Technische Universität Dresden
Institut für Mathematische Stochastik

Dresdner Schriften zur
Versicherungsmathematik
1/2010

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Herausgeber: Die Professoren des Instituts für Mathematische Stochastik
ISSN 0946–4727

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Abstract

For the multivariate additive model proposed by Hess et al. [2006] which provides an extension of the additive (or incremental loss ratio) method to correlated lines of business, we determine the Gauss–Markov predictors of the calendar year reserves and their mean squared errors of prediction. We also show that the mean squared errors of prediction can be estimated by using moment estimators for the parameters which determine the correlation structure. The result is of interest with regard to Solvency II, and it is new even in the univariate case considered by Mack [1991].

1 Introduction

Like the chain–ladder method, the *additive* (or *incremental loss ratio*) *method* is one of the most popular methods in loss reserving. Unlike the chain–ladder method, however, it uses not only the data of the run–off triangle of all observable losses but also certain volume measures for the accident years. As a volume measure one could take, for example, the number of policies or the premium income, and such objective measures of exposure could compensate for the volatility of the data of the run–off triangle.

Mack [1991] proposed a stochastic model which, combined with a natural principle of parameter estimation, produces the predictors of the non–observable incremental losses used in the additive method and which is now called the *additive model* of loss reserving. In fact, the additive model is a linear model, the estimators of the parameters turn out to be Gauss–Markov estimators, and the predictors of the non–observable losses turn out to be Gauss–Markov predictors. Because of this observation and inspired by a personal communication of Braun [2005], Hess et al. [2006] proposed a multivariate version of the additive model and the Gauss–Markov predictors in their *multivariate additive model* provide a multivariate version of the additive method which takes into account correlations between different lines of business; see also Schmidt [2006b].

In the (multivariate) additive model, the Gauss–Markov predictors of the non–observable incremental losses are completely determined by the volume measures and the correlation structure, and this is also true for their mean squared errors of prediction. Moreover, since Gauss–Markov prediction is linear, the Gauss–Markov predictor of a sum of non–observable incremental losses is just the sum of their Gauss–Markov predictors. Since reserves are sums of non–observable incremental losses, the Gauss–Markov predictors of reserves and their mean squared errors of prediction are easy to determine, and it turns out that the mean squared errors of prediction can be estimated by replacing the variances and covariances with moment estimators.[1]

For the accident year reserves and the total reserve, the natural estimators of the mean squared errors of prediction were determined by Mack [1991] for the univariate additive model and by Merz and Wüthrich [2009] for the multivariate additive model. For Solvency II purposes, however, calendar year reserves are more important, and this is particularly true for the reserve for the first non–observable calendar year. In the present paper, we determine the calendar year reserves in the multivariate additive model and their mean squared errors of prediction which, as noted before, can be estimated by the use of plug–in estimators.

[1]The use of plug–in estimators for estimating the mean squared errors of prediction is not possible in the chain–ladder model proposed by Mack [1993] and its multivariate extension proposed by Pröhl and Schmidt [2005]. In these models, certain approximations seem to be unavoidable in the construction of estimators of the mean squared errors of prediction and it appears to be difficult to quantify the approximation errors.

2 The Multivariate Additive Model

We consider m portfolios of risks all having the same number of development years. The m portfolios may be interpreted as subportfolios of an aggregate portfolio. For portfolio $p \in \{1, \ldots, m\}$, we denote by

$$Z_{i,k}^{(p)}$$

the incremental loss of accident year $i \in \{0, 1, \ldots, n\}$ and development year $k \in \{0, 1, \ldots, n\}$. For $i, k \in \{0, 1, \ldots, n\}$, we thus obtain the m–dimensional random vector of *incremental losses*

$$\mathbf{Z}_{i,k} \;:=\; \left(Z_{i,k}^{(p)}\right)_{p \in \{1,\ldots,m\}}$$

The observable incremental losses are represented by the following run–off triangle:

Accident	**Development Year**								
Year	0	1	…	k	…	$n-i$	…	$n-1$	n
0	$\mathbf{Z}_{0,0}$	$\mathbf{Z}_{0,1}$	…	$\mathbf{Z}_{0,k}$	…	$\mathbf{Z}_{0,n-i}$	…	$\mathbf{Z}_{0,n-1}$	$\mathbf{Z}_{0,n}$
1	$\mathbf{Z}_{1,0}$	$\mathbf{Z}_{1,1}$	…	$\mathbf{Z}_{1,k}$	…	$\mathbf{Z}_{1,n-i}$	…	$\mathbf{Z}_{1,n-1}$	
⋮	⋮	⋮		⋮		⋮			
i	$\mathbf{Z}_{i,0}$	$\mathbf{Z}_{i,1}$	…	$\mathbf{Z}_{i,k}$	…	$\mathbf{Z}_{i,n-i}$			
⋮	⋮	⋮		⋮					
$n-k$	$\mathbf{Z}_{n-k,0}$	$\mathbf{Z}_{n-k,1}$	…	$\mathbf{Z}_{n-k,k}$					
⋮	⋮	⋮							
$n-1$	$\mathbf{Z}_{n-1,0}$	$\mathbf{Z}_{n-1,1}$							
n	$\mathbf{Z}_{n,0}$								

Here the rows represent *accident years*, the columns represent *development years*, and the diagonals (with the sum $i + k$ being constant) represent *calendar years*.

