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**Siegfried Dietze, Thomas Riedrich and Klaus D. Schmidt**

# Marginal–Sum Equations and Related Fixed–Point Problems

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# Marginal–Sum Equations and Related Fixed–Point Problems

Siegfried Dietze, Thomas Riedrich and Klaus D. Schmidt

Fachrichtung Mathematik
Technische Universität Dresden

**Abstract**

A marginal–sum equation of order $p \geq 2$ is a system of nonlinear equations which in turn are linear equations for polynomials of degree $p$ in $p$ variables. Marginal–sum equations typically arise in the construction of a multiplicative tariff in actuarial mathematics.
In the present paper we study the existence and the radial uniqueness of solutions of marginal–sum equations and the possibility of computing solutions by iteration. To this end, we first show that the marginal–sum problem is equivalent with several fixed–point problems and we then study these fixed–point problems and the corresponding fixed–point iterations.
As a general result, we show that a marginal–sum equation always has a solution and that the solution cannot be unique. Moreover, for the case $p = 2$ we show that the solution is radially unique and can be computed by the fixed–point iteration with respect to a related fixed–point problem and arbitrary initial values. By contrast, for the case $p \geq 3$ we present a numerical example in which for certain initial values the fixed–point iteration is cyclic and hence divergent.

# 1 Introduction

To introduce the marginal–sum problem, we consider an example from actuarial mathematics; see Schmidt [2009] for further details.

**1.1 Example (Multiplicative tariff in motor–car liability insurance).** Consider a portfolio of risks in motor–car liability insurance and assume that the risks are classified with respect to $I$ classes of the power of the engine and $J$ classes of the annual mileage of the car insured. Then the portfolio consists of $I \times J$ cells according to the possible combinations of power and mileage. For every risk in the portfolio the insurance company has to determine an annual premium $\pi_{ij} \in (0, \infty)$ depending on the cell $(i, j) \in \{1, \ldots, I\} \times \{1, \ldots, J\}$ to which the risk belongs. According to the *equivalence principle* of actuarial mathematics, applied to every cell, the premiums should satisfy

$$N_{ij}\, \pi_{ij} \;=\; S_{ij}$$

where $N_{ij}$ is the number of risks in cell $(i, j)$ and $S_{ij}$ is the total claim amount produced by the risks in that cell.
For statistical reasons, it is desirable to reduce the number of the $I \times J$ parameters $\pi_{ij}$. This can be achieved by the requirement that the tariff has a multiplicative structure in the sense that the premiums have the form

$$\pi_{ij} \;=\; \mu\alpha_i\beta_j$$

with parameters $\mu, \alpha_i, \beta_j \in (0, \infty)$ such that the identity

$$N_{ij}\, \mu\alpha_i\beta_j \;=\; S_{ij}$$

holds for every cell $(i, j)$. Summation of these identities over $j$ resp. $i$ yields the *marginal–sum equations*

$$\begin{aligned} \sum_{j=1}^{J} N_{ij}\mu\alpha_i\beta_j &= \sum_{j=1}^{J} S_{ij} \qquad \text{for } i \in \{1, \ldots, I\} \\ \sum_{i=1}^{I} N_{ij}\mu\alpha_i\beta_j &= \sum_{i=1}^{I} S_{ij} \qquad \text{for } j \in \{1, \ldots, J\} \end{aligned}$$

and hence the equation

$$\mu \sum_{i=1}^{I}\sum_{j=1}^{J} N_{ij}\alpha_i\beta_j \;=\; \sum_{i=1}^{I}\sum_{j=1}^{J} S_{ij}\,.$$

Here $\mu$ is interpreted as the *basic premium* and $\alpha_i$ and $\beta_j$ are interpreted as *tariff factors* corresponding to the *tariff classes* $i$ and $j$ of the risks in cell $(i, j) \in \{1, \ldots, I\} \times \{1, \ldots, J\}$ and the last equation corresponds to the equivalence principle, applied not to the cells but to the full portfolio of risks.

The question arises whether or not the marginal–sum equations have a solution and whether or not a solution, if it exists, can be obtained by iteration from arbitrary initial values. Of course, uniqueness of a solution cannot be expected since multiplication of $\mu$ by a constant can be compensated by division of all $\alpha_i$ or all $\beta_j$ by the same constant. Thus, if a solution exists, then there also exist normalized solutions satisfying, e. g., $\mu = 1$ or

$$\sum_{i=1}^{I} \alpha_i = 1 = \sum_{j=1}^{J} \beta_j$$

or

$$\max_{i \in \{1,\dots,I\}} \alpha_i = 1 = \max_{j \in \{1,\dots,J\}} \beta_j \,.$$

The last of these normalizations is particularly popular since in that case the parameters $\alpha_i$ and $\beta_j$ represent a discount on the basic premium.

The marginal–sum problem of order $p = 2$ arising in the previous example has been studied by Dietze, Riedrich and Schmidt [2006]. In the present paper we shall study the marginal–sum problem for any order $p \geq 2$. This general version of the marginal–sum problem is of interest in actuarial mathematics since in most cases a multiplicative tariff involves more than two tariff factors.

We shall use the following notation: For $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote the vector space of all $n$–dimensional vectors of real numbers, written as row vectors $\mathbf{x} = (x_1, \dots, x_n)$, and, for $i \in \{1, \dots, n\}$, let $\mathbf{e}_i$ denote the $i$–th unit vector of $\mathbb{R}^n$. We denote by $\|\,.\,\|$ an arbitrary norm on $\mathbb{R}^n$ and by $\|\,.\,\|_L$ the norm given by $\|\mathbf{x}\|_L := \sum_{i=1}^n |x_i|$. We also denote by $\leq$ the natural order relation on $\mathbb{R}^n$ such that $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$ holds for all $i \in \{1, \dots, n\}$, and we define $\mathbb{R}^n_+ := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{0} \leq \mathbf{x}\}$.

## 2 The marginal–sum equation

Consider $p \in \mathbb{N}$ with $p \geq 2$ and $n_1, \dots, n_p \in \mathbb{N}$ and define

$$\mathbf{J} \;:=\; \prod_{k=1}^{p} \{1, \dots, n_k\} = \{1, \dots, n_1\} \times \cdots \times \{1, \dots, n_p\} \,.$$

The elements of $\mathbf{J}$ are referred to as *multiindices* and for a multiindex $\mathbf{j} \in \mathbf{J}$ we shall also write

$$\mathbf{j} \;=\; (j_1, \dots, j_p) \,.$$

For $k \in \{1, \dots, p\}$ and $i \in \{1, \dots, n_k\}$ we define

$$\mathbf{J}(k,i) \;:=\; \bigl\{\mathbf{j} \in \mathbf{J} \bigm| j_k = i\bigr\} \,.$$

We assume that for every multiindex $\mathbf{j} \in \mathbf{J}$ we are given real numbers $N_{\mathbf{j}} \in (0, \infty)$ and $S_{\mathbf{j}} \in \mathbb{R}_+$ and that for every choice of $k \in \{1, \dots, p\}$ and $i \in \{1, \dots, n_k\}$ there exists some $\mathbf{j} \in \mathbf{J}(k,i)$ such that $S_{\mathbf{j}} \in (0, \infty)$.

In what follows we study the existence and uniqueness of a solution

$$\Bigl(\mu, \{a_i^{(k)}\}_{k\in\{1,\dots,p\},i\in\{1,\dots,n_k\}}\Bigr)$$

with $\mu \in (0,\infty)$ and $a_i^{(k)} \in (0,\infty)$ for all $k \in \{1,\dots,p\}$ and $i \in \{1,\dots,n_k\}$ of the *marginal–sum equations*

$$\mu a_i^{(k)} \sum_{\mathbf{j}\in\mathbf{J}(k,i)} N_{\mathbf{j}} \prod_{r\in\{1,\dots,p\}\setminus\{k\}} a_{j_r}^{(r)} \;=\; \sum_{\mathbf{j}\in\mathbf{J}(k,i)} S_{\mathbf{j}}$$

as well as related fixed–point equations and the problem of whether or not a solution can be obtained by iteration.

