needs to be estimated first. The model then enables the prediction of non–observable random variables whose expectations are assumed to be a linear function of the same parameter vector. In general, the parameter vector may be estimated by means of several procedures. Least squares approaches, maximum likelihood as well as the generalized method of moments are common estimation techniques. They are described in detail in textbooks such as Rao and Toutenburg [1995], Koch [1997], and Greene [2002].

One of the most prominent fields of actuarial application of the linear model is loss reserving. Loss reserving tries to find patterns in the run–off of claim payments of insurance companies and uses them to predict future loss payments. In general, there are various approaches to loss reserving, a record of which is given in Schmidt [2006]. The linear model on which we focus in this work is a straightforward method, which makes prediction and the estimation of the prediction error easily feasible once the model has been set up. Moreover, it provides various desired properties of estimators and predictors. The use of linear models in loss reserving was firstly stressed by Halliwell [1997] as well as Radtke and Schmidt [2004] and Schmidt [2004].

In the Gauss–Markov framework applied in this work, a linear model gives linear and unbiased parameter estimators as well as predictors. They are determined by minimizing a common loss function which is the (conditional) expected squared prediction error in this work. We thus strive for embedding traditional as well as recent proposals for loss reserving into the linear model in order to obtain statistically justified predictors.

As mentioned above, predictors are forced to be linear, unbiased and to minimize the expected squared prediction error. Furthermore, we are able to determine variances of and covariances between prediction errors. Thus, we provide the basis to compare the uncertainty of predictions obtained by different models with respect to the given insurance portfolio.

Driven by a paper of Ajne [1994], who showed that the sum of chain–ladder predictors for different lines of businesses equals the chain–ladder predictor of the aggregated portfolio only in rare and rather academical cases, loss reserving researchers strived for taking the next step in modeling the insurer's portfolio adequately. In this respect, Braun [2004], Kremer [2005], Pröhl and Schmidt [2005] as well as Hess, Schmidt and Zocher [2006] were among the first to propose multivariate models which reflect the dependencies between an arbitrary number of lines of businesses. The problem of additivity hence found its appropriate answer by these models. This work aspires at the derivation of multivariate extensions of various loss reserving approaches as well as the computation of prediction errors of the accident year, calendar year and total reserve in these models where they have not been revealed so far.

Moreover, linear models enable the user to mitigate the paid/incurred problem. A single portfolio may be described by a triangle of paid losses, i.e. all payments of the insurance company, as well as a triangle of incurred losses, i.e. all payments and case reserves for future payments. Naturally, these triangles lead to different predictors. Using a linear constraint on the model parameters, Halliwell [1997] proposed a method to close the gap between predictors obtained from the paid and the incurred triangle, respectively. In this work, we examine the linear model under a linear constraint following Kloberdanz and Schmidt [2008a, 2008b] and utilize these results in connection with specific loss reserving models. Another approach to solving the paid/incurred problem, the so–called Munich chain–ladder model, was developed by Quarg and Mack [2004] and further motivated by Merz and Wüthrich [2006].

This work is organized as follows. We first present the elements of the general linear model and derive important conclusions as to the shape of predictors and their prediction errors (Chapter 1). Moreover, we show the uniqueness of the Gauss–Markov predictor obtained. Chapter 1 also extends the linear model in a way that linear constraints on the parameters may be introduced. This allows for the inclusion of a–priori information as to the behaviour of parameters as well as of random variables which are to be predicted. Hence, this enables us to reduce the gap between ultimate loss predictions for the paid and incurred triangles and thus helps mitigate the paid/incurred problem mentioned above.

Since we strive for the presentation of distribution–based models of loss reserving in this work as well, Chapter 2 provides the basic results for linear models with normal distribution assumed on the variable which is to be explained. To parallel the discussion of the first chapter, we also present the normal linear model under a linear constraint. Chapter 3 discusses the lognormal loglinear model in which the variables to be explained are assumed to be lognormally distributed. Again, the case with linear constraints is additionally discussed. In order to be able to embed the widely used chain–ladder model, amongst others, into our general setting, we need to introduce the conditional linear model in Chapter 4. It presents results for the conditional linear model under a linear constraint as well. Subsequently, we will be in a position to apply these results to specific models for loss reserving.