The *multivariate additive model* is defined as follows:[2]

The Multivariate Additive Model: There exist positive definite symmetric matrices $\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_n$ and $\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_n$ and unknown vectors $\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n$ such that

$$E[\mathbf{Z}_{i,k}] \;=\; \mathbf{V}_i \, \boldsymbol{\zeta}_k$$

and

$$\mathrm{cov}[\mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}] \;=\; \mathbf{V}_i^{1/2} \boldsymbol{\Sigma}_k \mathbf{V}_i^{1/2} \, \delta_{i,j} \, \delta_{k,l}$$

holds for all $i, j, k, l \in \{0, 1, \ldots, n\}$.

[2]We use the Kronecker symbol $\delta_{k,l}$ with $\delta_{k,l} := 1$ if $k = l$ and $\delta_{k,l} := 0$ else.

The multivariate additive model is a general but straightforward extension of the univariate additive model proposed by Mack [1991]. In particular, the matrices $\mathbf{V}_i$ may be chosen to be diagonal as to represent volume measures of the portfolios for accident year i.

We assume henceforth that the assumptions of the multivariate additive model are fulfilled.

Because of the assumption on the expectations of the incremental claims, the multivariate additive model is a linear model. This can be seen as follows: Define

$$\boldsymbol{\zeta} \;:=\; \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_{k-1} \\ \zeta_k \\ \zeta_{k+1} \\ \vdots \\ \zeta_n \end{pmatrix}$$

and, for all $i, k \in \{0, 1, \ldots, n\}$, define

$$\mathbf{A}_{i,k} \;:=\; \begin{pmatrix} \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{V}_i & \mathbf{O} & \ldots & \mathbf{O} \end{pmatrix}$$

where the matrix $\mathbf{V}_i$ occurs in position $k{+}1$. Then we have

$$E[\mathbf{Z}_{i,k}] \;=\; \mathbf{A}_{i,k}\boldsymbol{\zeta}$$

for all $i, k \in \{0, 1, \ldots, n\}$. Let $\mathbf{Z}_1$ and $\mathbf{A}_1$ denote a block vector and a block matrix consisting of the vectors $\mathbf{Z}_{i,k}$ and the matrices $\mathbf{A}_{i,k}$ with $i + k \leq n$ (arranged in the same order) and let $\mathbf{Z}_2$ and $\mathbf{A}_2$ denote a block vector and a block matrix consisting of the vectors $\mathbf{Z}_{i,k}$ and the matrices $\mathbf{A}_{i,k}$ with $i + k \geq n + 1$. Then we have

$$\begin{aligned} E[\mathbf{Z}_1] &= \mathbf{A}_1\boldsymbol{\zeta} \\ E[\mathbf{Z}_2] &= \mathbf{A}_2\boldsymbol{\zeta} \end{aligned}$$

Therefore, the multivariate additive model is indeed a linear model.

3 The Results

In the multivariate additive model, the problem is to predict the non–observable incremental losses $\mathbf{Z}_{i,k}$ with $i + k \geq n + 1$, certain sums of such losses or, even more generally, random vectors of the form $\mathbf{D}\mathbf{Z}_2$ with some matrix $\mathbf{D}$ of suitable dimension. A random vector $\mathbf{Y}$ is said to be an *admissible predictor* of $\mathbf{D}\mathbf{Z}_2$ if there exists a matrix $\mathbf{Q}$ satisfying $\mathbf{Y} = \mathbf{Q}\mathbf{Z}_1$ and $\mathbf{Q}\mathbf{A}_1 = \mathbf{D}\mathbf{A}_2$ (such that $\mathbf{Y}$ a linear and unbiased predictor of $\mathbf{D}\mathbf{Z}_2$), and it is said to be a *Gauss–Markov predictor* of $\mathbf{D}\mathbf{Z}_2$

if it minimizes the *mean squared error or prediction* (or *expected squared prediction error*)

$$E[(\mathbf{U}-\mathbf{D}\mathbf{Z}_2)'(\mathbf{U}-\mathbf{D}\mathbf{Z}_2)]$$

over all admissible predictors $\mathbf{U}$ of $\mathbf{D}\mathbf{Z}_2$. The Gauss–Markov Theorem asserts that there exists a unique Gauss–Markov predictor $(\mathbf{D}\mathbf{Z}_2)^{\mathrm{GM}}$ of $\mathbf{D}\mathbf{Z}_2$ and that it satisfies $(\mathbf{D}\mathbf{Z}_2)^{\mathrm{GM}} = \mathbf{D}\mathbf{Z}_2^{\mathrm{GM}}$; see Hess et al. [2006; Proposition 2.1]. This last identity is most useful since it implies that the Gauss–Markov predictor of $\mathbf{Z}_2$ is determined by the Gauss–Markov predictors of the non–observable losses $\mathbf{Z}_{i,k}$.