To simplify the notation, we shall henceforth use vectors and block vectors of suitable dimensions. A vector $\mathbf{a}^{(k)} \in \mathbb{R}^{n_k}$ with $k \in \{1,\dots,p\}$ will also be written as

$$\mathbf{a}^{(k)} \;=\; (a_1^{(k)},\dots,a_{n_k}^{(k)})\,.$$

Correspondingly, a vector $\mathbf{A} \in \prod_{r\in\{1,\dots,p\}} \mathbb{R}^{n_r}$ will also be written as

$$\mathbf{A} \;=\; (\mathbf{a}^{(1)},\dots,\mathbf{a}^{(p)})$$

and, for $k \in \{1,\dots,p\}$, a vector $\mathbf{C} \in \prod_{r\in\{1,\dots,p\}\setminus\{k\}} \mathbb{R}^{n_r}$ will also be written as

$$\mathbf{C} \;=\; (\mathbf{c}^{(1)},\dots,\mathbf{c}^{(k-1)},\mathbf{c}^{(k+1)},\dots,\mathbf{c}^{(p)})\,.$$

For $\mathbf{A} \in \prod_{r\in\{1,\dots,p\}} \mathbb{R}^{n_r}$ and $k \in \{1,\dots,p\}$ we also consider the reduced vector

$$\mathbf{A}^{(k)} \;:=\; (\mathbf{a}^{(1)},\dots,\mathbf{a}^{(k-1)},\mathbf{a}^{(k+1)},\dots,\mathbf{a}^{(p)})$$

which results from $\mathbf{A}$ by deleting the subvector $\mathbf{a}^{(k)}$. Define also

$$D \;:=\; \left\{\mathbf{A} \in \prod_{r=1}^{p} \mathbb{R}^{n_r} \;\middle|\; \mathbf{a}^{(r)} \in \mathbb{R}_+^{n_r} \setminus \{\mathbf{0}\} \text{ for all } r \in \{1,\dots,p\}\right\}$$

and, for $k \in \{1,\dots,p\}$, define

$$D^{(k)} \;:=\; \left\{\mathbf{C} \in \prod_{r\in\{1,\dots,p\}\setminus\{k\}} \mathbb{R}^{n_r} \;\middle|\; \mathbf{c}^{(r)} \in \mathbb{R}_+^{n_r} \setminus \{\mathbf{0}\} \text{ for all } r \in \{1,\dots,p\}\setminus\{k\}\right\}.$$

For $k \in \{1,\dots,p\}$ and $i \in \{1,\dots,n_k\}$ we define the map $\xi_i^{(k)} : D^{(k)} \to (0,\infty)$ by letting

$$\xi_i^{(k)}(\mathbf{C}) \;:=\; \frac{\sum_{\mathbf{j}\in\mathbf{J}(k,i)} S_{\mathbf{j}}}{\sum_{\mathbf{j}\in\mathbf{J}(k,i)} N_{\mathbf{j}} \prod_{r\in\{1,\dots,p\}\setminus\{k\}} c_{j_r}^{(r)}}\,.$$

For $k \in \{1,\dots,p\}$ this yields the map $\boldsymbol{\xi}^{(k)} : D \to (0,\infty)^{n_k}$ given by

$$\boldsymbol{\xi}^{(k)} \;:=\; (\xi_1^{(k)},\dots,\xi_{n_k}^{(k)})$$

and these maps are combined to yield the map $\boldsymbol{\Xi} : D \to \prod_{k=1}^p (0,\infty)^{n_k} \subseteq D$ given by

$$\boldsymbol{\Xi}(\mathbf{A}) \;:=\; (\boldsymbol{\xi}^{(1)}(\mathbf{A}^{(1)}),\dots,\boldsymbol{\xi}^{(p)}(\mathbf{A}^{(p)}))\,.$$

Then $(\mu,\mathbf{A})$ with $\mu \in (0,\infty)$ and $\mathbf{A} \in \prod_{r=1}^p (0,\infty)^{n_r}$ is a solution of the *marginal–sum equation*

$$\mu\mathbf{A} \;=\; \boldsymbol{\Xi}(\mathbf{A})$$

if and only if $(\mu, \{\alpha_i^{(k)}\}_{k\in\{1,\dots,p\},i\in\{1,\dots,n_k\}})$ is a solution of the *marginal–sum equations*

$$\mu\alpha_i^{(k)} \sum_{\mathbf{j}\in\mathbf{J}(k,i)} N_{\mathbf{j}} \prod_{r\in\{1,\dots,p\}\setminus\{k\}} a_{j_r}^{(r)} \;=\; \sum_{\mathbf{j}\in\mathbf{J}(k,i)} S_{\mathbf{j}}\,.$$

Summation of the previous identity over $i \in \{1,\dots,n_k\}$ yields the following lemma:

**2.1 Lemma.** *If $(\mu,\mathbf{A})$ is a solution of the marginal–sum equation, then*

$$\mu \;=\; \frac{\sum_{\mathbf{j}\in\mathbf{J}} S_{\mathbf{j}}}{\sum_{\mathbf{j}\in\mathbf{J}} N_{\mathbf{j}} \prod_{r=1}^p a_{j_r}^{(r)}}\,.$$

The following lemma provides some obvious but useful properties of the map $\boldsymbol{\Xi}$:

**2.2 Lemma.** *Consider $k \in \{1,\dots,p\}$.*
(1) *The map $\boldsymbol{\xi}^{(k)}$ is continuous.*
(2) *If $\mathbf{C},\mathbf{D} \in D^{(k)}$ are such that $\mathbf{C} \le \mathbf{D}$, then $\boldsymbol{\xi}^{(k)}(\mathbf{C}) \ge \boldsymbol{\xi}^{(k)}(\mathbf{D})$.*
(3) *The identity*

$$\boldsymbol{\xi}^{(k)}(t_1\mathbf{c}^{(1)},\dots,t_{k-1}\mathbf{c}^{(k-1)},t_{k+1}\mathbf{c}^{(k+1)},\dots,t_p\mathbf{c}^{(p)}) \;=\; \frac{1}{\prod_{r\in\{1,\dots,p\}\setminus\{k\}} t_r}\,\boldsymbol{\xi}^{(k)}(\mathbf{C})$$

*holds for every choice of $\mathbf{C} \in D^{(k)}$ and $t_1,\dots,t_{k-1},t_{k+1},\dots,t_p \in (0,\infty)$.*

As a first application of Lemma 2.2, we show that whenever the marginal–sum equation has a solution then it has infinitely many solutions:

**2.3 Theorem.** *Assume that the marginal–sum equation has a solution* $(\mu, \mathbf{A})$. *Then, for every choice of* $t_1, \ldots, t_p \in (0, \infty)$,

$$\left( \frac{\mu}{\prod_{r=1}^p t_r}, \; (t_1 \mathbf{a}^{(1)}, \ldots, t_p \mathbf{a}^{(p)}) \right)$$

*is a solution as well and, in particular, each of*

$$\left( 1, \; (\mu^{1/p} \mathbf{a}^{(1)}, \ldots, \mu^{1/p} \mathbf{a}^{(p)}) \right)$$

*and*

$$\left( \mu \prod_{r=1}^p \|\mathbf{a}^{(r)}\| \, , \; \left( \frac{\mathbf{a}^{(1)}}{\|\mathbf{a}^{(1)}\|}, \ldots, \frac{\mathbf{a}^{(p)}}{\|\mathbf{a}^{(p)}\|} \right) \right)$$

*is a solution.*

**Proof.** For every $k \in \{1, \ldots, p\}$, the assumption yields $\mu \mathbf{a}^{(k)} = \boldsymbol{\xi}^{(k)}(\mathbf{A})$. Using Lemma 2.2 we thus obtain

$$\begin{aligned} \frac{\mu}{\prod_{r=1}^p t_r} \, t_k \mathbf{a}^{(k)} &= \frac{1}{\prod_{r \in \{1,\ldots,p\} \setminus \{k\}} t_r} \, \boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) \\ &= \boldsymbol{\xi}^{(k)}(t_1 \mathbf{a}^{(1)}, \ldots, t_{k-1} \mathbf{a}^{(k-1)}, t_{k+1} \mathbf{a}^{(k+1)}, \ldots, t_p \mathbf{a}^{(p)}) \end{aligned}$$

which proves the assertion. □

By Theorem 2.3, every solution $(\mu, \mathbf{A})$ of the marginal–sum equation may be normalized such that, e. g., either $\mu = 1$ or $\|\mathbf{a}^{(k)}\| = 1$ holds for all $k \in \{1, \ldots, p\}$. Theorem 2.3 also implies that the marginal–sum equation cannot have a unique solution and it suggests the following definition: We say that the marginal–sum equation has a *radially unique solution* if it has a solution and if for any two solutions $(\mu, \mathbf{A})$ and $(\nu, \mathbf{B})$ there exist $t_1, \ldots, t_p \in (0, \infty)$ satisfying

$$\mathbf{b}^{(k)} = t_k \mathbf{a}^{(k)}$$

for all $k \in \{1, \ldots, p\}$ (and hence also $\nu = \mu / \prod_{r=1}^p t_r$, by equation (2.1) or Theorem 2.3).

# 3 Equivalent fixed–point problems

In the present section we show that the marginal–sum equation is closely related to several fixed–point equations.