Chapter 5 leads over from theoretical considerations regarding the linear model to models used in practice. Chapter 6 deals with the additive model, in which we model the expected value of incremental losses of the same development year as a constant multiple of the volume measure belonging to the respective accident year. Based on the results of the multivariate additive model, a special bivariate model in which volume measures do not differ among subportfolios is analyzed and subsequently results for the bivariate additive model under a linear constraint are given. In Chapter 7, we examine the lognormal logadditive model as a representative of a class of models in which assumptions on the distribution of incremental and cumulative claims, respectively, are drawn. This discussion is based on the textbooks of Mack [1997] as well as Radtke and Schmidt [2004].

A more recent model proposed by Panning [2006] is presented in Chapter 8. In a conditional setting, we use incremental losses of the first development year as the volume measure. The bivariate Panning model under a linear constraint gives another means to counter the paid/incurred problem. The most well–known model in loss reserving, the chain–ladder model, is the focal point of Chapter 9. In the traditional approach, development factors project cumulative claims from one development year to the following one. However, we also present a chain–ladder model for incremental losses and show the relation of its parameter estimates and reserve predictions to the ones obtained in the model for cumulative losses. When determining prediction errors, we utilize approximations suggested by Mack [1993].

Eventually, we present proposals for further research in Chapter 10. An attempt to solve the paid/incurred problem, the model of Dahms [2008] incorporates the basic ideas of the chain–ladder model applied on the outstanding case reserves. The closure–based regression method of Anhalt and Marsden [2007] provides another means to tackle the paid/incurred problem. Moreover, as the models mentioned above turn out to depend on only one explaining variable, we suggest to make use of two or more regressors. Analyzing the case of two regressors, we point out the mathematical pitfalls of such approaches. Exemplarily, we present the Panning model with premiums as a second regressor.

The additive model, the Panning model as well as the chain–ladder model were implemented in Excel and used for the calculation of practical examples. Chapter 11 presents results for reserves and their prediction errors obtained by these models. Moreover, we examine the ability of the bivariate Panning and additive model, respectively, under a linear constraint in order to close the gap between paid and incurred ultimate loss predictions. Our triangles are taken from real-world data provided by reinsurers.

In the entire work, all random variables, random vectors as well as random matrices are defined on the probability space $(\Omega, \mathcal{F}, P)$. Further assumptions such as measurability and integrability with respect to certain sub-σ-algebras of $\mathcal{F}$ are stated in the text. Furthermore, we assume that all equations, model assumptions and assertions refering to random variables hold P-almost surely.

References

Ajne, B. [1994]: *Additivity of chain–ladder projections.* ASTIN Bull. 24, 313–318.

Anhalt, P., and Marsden, S. [2007]: *A closure–based regression method.* 37th Int. ASTIN Colloquium Orlando 2007.

Braun, C. [2004]: *The prediction error of the chain–ladder method applied to correlated run–off triangles.* ASTIN Bull. 34, 399–423.

Dahms, R. [2008]: *A loss reserving method for incomplete claim data.* Bulletin SAV, 127–148.

Doray, L. G. [1995]: *UMVUE of the IBNR reserve in a lognormal linear regression model.* Insurance: Mathematics and Economics 18, 43–57.

Goldfarb, R. [2006]: *Risk–Adjusted Performance Measurement for P&C Insurers.* CAS Exam 8 Study Note 2006.

Greene, W. H. [2002]: *Econometric Analysis.* 5th edition, New Jersey: Prentice Hall.

Halliwell, L. J. [1997]: *Conjoint prediction of paid and incurred losses.* CAS Forum Summer 1997, 241–379.

Hamer, M. D. [1999]: *Loss prediction by generalized least squares – Discussion of Halliwell (1996).* Proc. CAS 86, 748–763.

Harville, D. A. [1997]: *Matrix Algebra from a Statistician's Perspective. New York: Springer.*

Hess, K. T., Schmidt, K. D., and Zocher, M. [2006]: *Multivariate loss prediction in the multivariate additive model.* Insurance Math. Econom. 39, 185–191.