It turns out that Gauss–Markov prediction of non–observable losses is closely related to Gauss–Markov estimation of the parameter. A random vector $\mathbf{Y}$ is said to be an *admissible estimator* of $\mathbf{C}\boldsymbol{\zeta}$ if there exists a matrix $\mathbf{Q}$ satisfying $\mathbf{Y} = \mathbf{Q}\mathbf{Z}_1$ and $\mathbf{Q}\mathbf{A}_1 = \mathbf{C}$ (such that $\mathbf{Y}$ a linear and unbiased estimator of $\mathbf{C}\boldsymbol{\zeta}$), and it is said to be a *Gauss–Markov estimator* of $\mathbf{C}\boldsymbol{\zeta}$ if it minimizes the *mean squared error of estimation* (or *expected squared estimation error*)

$$E[(\mathbf{U}-\mathbf{C}\boldsymbol{\zeta})'(\mathbf{U}-\mathbf{C}\boldsymbol{\zeta})]$$

over all admissible estimators $\mathbf{U}$ of $\mathbf{C}\boldsymbol{\zeta}$. Again, there exists a unique Gauss–Markov estimator $(\mathbf{C}\boldsymbol{\zeta})^{\mathrm{GM}}$ of $\mathbf{C}\boldsymbol{\zeta}$ and it satisfies $(\mathbf{C}\boldsymbol{\zeta})^{\mathrm{GM}} = \mathbf{C}\boldsymbol{\zeta}^{\mathrm{GM}}$ such that, in particular, the Gauss–Markov estimator of $\boldsymbol{\zeta}$ is determined by the Gauss–Markov estimators of the parameters $\boldsymbol{\zeta}_k$.

3.1 Lemma. *The Gauss–Markov estimators* $\boldsymbol{\zeta}_k^{\mathrm{GM}}$ *of* $\boldsymbol{\zeta}_k$ *satisfy*

$$\boldsymbol{\zeta}_k^{\mathrm{GM}} \;=\; \left(\sum_{h=0}^{n-k} \mathbf{V}_h^{1/2}\boldsymbol{\Sigma}_k^{-1}\mathbf{V}_h^{1/2}\right)^{-1} \sum_{h=0}^{n-k}\left(\mathbf{V}_h^{1/2}\boldsymbol{\Sigma}_k^{-1}\mathbf{V}_h^{1/2}\right)\mathbf{V}_h^{-1}\mathbf{Z}_{h,k}$$

for all $k \in \{0,1,\ldots,n\}$ *and*

$$\mathrm{cov}[\boldsymbol{\zeta}_k^{\mathrm{GM}}, \boldsymbol{\zeta}_l^{\mathrm{GM}}] \;=\; \left(\sum_{h=0}^{n-k} \mathbf{V}_h^{1/2}\boldsymbol{\Sigma}_k^{-1}\mathbf{V}_h^{1/2}\right)^{-1} \delta_{k,l}$$

for all $k, l \in \{0,1,\ldots,n\}$.

Proof. The first identity follows from Hess et al. [2006; proof of Theorem 3.1] and the second identity is immediate from the first. □

The Gauss–Markov estimators of the parameters determine the Gauss–Markov predictors of the non–observable incremental losses:

3.2 Lemma. *The Gauss–Markov predictors* $\mathbf{Z}_{i,k}^{\mathrm{GM}}$ *of* $\mathbf{Z}_{i,k}$ *satisfy*

$$\mathbf{Z}_{i,k}^{\mathrm{GM}} \;=\; \mathbf{V}_i\,\boldsymbol{\zeta}_k^{\mathrm{GM}}$$

for all $i, k \in \{0,1,\ldots,n\}$ *such that* $i+k \geq n+1$ *and*

$$\mathrm{cov}[\mathbf{Z}_{i,k}^{\mathrm{GM}} - \mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}^{\mathrm{GM}} - \mathbf{Z}_{j,l}] \;=\; \left(\mathbf{V}_i\,\mathrm{var}[\boldsymbol{\zeta}_k^{\mathrm{GM}}]\mathbf{V}_j' + \mathbf{V}_i^{1/2}\boldsymbol{\Sigma}_k\mathbf{V}_i^{1/2}\,\delta_{i,j}\right)\delta_{k,l}$$

for all $i, j, k, l \in \{0,1,\ldots,n\}$ *such that* $i+k \geq n+1$ *and* $j+l \geq n+1$.