### A reduced fixed–point problem

The marginal–sum equation $\mu \mathbf{A} = \boldsymbol{\Xi}(\mathbf{A})$ can be reduced by eliminating one of the components $\mathbf{a}^{(k)}$ of $\mathbf{A}$. Here we consider elimination of the last component $\mathbf{a}^{(p)}$. For $k \in \{1, \ldots, p-1\}$, define a map $\boldsymbol{\varphi}^{(k)} : D^{(p)} \to (0, \infty)^{n_k}$ by letting

$$\boldsymbol{\varphi}^{(k)}(\mathbf{C}) := \boldsymbol{\xi}^{(k)}(\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(k-1)}, \mathbf{c}^{(k+1)}, \ldots, \mathbf{c}^{(p-1)}, \boldsymbol{\xi}^{(p)}(\mathbf{C})) \, .$$

These maps are combined to yield the map $\boldsymbol{\Phi} : D^{(p)} \to \prod_{r=1}^{p-1}(0,\infty)^{n_r}$ given by

$$\boldsymbol{\Phi} \;:=\; (\boldsymbol{\varphi}^{(1)},\ldots,\boldsymbol{\varphi}^{(p-1)})\,.$$

The following lemma is an immediate consequence of Lemma 2.2:

**3.1 Lemma.** *Consider $k \in \{1,\ldots,p-1\}$.*
(1) *The map $\boldsymbol{\varphi}^{(k)}$ is continuous.*
(2) *If $\mathbf{C},\mathbf{D} \in D^{(p)}$ are such that $\mathbf{c}^{(k)} \le \mathbf{d}^{(k)}$, then $\boldsymbol{\varphi}^{(k)}(\mathbf{C}) \le \boldsymbol{\varphi}^{(k)}(\mathbf{D})$.*
(3) *If $\mathbf{C} \in D^{(p)}$ and $t_1,\ldots,t_{p-1} \in (0,\infty)$, then $\boldsymbol{\varphi}^{(k)}(t_1\mathbf{c}^{(1)},\ldots,t_{p-1}\mathbf{c}^{(p-1)}) = t_k\,\boldsymbol{\varphi}^{(k)}(\mathbf{C})$.*
(4) *If $\mathbf{C} \in D^{(p)}$ and $t \in (0,\infty)$, then $\boldsymbol{\varphi}^{(k)}(t\mathbf{C}) = t\boldsymbol{\varphi}^{(k)}(\mathbf{C})$.*
(5) *If $\mathbf{C}$ is fixed–point of $\boldsymbol{\Phi}$ and if $t_1,\ldots,t_{p-1} \in (0,\infty)$, then $(t_1\mathbf{c}^{(1)},\ldots,t_{p-1}\mathbf{c}^{(p-1)})$ is a fixed–point of $\boldsymbol{\Phi}$ as well.*

By Theorem 2.3, the marginal–sum equation has a solution if and only if it has a solution $(\mu,\mathbf{A})$ with $\mu = 1$. The following result provides a connection between the existence of a solution of the marginal–sum equation and that of a fixed–point of $\boldsymbol{\Phi}$:

**3.2 Theorem.** *For $\mathbf{A} \in D$ the following assertions are equivalent:*
(a) *$(1,\mathbf{A})$ is a solution of the marginal–sum equation.*
(b) *$\mathbf{A}^{(p)}$ is a fixed–point of $\boldsymbol{\Phi}$ and $\mathbf{A}$ satisfies $\boldsymbol{\xi}^{(p)}(\mathbf{A}^{(p)}) = \mathbf{a}^{(p)}$.*

**Proof.** Assume first that (a) holds. Then we have $\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) = \mathbf{a}^{(k)}$ for all $k \in \{1,\ldots,p\}$. In particular, we have $\boldsymbol{\xi}^{(p)}(\mathbf{A}^{(p)}) = \mathbf{a}^{(p)}$ and hence $\boldsymbol{\varphi}^{(k)}(\mathbf{A}^{(p)}) = \boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) = \mathbf{a}^{(k)}$ for all $k \in \{1,\ldots,p-1\}$. Therefore, (a) implies (b).
Assume now that (b) holds. Then we have $\boldsymbol{\xi}^{(p)}(\mathbf{A}^{(p)}) = \mathbf{a}^{(p)}$ and hence $\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) = \boldsymbol{\varphi}^{(k)}(\mathbf{A}^{(p)}) = \mathbf{a}^{(k)}$ for all $k \in \{1,\ldots,p-1\}$. Therefore, (b) implies (a). □

Lemma 3.1 implies that the map $\boldsymbol{\Phi}$ cannot have a unique fixed–point. We say that the map $\boldsymbol{\Phi}$ has a *radially unique fixed–point* if it has a fixed–point and if for any two fixed–points $\mathbf{C}$ and $\mathbf{D}$ of $\boldsymbol{\Phi}$ there exist $t_1,\ldots,t_{p-1} \in (0,\infty)$ satisfying

$$\mathbf{d}^{(k)} \;=\; t_k\mathbf{c}^{(k)}$$

for all $k \in \{1,\ldots,p-1\}$.

**3.3 Corollary.** *The following assertions are equivalent:*
(a) *The marginal–sum equation has a radially unique solution.*
(b) *The map $\boldsymbol{\Phi}$ has a radially unique fixed–point.*

**Proof.** Assume first that (a) holds. For every fixed–point $\mathbf{C}$ of $\boldsymbol{\Phi}$ there exists some $\mathbf{A}$ satisfying $\mathbf{A}^{(p)} = \mathbf{C}$ and $\mathbf{a}^{(p)} = \boldsymbol{\xi}^{(p)}(\mathbf{C})$ and Theorem 3.2 implies that $(1,\mathbf{A})$ is a solution of the marginal–sum equation. Since, by assumption, the marginal–sum equation has a radially unique solution, it follows that the map $\boldsymbol{\Phi}$ has a radially unique fixed–point. Therefore, (a) implies (b).

Assume now that (b) holds and consider two solutions $(\mu, \mathbf{A})$ and $(\nu, \mathbf{B})$ of the marginal–sum equation. Because of Theorem 2.3, we may assume that $\mu = 1 = \nu$. Now Theorem 3.2 implies that $\mathbf{A}^{(p)}$ and $\mathbf{B}^{(p)}$ are fixed–points of $\mathbf{\Phi}$ and satisfy $\boldsymbol{\xi}^{(p)}(\mathbf{A}^{(p)}) = \mathbf{a}^{(p)}$ and $\boldsymbol{\xi}^{(p)}(\mathbf{B}^{(p)}) = \mathbf{b}^{(p)}$. This yields the existence of $t_1, \dots, t_{p-1} \in (0, \infty)$ satisfying

$$\mathbf{b}^{(k)} \;=\; t_k \mathbf{a}^{(k)}$$

for all $k \in \{1, \dots, p-1\}$. Using Lemma 2.2, we also obtain

$$\begin{aligned}
\mathbf{b}^{(p)} &= \boldsymbol{\xi}^{(p)}(\mathbf{B}^{(p)}) \\
&= \boldsymbol{\xi}^{(p)}(t_1 \mathbf{a}^{(1)}, \dots, t_{p-1} \mathbf{a}^{(p-1)}) \\
&= \frac{1}{\prod_{r=1}^{p-1} t_r} \, \boldsymbol{\xi}^{(p)}(\mathbf{A}^{(p)}) \\
&= \frac{1}{\prod_{r=1}^{p-1} t_r} \, \mathbf{a}^{(p)}
\end{aligned}$$

which yields the existence of some $t_p \in (0, \infty)$ satisfying

$$\mathbf{b}^{(p)} \;=\; t_p \mathbf{a}^{(p)} \,.$$

Therefore, (b) implies (a). $\square$

**The canonical fixed–point problem**

As noted before, the marginal–sum equation $\mu \mathbf{A} = \Xi(\mathbf{A})$ has a solution if and only if it has a solution $(\mu, \mathbf{A})$ with $\mu = 1$. Therefore, the canonical fixed–point problem associated with the marginal–sum equation is the fixed–point problem for the map $\Xi$. The following result is obvious:

**3.4 Theorem.** *For $\mathbf{A} \in D$ the following assertions are equivalent:*
(a) $(1, \mathbf{A})$ *is a solution of the marginal–sum equation.*
(b) $\mathbf{A}$ *is a fixed–point of* $\Xi$.

We say that the map $\Xi$ has a *radially unique fixed–point* if it has a fixed–point and if for any two fixed–points $\mathbf{A}$ and $\mathbf{B}$ of $\Xi$ there exist $t_1, \dots, t_p \in (0, \infty)$ satisfying

$$\mathbf{b}^{(k)} \;=\; t_k \mathbf{a}^{(k)}$$

for all $k \in \{1, \dots, p\}$. The following result is obvious:

**3.5 Corollary.** *The following assertions are equivalent:*
(a) *The marginal–sum equation has a radially unique solution.*
(b) *The map* $\Xi$ *has a radially unique fixed–point.*

Because of Theorem 3.4, the map $\Xi$ cannot have a unique fixed–point.