Kaulfuss, S. [1999]: *Loglineares Modell zur Reservierung von Spätschäden.* Diplomarbeit, Technische Universität Dresden.

Kloberdanz, K., and Schmidt, K. D. [2008]: *Prediction in the linear model under a linear constraint.* AStA Advances in Statistical Analysis 92, 207–215.

Kloberdanz, K., and Schmidt, K. D. [2009]: *Loss prediction in a linear model under a linear constraint.* AStA Advances in Statistical Analysis 93, 221–233.

Koch, K. R. [1997]: *Parameterschätzung und Hypothesentests in linearen Modellen.* Bonn: Ferd. Dümmlers Verlag.

Kremer, E. [2005]: *The correlated chain–ladder method for reserving in case of correlated claims developments.* Blätter DGVFM 27, 315–322.

Mack, T. [1993]: *Distribution–free calculation of the standard error of chain–ladder reserve estimates.* ASTIN Bull. 23, 213–225.

Mack, T. [1997]: *Schadenversicherungsmathematik.* Karlsruhe: Verlag Versicherungswirtschaft.

Merz, M., and Wüthrich, M.V. [2006]: *A credibility approach to the Munich chain-ladder method.* Blätter DGVFM 27, 619–628.

Merz, M., and Wüthrich, M.V. [2008a]: *Prediction error of the multivariate additive loss reserving method for dependent lines of business.* Variance (forthcoming).

Merz M., and Wüthrich, M. V. [2008b] *Stochastic Claims Reserving Methods in Insurance.* Chichester: Wiley.

Panning, W. H. [2006]: *Measuring loss reserve uncertainty.* CAS Forum Fall 2006, 237–267.

Pröhl, C., and Schmidt, K. D. [2005]: *Multivariate chain–ladder.* Dresdner Schriften zur Versicherungsmathematik 3/2005.

Quarg, G., and Mack, T. [2004]: *Munich chain–ladder.* Blätter DGVM 26, 597–630.

Radtke, M., and Schmidt, K. D. (Hrsg.) [2004]: *Handbuch zur Schadenreservierung.* Karlsruhe: Verlag Versicherungswirtschaft.

Rao, C. R., and Toutenburg, H. [1995]: *Linear Models – Least Squares and Alternatives.* Berlin – Heidelberg – New York: Springer.

Rencher, A. C., and Schaalje, B. G. [2008]: *Linear Models in Statistics.* 2nd edition, New Jersey: Wiley.

Schmidt, K. D. [1997]: *Non–optimal prediction by the chain–ladder method.* Insurance: Mathematics and Economics 21, 17–24.

Schmidt, K. D. [1999]: *Chain ladder prediction and asset liability management.* Blätter DGVFM 24, 1–9.

Schmidt, K. D. [2004]: *Prediction.* In: *Encyclopedia of Actuarial Science*, vol. 3, 1317–1321. Chichester: Wiley.

Schmidt, K. D. [2006a]: *Methods and models of loss reserving based on run–off triangles – A unifying survey.* CAS Forum Fall 2006, 269–317.

Schmidt, K. D. [2006b]: *Optimal and additive loss reserving for dependent lines of business.* CAS Forum Fall 2006, 319–351

Schmidt, K. D. [2009a]: *Maß und Wahrscheinlichkeit.* Berlin – Heidelberg – New York: Springer.

Schmidt, K. D. [2009b]: *Versicherungsmathematik.* 3. Auflage. Berlin – Heidelberg – New York: Springer.

Schmidt, K. D., and Schnaus, A. [1996]: *An extension of Mack's model for the chain–ladder method.* ASTIN Bull. 26, 247–262.

Schmidt, K. D., and Zocher, M. [2008]: *The Bornhuetter–Ferguson principle.* Variance 2, 85–110.

Stapleton, J. H. [1995]: *Linear Statistical Models.* New York: Wiley.

Vanduffel, S., Dhaene, J., and Hoedemakers, T. [2005]: *Comparing approximations for risk measures of sums of non–independent lognormal random variables.* North American Actuarial Journal 9, 71–82.

Verrall, R. J. [1991]: *On the estimation of reserves from loglinear models.* Insurance: Mathematics and Economics 10, 75–80.