Proof. Because of Lemma 3.1, the first identity follows from Hess et al. [2006; Theorem 3.1]. Furthermore, Lemma 3.1 yields

$$\begin{aligned}\operatorname{cov}[\mathbf{Z}_{i,k}^{\mathrm{GM}}, \mathbf{Z}_{j,l}^{\mathrm{GM}}] &= \operatorname{cov}[\mathbf{V}_i \boldsymbol{\zeta}_k^{\mathrm{GM}}, \mathbf{V}_j \boldsymbol{\zeta}_l^{\mathrm{GM}}] \\ &= \mathbf{V}_i \operatorname{cov}[\boldsymbol{\zeta}_k^{\mathrm{GM}}, \boldsymbol{\zeta}_l^{\mathrm{GM}}]\mathbf{V}_j' \\ &= \mathbf{V}_i \operatorname{var}[\boldsymbol{\zeta}_k^{\mathrm{GM}}]\mathbf{V}_j' \,\delta_{k,l}\end{aligned}$$

and we also have

$$\operatorname{cov}[\mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}] = \mathbf{V}_i^{1/2}\boldsymbol{\Sigma}_k \mathbf{V}_i^{1/2}\,\delta_{i,j}\,\delta_{k,l}$$

Since $\mathbf{Z}_{i,k}^{\mathrm{GM}}$ and $\mathbf{Z}_{j,l}^{\mathrm{GM}}$ are linear combinations of observable incremental losses whereas $\mathbf{Z}_{i,k}$ and $\mathbf{Z}_{j,l}$ are non–observable incremental losses, we have $\operatorname{cov}[\mathbf{Z}_{i,k}^{\mathrm{GM}}, \mathbf{Z}_{j,l}] = \mathbf{O} = \operatorname{cov}[\mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}^{\mathrm{GM}}]$ and hence

$$\begin{aligned}\operatorname{cov}[\mathbf{Z}_{i,k}^{\mathrm{GM}} - \mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}^{\mathrm{GM}} - \mathbf{Z}_{j,l}] &= \operatorname{cov}[\mathbf{Z}_{i,k}^{\mathrm{GM}}, \mathbf{Z}_{j,l}^{\mathrm{GM}}] + \operatorname{cov}[\mathbf{Z}_{i,k}, \mathbf{Z}_{j,l}] \\ &= \mathbf{V}_i \operatorname{var}[\boldsymbol{\zeta}_k^{\mathrm{GM}}]\mathbf{V}_j'\,\delta_{k,l} + \mathbf{V}_i^{1/2}\boldsymbol{\Sigma}_k \mathbf{V}_i^{1/2}\,\delta_{i,j}\,\delta_{k,l}\end{aligned}$$

which gives the second identity. □

For the non–observable calendar years $c \in \{n+1, \dots, 2n\}$, the *calendar year reserve* $\mathbf{R}_{(c)}$ is defined as

$$\mathbf{R}_{(c)} := \sum_{i=c-n}^{n} \mathbf{Z}_{i,c-i}$$

The Gauss–Markov estimators of the parameters also determine the Gauss–Markov predictors of the calendar year reserves:

3.3 Theorem. *For every $c \in \{n+1, \dots, 2n\}$, the Gauss–Markov predictor $\mathbf{R}_{(c)}^{\mathrm{GM}}$ of $\mathbf{R}_{(c)}$ satisfies*

$$\mathbf{R}_{(c)}^{\mathrm{GM}} = \sum_{i=c-n}^{n} \mathbf{V}_i \left(\sum_{h=0}^{n-c+i} \mathbf{V}_h^{1/2}\boldsymbol{\Sigma}_{c-i}^{-1}\mathbf{V}_h^{1/2}\right)^{-1} \sum_{h=0}^{n-c+i} \left(\mathbf{V}_h^{1/2}\boldsymbol{\Sigma}_{c-i}^{-1}\mathbf{V}_h^{1/2}\right)\mathbf{V}_h^{-1}\mathbf{Z}_{h,c-i}$$

and

$$\operatorname{var}[\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}] = \sum_{i=c-n}^{n} \mathbf{V}_i \left(\left(\sum_{h=0}^{n-c+i} \mathbf{V}_h^{1/2}\boldsymbol{\Sigma}_{c-i}^{-1}\mathbf{V}_h^{1/2}\right)^{-1} + \left(\mathbf{V}_i^{1/2}\boldsymbol{\Sigma}_{c-i}^{-1}\mathbf{V}_i^{1/2}\right)^{-1}\right)\mathbf{V}_i$$

Moreover, its mean squared error of prediction satisfies

$$E[(\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)})'(\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)})] = \operatorname{trace}\Big(\operatorname{var}[\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}]\Big)$$

Proof. Since Gauss–Markov prediction is linear, Lemma 3.2 yields

$$\begin{aligned}
\mathbf{R}_{(c)}^{\mathrm{GM}} &= \sum_{i=c-n}^{n} \mathbf{Z}_{i,c-i}^{\mathrm{GM}} \\
&= \sum_{i=c-n}^{n} \mathbf{V}_i \boldsymbol{\zeta}_{c-i}^{\mathrm{GM}} \\
&= \sum_{i=c-n}^{n} \mathbf{V}_i \left(\sum_{h=0}^{n-c+i} \mathbf{V}_h^{1/2} \boldsymbol{\Sigma}_{c-i}^{-1} \mathbf{V}_h^{1/2} \right)^{-1} \sum_{h=0}^{n-c+i} \left(\mathbf{V}_h^{1/2} \boldsymbol{\Sigma}_{c-i}^{-1} \mathbf{V}_h^{1/2} \right) \mathbf{V}_h^{-1} \mathbf{Z}_{h,c-i}
\end{aligned}$$

which is the first identity. Furthermore, we have

$$\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)} = \sum_{i=c-n}^{n} (\mathbf{Z}_{i,c-i}^{\mathrm{GM}} - \mathbf{Z}_{i,c-i})$$