**The normalized fixed–point problem**

For $k \in \{1, \dots, p\}$ define a map $\bar{\boldsymbol{\xi}}^{(k)} : D^{(k)} \to (0, \infty)^{n_k}$ by letting

$$\bar{\boldsymbol{\xi}}^{(k)}(\mathbf{C}) \quad := \quad \frac{\boldsymbol{\xi}^{(k)}(\mathbf{C})}{\|\boldsymbol{\xi}^{(k)}(\mathbf{C})\|}$$

and these maps are combined to yield the map $\bar{\boldsymbol{\Xi}} : D \to \prod_{k=1}^{p} (0, \infty)^{n_k} \subseteq D$ given by

$$\bar{\boldsymbol{\Xi}}(\mathbf{A}) \quad := \quad (\bar{\boldsymbol{\xi}}^{(1)}(\mathbf{A}^{(1)}), \dots, \bar{\boldsymbol{\xi}}^{(p)}(\mathbf{A}^{(p)}))\,.$$

As $\bar{\boldsymbol{\Xi}}$ depends on the norm $\|\,.\,\|$, we shall write $\bar{\boldsymbol{\Xi}}_L$ in the case where $\|\,.\,\| = \|\,.\,\|_L$. Because of Lemma 2.2, the map $\bar{\boldsymbol{\Xi}}$ is continuous. We have the following result:

**3.6 Theorem.** *For $\mathbf{A} \in D$ satisfying $\|\mathbf{a}^{(k)}\| = 1$ for all $k \in \{1, \dots, p\}$, the following assertions are equivalent:*
(a) $(\mu, \mathbf{A})$ *with*

$$\mu \quad = \quad \frac{\sum_{\mathbf{j} \in \mathbf{J}} S_{\mathbf{j}}}{\sum_{\mathbf{j} \in \mathbf{J}} N_{\mathbf{j}} \prod_{r=1}^{p} a_{j_r}^{(r)}}$$

*is a solution of the marginal–sum equation.*
(b) $\mathbf{A}$ *is a fixed–point of* $\bar{\boldsymbol{\Xi}}$.

**Proof.** Assume first that (a) holds. Then we have, for every $k \in \{1, \dots, p\}$,

$$\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) \quad = \quad \mu \mathbf{a}^{(k)}$$

and hence, since $\|\mathbf{a}^{(k)}\| = 1$,

$$\bar{\boldsymbol{\xi}}^{(k)}(\mathbf{A}^{(k)}) \quad = \quad \mathbf{a}^{(k)}$$

which means that $\mathbf{A}$ is a fixed–point of $\bar{\boldsymbol{\Xi}}$. Therefore, (a) implies (b).
Assume now that (b) holds. Then we have, for every $k \in \{1, \dots, p\}$,

$$\bar{\boldsymbol{\xi}}^{(k)}(\mathbf{A}^{(k)}) \quad = \quad \mathbf{a}^{(k)}$$

and hence

$$\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) \quad = \quad \|\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)})\|\, \mathbf{a}^{(k)}$$

which yields, for every $i \in \{1, \dots, n_k\}$,

$$\frac{\sum_{\mathbf{j} \in \mathbf{J}(k,i)} S_{\mathbf{j}}}{\sum_{\mathbf{j} \in \mathbf{J}(k,i)} N_{\mathbf{j}} \prod_{r \in \{1,\dots,p\} \setminus \{k\}} a_{j_r}^{(r)}} \quad = \quad \xi_i^{(k)}(\mathbf{A}^{(k)}) \quad = \quad \|\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)})\|\, a_i^{(k)}$$

and hence

$$\frac{\sum_{\mathbf{j} \in \mathbf{J}(k,i)} S_{\mathbf{j}}}{\sum_{\mathbf{j} \in \mathbf{J}(k,i)} N_{\mathbf{j}} \prod_{r=1}^{p} a_{j_r}^{(r)}} \quad = \quad \|\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)})\|\,.$$

Multiplication with the denominator and subsequent summation over $i \in \{1, \ldots, n_k\}$ and putting

$$\mu \quad := \quad \frac{\sum_{\mathbf{j} \in \mathbf{J}} S_{\mathbf{j}}}{\sum_{\mathbf{j} \in \mathbf{J}} N_{\mathbf{j}} \prod_{r=1}^{p} a_{j_r}^{(r)}}$$

yields $\mu = \|\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)})\|$ and hence

$$\boldsymbol{\xi}^{(k)}(\mathbf{A}^{(k)}) \quad = \quad \mu \mathbf{a}^{(k)}$$

for all $k \in \{1, \ldots, p\}$, which means that $(\mu, \mathbf{A})$ is a solution of the marginal–sum equation. Therefore, (b) implies (a). $\square$

We thus obtain the following result:

**3.7 Corollary.** *The following assertions are equivalent:*
(a) *The marginal–sum equation has a radially unique solution.*
(b) *The map $\bar{\boldsymbol{\Xi}}$ has a unique fixed–point.*

**Proof.** The marginal–sum equation has a radially unique solution if and only if it has a unique solution $(\mu, \mathbf{A})$ satisfying $\|\mathbf{a}^{(k)}\| = 1$ for all $k \in \{1, \ldots, p\}$. The assertion now follows from Theorem 3.6. $\square$

## 4 Existence of solutions and fixed–points

In the case $p = 2$, the existence of a solution of the marginal–sum equation has been proven by Dietze, Riedrich and Schmidt [2006] for the first time. The proof given there proceeds via the fixed–point equation for the map $\boldsymbol{\Phi}$ and is based on Brouwer's fixed–point theorem; it is in fact analogous to a proof given by Morishima (1964; Appendix, Section 2) for the existence in certain nonlinear eigenvalue problems; see also Krasnoselskii (1964; Theorem 5.5). A slightly different proof proceeding via the fixed–point equation for $\bar{\boldsymbol{\Xi}}$ may be found in Göpfert, Riedrich and Tammer [2009]. Each of these proofs can be extended to the general case and we present here an extension of the second one.

**4.1 Theorem.** *The marginal–sum equation has a solution and each of the maps $\boldsymbol{\Phi}$, $\boldsymbol{\Xi}$ and $\bar{\boldsymbol{\Xi}}$ has a fixed–point.*

**Proof.** Due to the results of Section 3 and Theorem 2.3, it is sufficient to show that the map $\bar{\boldsymbol{\Xi}}_L$ has a fixed–point. To this end we consider the set

$$D_L \quad := \quad \left\{ \mathbf{A} \in \prod_{r=1}^{p} \mathbb{R}^{n_r} \;\middle|\; \mathbf{a}^{(r)} \in \mathbb{R}_+^{n_r} \text{ and } \|\mathbf{a}^{(r)}\|_L = 1 \text{ for all } r \in \{1, \ldots, p\} \right\}.$$

Then we have $D_L \subseteq D$ and $\bar{\boldsymbol{\Xi}}_L$ maps the set $D_L$ into itself. Moreover, the set $D_L$ is convex and compact, and Lemma 2.2 implies that the map $\bar{\boldsymbol{\Xi}}_L$ is continuous. Now Brouwer's fixed–point theorem ensures the existence of a fixed–point of $\bar{\boldsymbol{\Xi}}_L$; see e. g. Granas and Dugundji [2003]. $\square$

## 5 The case $p = 2$: Radial uniqueness and iteration

Throughout this section we assume that $p = 2$. We shall show that

- the marginal–sum equation and each of the fixed–point equations for $\mathbf{\Phi}$ resp. $\mathbf{\Xi}$ has a radially unique solution,
- the map $\bar{\mathbf{\Xi}}$ has a unique fixed–point, and
- the fixed–point iteration for $\mathbf{\Phi}$ resp. $\bar{\mathbf{\Xi}}$ converges, for every initial value, to a fixed–point.