and because of Lemma 3.2 and Lemma 3.1 we obtain

$$\begin{aligned}
\mathrm{var}[\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}] &= \mathrm{var}\left[\sum_{i=c-n}^{n} (\mathbf{Z}_{i,c-i}^{\mathrm{GM}} - \mathbf{Z}_{i,c-i}) \right] \\
&= \sum_{i=c-n}^{n} \sum_{j=c-n}^{n} \mathrm{cov}\Big[(\mathbf{Z}_{i,c-i}^{\mathrm{GM}} - \mathbf{Z}_{i,c-i}), (\mathbf{Z}_{j,c-j}^{\mathrm{GM}} - \mathbf{Z}_{j,c-j}) \Big] \\
&= \sum_{i=c-n}^{n} \sum_{j=c-n}^{n} \Big(\mathbf{V}_i \, \mathrm{var}[\boldsymbol{\zeta}_{c-i}^{\mathrm{GM}}] \mathbf{V}_j' + \mathbf{V}_i^{1/2} \boldsymbol{\Sigma}_{c-i} \mathbf{V}_i^{1/2} \, \delta_{i,j} \Big) \, \delta_{c-i,c-j} \\
&= \sum_{i=c-n}^{n} \left(\mathbf{V}_i \left(\sum_{h=0}^{n-c+i} \mathbf{V}_h^{1/2} \boldsymbol{\Sigma}_{c-i}^{-1} \mathbf{V}_h^{1/2} \right)^{-1} \mathbf{V}_i' + \mathbf{V}_i^{1/2} \boldsymbol{\Sigma}_{c-i} \mathbf{V}_i^{1/2} \right) \\
&= \sum_{i=c-n}^{n} \mathbf{V}_i \left(\left(\sum_{h=0}^{n-c+i} \mathbf{V}_h^{1/2} \boldsymbol{\Sigma}_{c-i}^{-1} \mathbf{V}_h^{1/2} \right)^{-1} + \left(\mathbf{V}_i^{1/2} \boldsymbol{\Sigma}_{c-i}^{-1} \mathbf{V}_i^{1/2} \right)^{-1} \right) \mathbf{V}_i
\end{aligned}$$

which is the second identity. Since Gauss–Markov predictors are unbiased, this yields

$$\begin{aligned}
E\Big[(\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)})' (\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}) \Big] &= E\Big[\mathrm{trace}\Big((\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}) (\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)})' \Big) \Big] \\
&= \mathrm{trace}\Big(E\Big[(\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}) (\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)})' \Big] \Big) \\
&= \mathrm{trace}\Big(\mathrm{var}[\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}] \Big)
\end{aligned}$$

which is the final identity. □

Since Gauss–Markov prediction is linear, the precious result also yields, for a given calendar year $c \in \{n{+}1, \ldots, 2n\}$, the Gauss–Markov predictor of the sum $\mathbf{1}'\mathbf{R}_{(c)}$ of the calendar year reserves over all subportfolios $p \in \{1, \ldots, m\}$:

3.4 Corollary. *For every $c \in \{n+1, \ldots, 2n\}$, the Gauss–Markov predictor $(\mathbf{1}'\mathbf{R}_{(c)})^{\mathrm{GM}}$ of $\mathbf{1}'\mathbf{R}_{(c)}$ satisfies*

$$(\mathbf{1}'\mathbf{R}_{(c)})^{\mathrm{GM}} \;=\; \mathbf{1}'\mathbf{R}_{(c)}^{\mathrm{GM}}$$

and its mean squared error of prediction satisfies

$$E\Big[\Big((\mathbf{1}'\mathbf{R}_{(c)})^{\mathrm{GM}} - \mathbf{1}'\mathbf{R}_{(c)}\Big)^2\Big] \;=\; \mathbf{1}'\,\mathrm{var}[\mathbf{R}_{(c)}^{\mathrm{GM}} - \mathbf{R}_{(c)}]\,\mathbf{1}$$

Prediction of the aggregated reserves $\mathbf{1}'\mathbf{R}_{(c)}$ is of considerable interest for Solvency II purposes. This is particularly true for the reserve $\mathbf{1}'\mathbf{R}_{(n+1)}$ of the first non–observable calendar year. When combined with Theorem 3.3, Corollary 3.4 provides the Gauss–Markov predictors of the aggregated reserves and their mean squared errors of prediction. Then the *standard errors of prediction* which is defined as the square root of the mean squared error of prediction measures the prediction error in the monetary unit. Another measure would be the *coefficient of variation* which is defined as the ratio of the standard error of prediction and the (absolute value of) the predictor and is dimension–free.

4 Estimation of the Variance Parameters

The Gauss–Markov estimators of the parameters, the Gauss–Markov predictors of calendar year reserves and the mean squared errors of prediction involve inverses of the variance parameters $\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_n$ which have to be estimated.