To simplify the notation, we write $I, J$ instead of $n_1, n_2$ and we also write $\boldsymbol{\alpha}, \boldsymbol{\beta}$ instead of $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}$ and $\mathbf{H}, \mathbf{G}$ instead of $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}$. Then the maps $\mathbf{H}$ and $\mathbf{G}$ are continuous and homogeneous of degree $-1$ and the maps $\mathbf{\Xi}$ and $\bar{\mathbf{\Xi}}$ satisfy

$$\mathbf{\Xi}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \;=\; (\mathbf{H}(\boldsymbol{\beta}), \mathbf{G}(\boldsymbol{\alpha}))$$

and

$$\bar{\mathbf{\Xi}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \;=\; \left( \frac{\mathbf{H}(\boldsymbol{\beta})}{\|\mathbf{H}(\boldsymbol{\beta})\|}, \frac{\mathbf{G}(\boldsymbol{\alpha})}{\|\mathbf{G}(\boldsymbol{\alpha})\|} \right)$$

and the marginal–sum equation $\mu(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{\Xi}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ becomes

$$\mu(\boldsymbol{\alpha}, \boldsymbol{\beta}) \;=\; (\mathbf{H}(\boldsymbol{\beta}), \mathbf{G}(\boldsymbol{\alpha}))\,.$$

We also have $\mathbf{\Phi} = \boldsymbol{\varphi}^{(1)}$ and hence

$$\mathbf{\Phi} \;=\; \mathbf{H} \circ \mathbf{G}$$

and we define

$$\mathbf{\Psi} \;:=\; \mathbf{G} \circ \mathbf{H}\,.$$

Then $\mathbf{\Psi}$ is the map of the fixed–point problem resulting from the marginal–sum equation by elimination of the first variable instead of the last. By Lemma 3.1, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are continuous, monotone increasing and homogeneous of degree 1. The fixed–point equations for $\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Xi}, \bar{\mathbf{\Xi}}$ read

$$\begin{aligned} \mathbf{\Phi}(\boldsymbol{\alpha}) &= \boldsymbol{\alpha} \\ \mathbf{\Psi}(\boldsymbol{\beta}) &= \boldsymbol{\beta} \\ \mathbf{\Xi}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= (\boldsymbol{\alpha}, \boldsymbol{\beta}) \\ \bar{\mathbf{\Xi}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= (\boldsymbol{\alpha}, \boldsymbol{\beta}) \end{aligned}$$

and for each of these fixed–point problems we shall now study the radial uniqueness of the solution and the question whether or not the respective fixed–point iteration converged to a fixed–point. Let us first consider the fixed–point iteration for $\mathbf{\Phi}$.

**5.1 Theorem.** *The map $\mathbf{\Phi}$ has a radially unique fixed–point and for every $\boldsymbol{\alpha}^{(0)}$ the sequence $\{\mathbf{\Phi}^n(\boldsymbol{\alpha}^{(0)})\}_{n \in \mathbb{N}_0}$ converges to a fixed–point of $\mathbf{\Phi}$.*

**Proof.** Because of Theorem 4.1, the map $\mathbf{\Phi}$ has a fixed–point $\boldsymbol{\alpha}^*$ and radial uniqueness of the fixed–point will follow from the convergence of the fixed–point iteration. For $n \in \mathbb{N}_0$, define

$$\mathbf{x}^{(n)} \; := \; \mathbf{\Phi}^n(\boldsymbol{\alpha}^{(0)})$$

as well as

$$\mathbf{u}^{(n)} \; := \; \lambda^{(n)}\boldsymbol{\alpha}^* \qquad \text{and} \qquad \mathbf{z}^{(n)} \; := \; \mu^{(n)}\boldsymbol{\alpha}^*$$

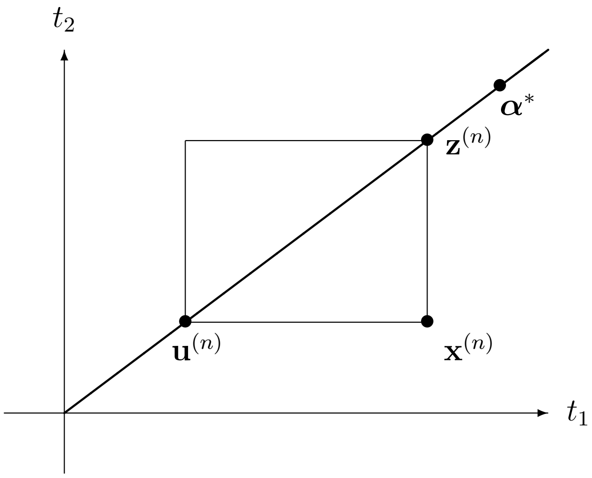
where

$$\lambda^{(n)} \; := \; \min_{i\in\{1,\ldots,I\}} \frac{x_i^{(n)}}{\alpha_i^*} \qquad \text{and} \qquad \mu^{(n)} \; := \; \max_{i\in\{1,\ldots,I\}} \frac{x_i^{(n)}}{\alpha_i^*} \, .$$

Then we have

$$\mathbf{u}^{(n)} \; \leq \; \mathbf{x}^{(n)} \; \leq \; \mathbf{z}^{(n)} \, .$$

This construction is illustrated by the following picture (for the case $I = 2$):




Since $\mathbf{u}^{(n)}$ and $\mathbf{z}^{(n)}$ are fixed–points of $\mathbf{\Phi}$ and since $\mathbf{\Phi}$ is monotone increasing, we obtain $\lambda^{(n)}\boldsymbol{\alpha}^* = \mathbf{u}^{(n)} = \mathbf{\Phi}(\mathbf{u}^{(n)}) \leq \mathbf{\Phi}(\mathbf{x}^{(n)}) \leq \mathbf{\Phi}(\mathbf{z}^{(n)}) = \mathbf{z}^{(n)} = \mu^{(n)}\boldsymbol{\alpha}^*$, hence

$$\lambda^{(n)}\boldsymbol{\alpha}^* \; \leq \; \mathbf{x}^{(n+1)} \; \leq \; \mu^{(n)}\boldsymbol{\alpha}^*$$

and thus

$$\lambda^{(n)} \; \leq \; \lambda^{(n+1)} \; \leq \; \mu^{(n+1)} \; \leq \; \mu^{(n)} \, .$$

Therefore, there exist some $\lambda, \mu \in (0, \infty)$ satisfying

$$\lambda \; = \; \lim_{n\to\infty} \lambda^{(n)} \; \leq \; \lim_{n\to\infty} \mu^{(n)} \; = \; \mu$$

and we shall prove below that $\lambda = \mu$. Then we have

$$\lim_{n\to\infty} \mathbf{u}^{(n)} \; = \; \lambda\,\boldsymbol{\alpha}^* \; = \; \mu\,\boldsymbol{\alpha}^* \; = \; \lim_{n\to\infty} \mathbf{z}^{(n)}$$

and it follows that also the sequence $\{\mathbf{x}^{(n)}\}_{n\in\mathbb{N}_0}$ converges to $\lambda\,\boldsymbol{\alpha}^* = \mu\,\boldsymbol{\alpha}^*$, which is a fixed–point of $\boldsymbol{\Phi}$ since $\boldsymbol{\alpha}^*$ is a fixed–point of $\boldsymbol{\Phi}$ and since $\boldsymbol{\Phi}$ is homogeneous of degree 1. Since $\boldsymbol{\alpha}^*$ is an arbitrary fixed–point of $\boldsymbol{\Phi}$, it then follows that $\boldsymbol{\Phi}$ has a radially unique fixed–point.
Let us thus prove that $\lambda = \mu$. For $n \in \mathbb{N}_0$ we define

$$\begin{aligned}
\mathbf{v}^{(n)} &:= \mathbf{u}^{(n)} + \Big(\mu^{(n)} - \lambda^{(n)}\Big)\alpha^*_{i(n)}\,\mathbf{e}_{i(n)} \\
\mathbf{y}^{(n)} &:= \mathbf{z}^{(n)} + \Big(\lambda^{(n)} - \mu^{(n)}\Big)\alpha^*_{j(n)}\,\mathbf{e}_{j(n)}
\end{aligned}$$

where $i(n), j(n) \in \{1,\ldots,I\}$ satisfy

$$\frac{x^{(n)}_{j(n)}}{\alpha^*_{j(n)}} = \lambda^{(n)} \qquad \text{and} \qquad \frac{x^{(n)}_{i(n)}}{\alpha^*_{i(n)}} = \mu^{(n)}\,.$$

Then we have

$$\mathbf{u}^{(n)} \;\le\; \mathbf{v}^{(n)} \;\le\; \mathbf{x}^{(n)} \;\le\; \mathbf{y}^{(n)} \;\le\; \mathbf{z}^{(n)}\,.$$