For $k \in \{0, 1, \ldots, n-1\}$, Hess et al. [2006] proposed the estimators

$$\widehat{\boldsymbol{\Sigma}}_k \;:=\; \frac{1}{n-k} \sum_{j=0}^{n-k} \mathbf{V}_j^{-1/2} \Big(\mathbf{Z}_{j,k} - \mathbf{V}_j \widehat{\boldsymbol{\zeta}}_k\Big)\Big(\mathbf{Z}_{j,k} - \mathbf{V}_j \widehat{\boldsymbol{\zeta}}_k\Big)' \mathbf{V}_j^{-1/2}$$

where

$$\widehat{\boldsymbol{\zeta}}_k \;:=\; \left(\sum_{h=0}^{n-k} \mathbf{V}_h\right)^{-1} \sum_{h=0}^{n-k} \mathbf{Z}_{h,k}$$

is an unbiased linear estimator of $\boldsymbol{\zeta}_k$. The estimator $\widehat{\boldsymbol{\Sigma}}_k$ is a positive semidefinite estimator of the positive definite matrix $\boldsymbol{\Sigma}_k$; moreover, the diagonal elements of $\widehat{\boldsymbol{\Sigma}}_k$ are unbiased estimators of the diagonal elements of $\boldsymbol{\Sigma}_k$ but the nondiagonal elements of $\widehat{\boldsymbol{\Sigma}}_k$ slightly underestimate the corresponding elements of $\boldsymbol{\Sigma}_k$. The lack of unbiasedness of these estimators outside the diagonal is inessential since estimators of the variance parameters have to be inverted and unbiasedness of an estimator is usually not inherited by its inverse.

Inspite of the nice properties of these estimators, estimation of the variance parameters is by no means automatic:

– Although there is no need for an estimator of $\mathbf{\Sigma}_n$ with regard to the Gauss–Markov predictors, such an estimator is needed for the mean squared errors of prediction.
– For $k \in \{0, 1, \ldots, n-1\}$, we have $\text{rank}(\widehat{\mathbf{\Sigma}}_k) \leq n-k+1$; this is obvious from the fact that $\text{rank}(\mathbf{x}\mathbf{x}') \leq 1$ holds every $\mathbf{x} \in \mathbb{R}^m$. Since the estimator $\widehat{\mathbf{\Sigma}}_k$ is invertible if and only if $\text{rank}(\widehat{\mathbf{\Sigma}}_k) = m$, it cannot be invertible for $k \geq n - m + 2$.
– Even for $k \leq n-m+1$, the realization of the estimator $\widehat{\mathbf{\Sigma}}_k$ fails to be invertible if it happens that the realizations of the family $\{\mathbf{V}_j^{-1/2}(\mathbf{Z}_{j,k} - \mathbf{V}_j \widehat{\boldsymbol{\zeta}}_k)\}_{j \in \{0,1,\ldots,n-k\}}$ are linearly dependent.

For $k = n$ and also for those $k \in \{0, 1, \ldots, n-1\}$ for which the realization of $\widehat{\mathbf{\Sigma}}_k$ fails to be invertible, an additional effort has to be made to construct an estimator of $\mathbf{\Sigma}_k$. For every choice of such an estimator, one has to make sure that it is positive definite and hence invertible, and this includes the requirement that its diagonal elements are greater than zero. In principle, interpolation or extrapolation methods could be used but the resulting estimator has to be adjusted when it fails to be invertible. After all, estimation of the variance parameters requires not only statistical skills but also profound understanding of the data and actuarial judgement.

5 A Numerical Example

As an example, we apply our results to the case of a portfolio consisting of two subportfolios where portfolio 1 concerns general liability and portfolio 2 concerns auto liability. We use the incremental losses provided by Braun [2004] and the volume measures proposed by Merz and Wüthrich [2009]. These data are given in Tables 3 and 4 of the Appendix.

Table 1 presents the Gauss–Markov predictors of the calendar year reserves which are obtained from the respective univariate additive models ($m = 1$) for each of the two subportfolios and from the bivariate additive model ($m = 2$) as well as their sums over both subportfolios.

Calendar Year	Subportfolio 1 Univariate	Subportfolio 1 Bivariate	Subportfolio 2 Univariate	Subportfolio 2 Bivariate	Joint Portfolio Univariate	Joint Portfolio Bivariate
14	1860715	1860396	941851	942713	2802565	2803109
15	1511377	1510865	495808	496655	2007185	2007520
16	1093135	1092589	282928	283126	1376063	1375715
17	716349	716674	161417	161561	877766	878236
18	448055	448488	85786	86021	533841	534509
19	257669	258625	41579	41772	299248	300397
20	150088	151440	19425	19634	169513	171074
21	97640	98973	7161	7383	104801	106357
22	68762	69195	7258	7308	76021	76503
23	44049	44259	3328	3308	47377	47567
24	32059	32055	3430	3437	35489	35492
25	22490	22486	-1899	-1879	20591	20607
26	9114	9114	-392	-392	8722	8722

Table 1: Gauss–Markov Predictors of Calendar Year Reserves

Table 1 shows that the differences between univariate and bivariate Gauss–Markov predictors are negligible. However, the difference between univariate and bivariate Gauss–Markov prediction become sensible if one considers the standard errors of prediction or even the coefficients of variation which are presented in Table 2:

Calendar Year	Subportfolio 1 Univariate		Subportfolio 2 Univariate		Joint Portfolio Bivariate	
14	79886	4.29%	52218	5.54%	107172	3.82%
15	63184	4.18%	39907	8.05%	90974	4.53%
16	57529	5.26%	34513	12.20%	77282	5.62%
17	54216	7.57%	29190	18.08%	68701	7.82%
18	45267	10.10%	20026	23.34%	53529	10.01%
19	40651	15.78%	12430	29.90%	43977	14.64%
20	34617	23.06%	10730	55.24%	36774	21.50%
21	35872	36.74%	9577	133.73%	38003	35.73%
22	34753	50.54%	9308	128.23%	34886	45.60%
23	26214	59.51%	6309	189.59%	20878	43.89%
24	25848	80.63%	6129	178.67%	16483	46.44%
25	25884	115.09%	3767	198.34%	16379	79.48%
26	13766	151.04%	2716	692.22%	11951	137.02%

Table 2: Standard Errors of Prediction and Coefficients of Variation

Table 2 shows that

- the standard error of prediction of the calendar year reserve of the joint portfolio is always smaller than the sum of the standard errors of the calendar year reserves of the two subportfolios and that
- for the majority of calendar years and, in particular, for the first non–observable calendar year 14, the coefficient of variation of the joint portfolio is smaller than the coefficients of variation of both subportfolios.

This indicates that, at least with respect to the standard error of prediction and the coefficient of variation, Gauss–Markov prediction in the multivariate additive model for a joint portfolio presents an advantage over Gauss–Markov prediction in the univariate additive models for the single subportfolios which typically neglects the correlation structure between the subportfolios.

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January 15, 2010.

Appendix

Tables 3 and 4 present the volume measures and the incremental losses of portfolios 1 and 2, respectively.

Table 5 presents the coordinates of the auxiliary estimator $\widehat{\boldsymbol{\zeta}}_k$ of the parameter $\boldsymbol{\zeta}_k$ which is used in the definition of the estimator $\widehat{\boldsymbol{\Sigma}}_k$ of the variance parameter $\boldsymbol{\Sigma}_k$. Representing the estimator $\widehat{\boldsymbol{\Sigma}}_k$ as

$$\widehat{\boldsymbol{\Sigma}}_k \quad = \quad \begin{pmatrix} \widehat{\sigma}_k^{(1,1)} & \widehat{\sigma}_k^{(1,2)} \\ \widehat{\sigma}_k^{(2,1)} & \widehat{\sigma}_k^{(2,2)} \end{pmatrix}$$

we have $\widehat{\sigma}_k^{(2,1)} = \widehat{\sigma}_k^{(1,2)}$ and we define the standard deviations

$$\begin{aligned} \widehat{\sigma}_k^{(1)} &:= (\widehat{\sigma}_k^{(1,1)})^{1/2} \\ \widehat{\sigma}_k^{(2)} &:= (\widehat{\sigma}_k^{(2,2)})^{1/2} \end{aligned}$$

and the coefficient of correlation

$$\widehat{\varrho}_k^{(1,2)} \quad := \quad \frac{\widehat{\sigma}_k^{(1,2)}}{\widehat{\sigma}_k^{(1)}\widehat{\sigma}_k^{(2)}}$$

This correlation structure in turn determines $\widehat{\boldsymbol{\Sigma}}_k$ and is given in Table 6 in which for $k = 12$ and $k = 13$ the estimators of the standard deviations have been obtained by exponential extrapolation and those of the coefficients of correlation have estimated by the arithmetic mean over development years $k \in \{0, 1, \ldots, 11\}$ (which appears to be reasonable because of the volatility of the estimated coefficients of correlation in these development years). Table 7 presents the Gauss–Markov estimators of the parameters.

For every subportfolio $p \in \{1, 2\}$, the assumption of the additive model implies that the *incremental loss ratios*

$$\zeta_k^{(p)} \quad = \quad E[Z_{i,k}^{(p)}/v_i^{(p)}]$$

are independent of $i \in \{0, 1, \ldots, n\}$. Thus, the normalized incremental loss ratios

$$\vartheta_k^{(p)} \quad := \quad \frac{\zeta_k^{(p)}}{\sum_{l=0}^{n} \zeta_l^{(p)}}$$

form a *development pattern for incremental quotas* and their sums

$$\gamma_k^{(p)} \quad := \quad \sum_{l=0}^{k} \vartheta_l^{(p)}$$

form a *development pattern for cumulative quotas*; see Schmidt [2006a] as well as Schmidt and Zocher [2008] for a general discussion of development patterns. Since Gauss–Markov estimation is linear, the Gauss–Markov estimator of the incremental loss ratio of subportfolio p is given by the p–th coordinate of the Gauss–Markov estimator $\boldsymbol{\zeta}_k^{\mathrm{GM}}$. Now normalization yields the estimators of the development pattern for incremental quotas presented in Table 8, and summation yields the estimators of the development pattern for cumulative quotas presented in Table 9. As can be seen from Table 9, loss development in auto liability (portfolio 2) is much faster than in general liability (portfolio 1).