Since every coordinate of $\boldsymbol{\Phi}$ is continuously partially differentiable with

$$\frac{\partial \Phi_i}{\partial t_j}(\mathbf{t}) \;>\; 0$$

and since the set $[\mathbf{u}^{(0)}, \mathbf{z}^{(0)}]$ is compact, there exists some $\gamma \in (0,\infty)$ such that

$$\frac{\partial \Phi_i}{\partial t_j}(\mathbf{t}) \;\ge\; \gamma$$

holds for all $i, j \in \{1,\ldots,I\}$ and $\mathbf{t} \in [\mathbf{u}^{(0)}, \mathbf{z}^{(0)}]$. Since $\mathbf{z}^{(n)}$ is a fixed–point of $\boldsymbol{\Phi}$ and since $\boldsymbol{\Phi}$ is monotone increasing, we obtain

$$\begin{aligned}
\mathbf{z}^{(n)} - \mathbf{x}^{(n+1)} &= \boldsymbol{\Phi}(\mathbf{z}^{(n)}) - \boldsymbol{\Phi}(\mathbf{x}^{(n)}) \\
&\ge \boldsymbol{\Phi}(\mathbf{z}^{(n)}) - \boldsymbol{\Phi}(\mathbf{y}^{(n)})
\end{aligned}$$

and for every $i \in \{1,\ldots,I\}$ the mean value theorem yields the existence of some $\mathbf{t}^{(n,i)} \in [\mathbf{y}^{(n)}, \mathbf{z}^{(n)}] \subseteq [\mathbf{u}^{(0)}, \mathbf{z}^{(0)}]$ such that

$$\begin{aligned}
z^{(n)}_i - x^{(n+1)}_i &\ge \varphi^{(i)}(\mathbf{z}^{(n)}) - \varphi^{(i)}(\mathbf{y}^{(n)}) \\
&= \left(\frac{\partial \varphi^{(i)}}{\partial t_{j(n)}}(\mathbf{t}^{(n,i)})\right)\Big(z^{(n)}_{j(n)} - y^{(n)}_{j(n)}\Big) \\
&= \left(\frac{\partial \varphi^{(i)}}{\partial t_{j(n)}}(\mathbf{t}^{(n,i)})\right)\Big(\mu^{(n)} - \lambda^{(n)}\Big)\alpha^*_{j(n)} \\
&\ge \gamma\Big(\mu^{(n)} - \lambda^{(n)}\Big)\min_{j\in\{1,\ldots,I\}} \alpha^*_j
\end{aligned}$$

which yields

$$
\begin{aligned}
\mathbf{z}^{(n+1)} &= \mu^{(n+1)}\,\boldsymbol{\alpha}^* \\
&= \frac{x_{i(n+1)}^{(n+1)}}{\alpha_{i(n+1)}}\,\boldsymbol{\alpha}^* \\
&\le \frac{1}{\alpha_{i(n+1)}^*}\left(z_{i(n+1)}^{(n)} - \gamma\Big(\mu^{(n)} - \lambda^{(n)}\Big)\min_{j\in\{1,\dots,I\}} \alpha_j^*\right)\boldsymbol{\alpha}^* \\
&\le \left(\mu^{(n)} - \gamma\Big(\mu^{(n)} - \lambda^{(n)}\Big)\frac{\min_{j\in\{1,\dots,I\}}\alpha_j^*}{\max_{j\in\{1,\dots,I\}}\alpha_j^*}\right)\boldsymbol{\alpha}^*\,.
\end{aligned}
$$

Repeating the argument for the fixed–point $\mathbf{u}^{(n)}$ of $\boldsymbol{\Phi}$ yields

$$
\mathbf{u}^{(n+1)} \;\ge\; \left(\lambda^{(n)} + \gamma\Big(\mu^{(n)} - \lambda^{(n)}\Big)\frac{\min_{j\in\{1,\dots,I\}}\alpha_j^*}{\max_{j\in\{1,\dots,I\}}\alpha_j^*}\right)\boldsymbol{\alpha}^*\,.
$$

Combining these two inequalities we obtain

$$
\begin{aligned}
\Big(\mu^{(n+1)} - \lambda^{(n+1)}\Big)\boldsymbol{\alpha}^* &= \mathbf{z}^{(n+1)} - \mathbf{u}^{(n+1)} \\
&\le \Big(\mu^{(n)} - \lambda^{(n)}\Big)\left(1 - 2\,\gamma\,\frac{\min_{j\in\{1,\dots,I\}}\alpha_j^*}{\max_{j\in\{1,\dots,I\}}\alpha_j^*}\right)\boldsymbol{\alpha}^*
\end{aligned}
$$

and hence

$$
\mu^{(n+1)} - \lambda^{(n+1)} \;\le\; \Big(\mu^{(n)} - \lambda^{(n)}\Big)\left(1 - 2\,\gamma\,\frac{\min_{j\in\{1,\dots,I\}}\alpha_j^*}{\max_{j\in\{1,\dots,I\}}\alpha_j^*}\right)
$$

for all $n \in \mathbb{N}_0$. Since $\mu^{(n+1)} - \lambda^{(n+1)} \ge 0$ and $\gamma > 0$, we have

$$
0 \;\le\; 1 - 2\,\gamma\,\frac{\min_{j\in\{1,\dots,I\}}\alpha_j^*}{\max_{j\in\{1,\dots,I\}}\alpha_j^*} \;<\; 1
$$

and it now follows that $\lambda = \lim_{n\to\infty}\lambda^{(n)} = \lim_{n\to\infty}\mu^{(n)} = \mu$, as was to be shown. $\square$

In the previous proof, radial uniqueness of the fixed–point is obtained as a consequence of the convergence of the fixed–point iteration. We note that radial uniqueness may also be obtained by elementary arguments, or by using Morishima (1964; Appendix, Section 2).

Correspondingly, we have the following result on the fixed–point iteration for $\boldsymbol{\Psi}$:

**5.2 Corollary.** *The map $\boldsymbol{\Psi}$ has a radially unique fixed–point and for every $\boldsymbol{\beta}^{(0)}$ the sequence $\{\boldsymbol{\Psi}^n(\boldsymbol{\beta}^{(0)})\}_{n\in\mathbb{N}_0}$ converges to a fixed–point of $\boldsymbol{\Psi}$.*

Combining Theorem 5.1 and Corollary 3.3 yields the following result:

**5.3 Corollary.** *The marginal–sum equation $\mu(\boldsymbol{\alpha},\boldsymbol{\beta}) = \boldsymbol{\Xi}(\boldsymbol{\alpha},\boldsymbol{\beta})$ has a radially unique solution.*

Let us now consider the fixed–point iteration for $\boldsymbol{\Xi}$:

**5.4 Corollary.** *The map $\boldsymbol{\Xi}$ has a radially unique fixed–point and for every $(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)})$ there exist fixed–points $\boldsymbol{\alpha}^*$ of $\boldsymbol{\Phi}$ and $\boldsymbol{\beta}^*$ of $\boldsymbol{\Psi}$ such that the sequence $\{\boldsymbol{\Xi}^n(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)})\}_{n\in\mathbb{N}_0}$ satisfies*

$$\lim_{j\to\infty} \boldsymbol{\Xi}^{2j}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}) \;=\; (\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$$

*and*

$$\lim_{j\to\infty} \boldsymbol{\Xi}^{2j+1}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}) \;=\; (\mathbf{H}(\boldsymbol{\beta}^*), \mathbf{G}(\boldsymbol{\alpha}^*))$$

*and such that $(\boldsymbol{\alpha}^*, \mathbf{G}(\boldsymbol{\alpha}^*))$ and $(\mathbf{H}(\boldsymbol{\beta}^*), \boldsymbol{\beta}^*)$ are fixed–points of $\boldsymbol{\Xi}$.*

**Proof.** By Corollary 5.3 and Corollary 3.5 the map $\boldsymbol{\Xi}$ has a radially unique fixed–point. Furthermore, for every $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ we have

$$\boldsymbol{\Xi}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \;=\; (\mathbf{H}(\boldsymbol{\beta}), \mathbf{G}(\boldsymbol{\alpha}))$$

and hence

$$\begin{aligned} \boldsymbol{\Xi}^2(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \boldsymbol{\Xi}((\mathbf{H}(\boldsymbol{\beta}), \mathbf{G}(\boldsymbol{\alpha}))) \\ &= (\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha})), \mathbf{G}(\mathbf{H}(\boldsymbol{\beta}))) \\ &= (\boldsymbol{\Phi}(\boldsymbol{\alpha}), \boldsymbol{\Psi}(\boldsymbol{\beta})) \end{aligned}$$

which by induction yields

$$\boldsymbol{\Xi}^{2j}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \;=\; (\boldsymbol{\Phi}^j(\boldsymbol{\alpha}), \boldsymbol{\Psi}^j(\boldsymbol{\beta}))$$

and hence

$$\begin{aligned} \boldsymbol{\Xi}^{2j+1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \boldsymbol{\Xi}((\boldsymbol{\Phi}^j(\boldsymbol{\alpha}), \boldsymbol{\Psi}^j(\boldsymbol{\beta}))) \\ &= (\mathbf{H}(\boldsymbol{\Psi}^j(\boldsymbol{\beta})), \mathbf{G}(\boldsymbol{\Phi}^j(\boldsymbol{\alpha}))) \,. \end{aligned}$$