Calendar Year	Volume Measure	Development Year 0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	510301	59966	103186	91360	95012	83741	42513	37882	6649	7669	11061	-1738	3572	6823	1893
1	632897	49685	103659	119592	110413	75442	44567	29257	18822	4355	879	4173	2727	-776	
2	658133	51914	118134	149156	105825	78970	40770	14706	17950	10917	2643	10311	1414		
3	723456	84937	188246	134135	139970	74450	65401	49165	21136	596	24048	2548			
4	709312	98921	179408	170201	113161	79641	80364	20414	10324	16204	-265				
5	845673	71708	173879	171295	144076	93694	72161	41545	25245	17497					
6	904378	92350	193157	180707	153816	121196	86753	45547	23202						
7	1156778	95731	217413	240558	202276	101881	104966	59416							
8	1214569	97518	245700	232223	193576	165086	85200								
9	1397123	173686	285730	262920	232999	186415									
10	1832676	139821	297137	372968	364270										
11	2156781	15496	373115	504604											
12	2559345	196124	576847												
13	2456991	204435													

Table 3: Volume Measures and Incremental Losses of Subportfolio 1 (General Liability)

Calendar Year	Volume Measure	Development Year 0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	413213	114423	133538	65021	31358	27139	-377	9889	4477	-316	7108	-1035	103	209	-109
1	537988	152296	152879	71438	41686	22009	25315	7961	4843	-113	1593	848	4383	-1164	
2	589145	144325	162919	106365	50432	55224	7951	8234	1409	2061	669	176	977		
3	523419	145904	161732	79458	46642	29384	15811	3598	5527	-2484	462	-1018			
4	501498	170333	171168	92601	36227	11872	18760	3180	3538	948	-875				
5	598345	189643	171480	85734	61226	18479	13556	7523	1964	88					
6	608376	179022	217202	101080	56183	28362	29791	11244	12568						
7	698993	205908	210139	104397	45277	34888	30193	17563							
8	704129	210951	215478	98618	62846	52435	22824								
9	903557	213426	295796	140211	82259	59209									
10	947326	249508	330502	142126	122023										
11	1134129	258425	427587												
12	1538916	368762	540304												
13	1487234	394997													

Table 4: Volume Measures and Incremental Losses of Subportfolio 2 (Auto Liability)

Auxiliary Estimators	**Development Year k**													
Parameters	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\widehat{\zeta}_k^{(1)}$	0.0885	0.1997	0.2064	0.1753	0.1212	0.0847	0.0485	0.0248	0.0140	0.0119	0.0061	0.0043	0.0052	0.0037
$\widehat{\zeta}_k^{(2)}$	0.2680	0.3290	0.1613	0.0905	0.0558	0.0317	0.0155	0.0091	0.0001	0.0035	-0.0005	0.0035	-0.0010	-0.0003

Table 5: Auxiliary Estimators of the Parameters

Auxiliary Estimators	**Development Year k**													
Correlation Structure	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sigma_k^{(1)}$	22.174	31.575	20.032	14.417	18.923	13.639	13.909	5.787	7.155	12.206	6.091	1.839	3.396	2.825
$\sigma_k^{(2)}$	27.957	27.739	18.195	15.171	15.997	11.737	5.173	4.699	2.052	4.960	1.348	2.998	1.379	1.060
$\rho_k^{(1,2)}$	0.351	-0.026	0.849	0.591	0.371	0.340	0.312	-0.105	0.753	0.332	0.666	-0.139	0.358	0.358

Table 6: Auxiliary Estimators of the Correlation Structure

Gauss–Markov	**Development Year**													
Estimators	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Portfolio 1	0.0886	0.1997	0.2064	0.1749	0.1212	0.0845	0.0484	0.0248	0.0144	0.0120	0.0061	0.0043	0.0053	0.0037
Portfolio 2	0.2677	0.3290	0.1617	0.0906	0.0557	0.0317	0.0155	0.0091	0.0002	0.0035	-0.0005	0.0035	-0.0010	-0.0003

Table 7: Gauss–Markov Estimators of the Parameters

Estimated D.P.	Development Year													
Incremental	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Portfolio 1	0.0891	0.2008	0.2076	0.1759	0.1219	0.0850	0.0487	0.0249	0.0145	0.0121	0.0061	0.0043	0.0053	0.0037
Portfolio 2	0.2770	0.3404	0.1673	0.0938	0.0576	0.0328	0.0160	0.0094	0.0002	0.0036	-0.0005	0.0036	-0.0010	-0.0003

Table 8: Estimators of the Development Pattern for Incremental Quotas

Estimated D.P.	Development Year													
Cumulative	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Portfolio 1	0.0891	0.2900	0.4975	0.6734	0.7953	0.8803	0.9290	0.9539	0.9684	0.9805	0.9866	0.9909	0.9963	1.0000
Portfolio 2	0.2770	0.6174	0.7848	0.8785	0.9362	0.9690	0.9850	0.9944	0.9946	0.9982	0.9977	1.0013	1.0003	1.0000

Table 9: Estimators of the Development Pattern for Cumulative Quotas