By Theorem 5.1 and Corollary 5.2, the sequence $\{\boldsymbol{\Phi}^j(\boldsymbol{\alpha}^{(0)})\}_{j\in\mathbb{N}_0}$ converges to a fixed–point $\boldsymbol{\alpha}^*$ of $\boldsymbol{\Phi}$ and the sequence $\{\boldsymbol{\Psi}^j(\boldsymbol{\beta}^{(0)})\}_{j\in\mathbb{N}_0}$ converges to a fixed–point $\boldsymbol{\beta}^*$ of $\boldsymbol{\Psi}$, and this yields

$$\begin{aligned} \lim_{j\to\infty} \boldsymbol{\Xi}^{2j}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}) &= (\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) \\ \lim_{j\to\infty} \boldsymbol{\Xi}^{2j+1}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}) &= (\mathbf{H}(\boldsymbol{\beta}^*), \mathbf{G}(\boldsymbol{\alpha}^*)) \,. \end{aligned}$$

Finally, we have

$$\begin{aligned} \boldsymbol{\Xi}(\boldsymbol{\alpha}^*, \mathbf{G}(\boldsymbol{\alpha}^*)) &= (\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha}^*)), \mathbf{G}(\boldsymbol{\alpha}^*)) \\ &= (\boldsymbol{\Phi}(\boldsymbol{\alpha}^*), \mathbf{G}(\boldsymbol{\alpha}^*)) \\ &= (\boldsymbol{\alpha}^*, \mathbf{G}(\boldsymbol{\alpha}^*)) \end{aligned}$$

as well as

$$\boldsymbol{\Xi}(\mathbf{H}(\boldsymbol{\beta}^*), \boldsymbol{\beta}^*) \;=\; (\mathbf{H}(\boldsymbol{\beta}^*), \boldsymbol{\beta}^*)$$

such that $(\boldsymbol{\alpha}^*, \mathbf{G}(\boldsymbol{\alpha}^*))$ and $(\mathbf{H}(\boldsymbol{\beta}^*), \boldsymbol{\beta}^*)$ are fixed–points of $\boldsymbol{\Xi}$ (which also follows from Theorems 3.2 and 3.4). $\square$

The previous result is quite interesting: The even and the odd iterates converge, but it is uncertain whether the limits are fixed–points. Nevertheless, crossing–over the components of these limits yields fixed–points, and if the two limits are identical then of course they provide a fixed–point.

Let us finally consider the fixed–point iteration for $\bar{\boldsymbol{\Xi}}$:

**5.5 Corollary.** *The map $\bar{\boldsymbol{\Xi}}$ has a unique fixed–point and for every $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ the sequence $\{\bar{\boldsymbol{\Xi}}^n(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)\}_{n\in\mathbb{N}_0}$ converges to the fixed–point of $\bar{\boldsymbol{\Xi}}$.*

**Proof.** By Corollary 5.3 and Corollary 3.7 the map $\bar{\boldsymbol{\Xi}}$ has a unique fixed–point. Furthermore, for every $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ we have

$$\bar{\boldsymbol{\Xi}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left( \frac{\mathbf{H}(\boldsymbol{\beta})}{\|\mathbf{H}(\boldsymbol{\beta})\|}, \frac{\mathbf{G}(\boldsymbol{\alpha})}{\|\mathbf{G}(\boldsymbol{\alpha})\|} \right)$$

and hence, due to the fact that $\mathbf{H}$ and $\mathbf{G}$ are homogeneous of degree $-1$,

$$\begin{aligned}
\bar{\boldsymbol{\Xi}}^2(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \bar{\boldsymbol{\Xi}}\left( \frac{\mathbf{H}(\boldsymbol{\beta})}{\|\mathbf{H}(\boldsymbol{\beta})\|}, \frac{\mathbf{G}(\boldsymbol{\alpha})}{\|\mathbf{G}(\boldsymbol{\alpha})\|} \right) \\
&= \left( \frac{\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha}))}{\|\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha}))\|}, \frac{\mathbf{G}(\mathbf{H}(\boldsymbol{\beta}))}{\|\mathbf{G}(\mathbf{H}(\boldsymbol{\beta}))\|} \right) \\
&= \left( \frac{\boldsymbol{\Phi}(\boldsymbol{\alpha})}{\|\boldsymbol{\Phi}(\boldsymbol{\alpha})\|}, \frac{\boldsymbol{\Psi}(\boldsymbol{\beta})}{\|\boldsymbol{\Psi}(\boldsymbol{\beta})\|} \right)
\end{aligned}$$

which by induction, and due to the fact that $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are homogeneous of degree 1, yields

$$\bar{\boldsymbol{\Xi}}^{2j}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left( \frac{\boldsymbol{\Phi}^j(\boldsymbol{\alpha})}{\|\boldsymbol{\Phi}^j(\boldsymbol{\alpha})\|}, \frac{\boldsymbol{\Psi}^j(\boldsymbol{\beta})}{\|\boldsymbol{\Psi}^j(\boldsymbol{\beta})\|} \right)$$

and hence

$$\begin{aligned}
\bar{\boldsymbol{\Xi}}^{2j+1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \bar{\boldsymbol{\Xi}}\left( \frac{\boldsymbol{\Phi}^j(\boldsymbol{\alpha})}{\|\boldsymbol{\Phi}^j(\boldsymbol{\alpha})\|}, \frac{\boldsymbol{\Psi}^j(\boldsymbol{\beta})}{\|\boldsymbol{\Psi}^j(\boldsymbol{\beta})\|} \right) \\
&= \left( \frac{\mathbf{H}(\boldsymbol{\Psi}^j(\boldsymbol{\beta}))}{\|\mathbf{H}(\boldsymbol{\Psi}^j(\boldsymbol{\beta}))\|}, \frac{\mathbf{G}(\boldsymbol{\Phi}^j(\boldsymbol{\alpha}))}{\|\mathbf{G}(\boldsymbol{\Phi}^j(\boldsymbol{\alpha}))\|} \right).
\end{aligned}$$

By Theorem 5.1 and Corollary 5.2, the sequence $\{\boldsymbol{\Phi}^j(\boldsymbol{\alpha}^{(0)})\}_{j\in\mathbb{N}_0}$ converges to a fixed–point $\boldsymbol{\alpha}^*$ of $\boldsymbol{\Phi}$ and the sequence $\{\boldsymbol{\Psi}^j(\boldsymbol{\beta}^{(0)})\}_{j\in\mathbb{N}_0}$ converges to a fixed–point $\boldsymbol{\beta}^*$ of $\boldsymbol{\Psi}$. For these fixed–points we have

$$\begin{aligned}
\bar{\boldsymbol{\Xi}}\left( \frac{\boldsymbol{\alpha}^*}{\|\boldsymbol{\alpha}^*\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}^*)}{\|\mathbf{G}(\boldsymbol{\alpha}^*)\|} \right) &= \left( \frac{\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha}^*))}{\|\mathbf{H}(\mathbf{G}(\boldsymbol{\alpha}^*))\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}^*)}{\|\mathbf{G}(\boldsymbol{\alpha}^*)\|} \right) \\
&= \left( \frac{\boldsymbol{\Phi}(\boldsymbol{\alpha}^*)}{\|\boldsymbol{\Phi}(\boldsymbol{\alpha}^*)\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}^*)}{\|\mathbf{G}(\boldsymbol{\alpha}^*)\|} \right) \\
&= \left( \frac{\boldsymbol{\alpha}^*}{\|\boldsymbol{\alpha}^*\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}^*)}{\|\mathbf{G}(\boldsymbol{\alpha}^*)\|} \right)
\end{aligned}$$

as well as

$$\bar{\boldsymbol{\Xi}}\left(\frac{\mathbf{H}(\boldsymbol{\beta}^*)}{\|\mathbf{H}(\boldsymbol{\beta}^*)\|}, \frac{\boldsymbol{\beta}^*}{\|\boldsymbol{\beta}^*\|}\right) \;=\; \left(\frac{\mathbf{H}(\boldsymbol{\beta}^*)}{\|\mathbf{H}(\boldsymbol{\beta}^*)\|}, \frac{\boldsymbol{\beta}^*}{\|\boldsymbol{\beta}^*\|}\right)$$

such that $(\boldsymbol{\alpha}^*/\|\boldsymbol{\alpha}^*\|, \mathbf{G}(\boldsymbol{\alpha}^*)/\|\mathbf{G}(\boldsymbol{\alpha}^*)\|)$ and $(\mathbf{H}(\boldsymbol{\beta}^*)/\|\mathbf{H}(\boldsymbol{\beta}^*)\|, \boldsymbol{\beta}^*/\|\boldsymbol{\beta}^*\|)$ are fixed–points of $\bar{\boldsymbol{\Xi}}$ (which also follows from Theorems 3.2, 2.3 and 3.6). Since $\bar{\boldsymbol{\Xi}}$ has a unique fixed–point, this implies

$$\left(\frac{\boldsymbol{\alpha}^*}{\|\boldsymbol{\alpha}^*\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}^*)}{\|\mathbf{G}(\boldsymbol{\alpha}^*)\|}\right) \;=\; \left(\frac{\mathbf{H}(\boldsymbol{\beta}^*)}{\|\mathbf{H}(\boldsymbol{\beta}^*)\|}, \frac{\boldsymbol{\beta}^*}{\|\boldsymbol{\beta}^*\|}\right).$$

We thus obtain

$$\begin{aligned}
\lim_{j\to\infty} \bar{\boldsymbol{\Xi}}^{2j}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}) &= \lim_{j\to\infty}\left(\frac{\boldsymbol{\Phi}^j(\boldsymbol{\alpha}^{(0)})}{\|\boldsymbol{\Phi}^j(\boldsymbol{\alpha}^{(0)})\|}, \frac{\boldsymbol{\Psi}^j(\boldsymbol{\beta}^{(0)})}{\|\boldsymbol{\Psi}^j(\boldsymbol{\beta}^{(0)})\|}\right)\\
&= \left(\frac{\boldsymbol{\alpha}^*}{\|\boldsymbol{\alpha}^*\|}, \frac{\boldsymbol{\beta}^*}{\|\boldsymbol{\beta}^*\|}\right)\\
&= \left(\frac{\mathbf{H}(\boldsymbol{\beta}^*)}{\|\mathbf{H}(\boldsymbol{\beta}^*)\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}^*)}{\|\mathbf{G}(\boldsymbol{\alpha}^*)\|}\right)\\
&= \lim_{j\to\infty}\left(\frac{\mathbf{H}(\boldsymbol{\Psi}^j(\boldsymbol{\beta}^{(0)}))}{\|\mathbf{H}(\boldsymbol{\Psi}^j(\boldsymbol{\beta}^{(0)}))\|}, \frac{\mathbf{G}(\boldsymbol{\Phi}^j(\boldsymbol{\alpha}^{(0)}))}{\|\mathbf{G}(\boldsymbol{\Phi}^j(\boldsymbol{\alpha}^{(0)}))\|}\right)\\
&= \lim_{j\to\infty} \bar{\boldsymbol{\Xi}}^{2j+1}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)})
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \bar{\boldsymbol{\Xi}}^{n}(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}) \;=\; \left(\frac{\boldsymbol{\alpha}^*}{\|\boldsymbol{\alpha}^*\|}, \frac{\boldsymbol{\beta}^*}{\|\boldsymbol{\beta}^*\|}\right)$$

as was to be shown. □

## 6 The case $p = 3$: A counterexample

Throughout this section we assume that $p = 3$. We shall present an example in which

– the marginal–sum equation has a radially unique solution which can be represented in parametric form,
– the map $\bar{\boldsymbol{\Xi}}$ has a unique fixed–point, and
– there exist initial values for which the fixed–point iteration for $\bar{\boldsymbol{\Xi}}$ is divergent (which would be impossible in the case $p = 2$).

To simplify the notation, we write $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ instead of $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ and $\mathbf{H}, \mathbf{G}, \mathbf{F}$ instead of $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(3)}$. Then the maps $\boldsymbol{\Xi}$ and $\bar{\boldsymbol{\Xi}}$ satisfy

$$\boldsymbol{\Xi}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \;=\; (\mathbf{H}(\boldsymbol{\beta}, \boldsymbol{\gamma}), \mathbf{G}(\boldsymbol{\alpha}, \boldsymbol{\gamma}), \mathbf{F}(\boldsymbol{\alpha}, \boldsymbol{\beta}))$$

and

$$\bar{\boldsymbol{\Xi}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \;=\; \left(\frac{\mathbf{H}(\boldsymbol{\beta}, \boldsymbol{\gamma})}{\|\mathbf{H}(\boldsymbol{\beta}, \boldsymbol{\gamma})\|}, \frac{\mathbf{G}(\boldsymbol{\alpha}, \boldsymbol{\gamma})}{\|\mathbf{G}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\|}, \frac{\mathbf{F}(\boldsymbol{\alpha}, \boldsymbol{\beta})}{\|\mathbf{F}(\boldsymbol{\alpha}, \boldsymbol{\beta})\|}\right)$$

and the marginal–sum equation $\mu(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \boldsymbol{\Xi}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ becomes

$$\mu(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) \quad = \quad (\mathbf{H}(\boldsymbol{\beta},\boldsymbol{\gamma}), \mathbf{G}(\boldsymbol{\alpha},\boldsymbol{\gamma}), \mathbf{F}(\boldsymbol{\alpha},\boldsymbol{\beta}))\,.$$

We shall study the following example:

**6.1 Example.** Assume that $n_1 = 2, n_2 = 2, n_3 = 3$ and assume further that $S_{ijk} = 1$ holds for all $(i,j,k) \in \{1,2\} \times \{1,2\} \times \{1,2,3\}$ and that

$$\begin{array}{l} N_{111} = 31,\ N_{112} = 1,\ N_{113} = 1,\ N_{121} = 1,\ N_{122} = 1,\ N_{123} = 1, \\ N_{211} = 1,\ N_{212} = 1,\ N_{213} = 1,\ N_{221} = 1,\ N_{222} = 31,\ N_{223} = 1 \end{array}$$

Then the marginal–sum equations read

$$\begin{aligned}
\mu\alpha_1 &= H_1(\boldsymbol{\beta},\boldsymbol{\gamma}) = 6/(\beta_1(31\gamma_1+\gamma_2+\gamma_3)+\beta_2(\gamma_1+\gamma_2+\gamma_3)) \\
\mu\alpha_2 &= H_2(\boldsymbol{\beta},\boldsymbol{\gamma}) = 6/(\beta_1(\gamma_1+\gamma_2+\gamma_3)+\beta_2(\gamma_1+31\gamma_2+\gamma_3)) \\
\mu\beta_1 &= G_1(\boldsymbol{\alpha},\boldsymbol{\gamma}) = 6/(\alpha_1(31\gamma_1+\gamma_2+\gamma_3)+\alpha_2(\gamma_1+\gamma_2+\gamma_3)) \\
\mu\beta_2 &= G_2(\boldsymbol{\alpha},\boldsymbol{\gamma}) = 6/(\alpha_1(\gamma_1+\gamma_2+\gamma_3)+\alpha_2(\gamma_1+31\gamma_2+\gamma_3)) \\
\mu\gamma_1 &= F_1(\boldsymbol{\alpha},\boldsymbol{\beta}) = 4/(\alpha_1(31\beta_1+\beta_2)+\alpha_2(\beta_1+\beta_2)) \\
\mu\gamma_2 &= F_2(\boldsymbol{\alpha},\boldsymbol{\beta}) = 4/(\alpha_1(\beta_1+\beta_2)+\alpha_2(\beta_1+31\beta_2)) \\
\mu\gamma_3 &= F_3(\boldsymbol{\alpha},\boldsymbol{\beta}) = 4/(\alpha_1(\beta_1+\beta_2)+\alpha_2(\beta_1+\beta_2))
\end{aligned}$$

For this example, we study the radial uniqueness of a solution of the marginal–sum equations, the uniqueness of a fixed–point of $\bar{\boldsymbol{\Xi}}$, and the question whether or not the fixed–point iteration with respect to $\bar{\boldsymbol{\Xi}}$ and arbitrary initial values converges to a fixed–point of $\bar{\boldsymbol{\Xi}}$. We obtain the following results:

*Claim 1. The marginal–sum equation has a radially unique solution*

$$(\mu^\circ, (\boldsymbol{\alpha}^\circ, \boldsymbol{\beta}^\circ, \boldsymbol{\gamma}^\circ))$$

*with*

$$\mu^\circ = \frac{84}{17rst} \qquad \boldsymbol{\alpha}^\circ = r\left(\frac{1}{2},\frac{1}{2}\right) \qquad \boldsymbol{\beta}^\circ = s\left(\frac{1}{2},\frac{1}{2}\right) \qquad \boldsymbol{\gamma}^\circ = t\left(\frac{2}{21},\frac{2}{21},\frac{17}{21}\right)$$

*and arbitrary $r,s,t \in (0,\infty)$.*

Indeed: By Theorem 2.3, the marginal–sum equations have a radially unique solution if and only if they have exactly one solution $(\mu,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}))$ satisfying $\|\boldsymbol{\alpha}\|_L = \|\boldsymbol{\beta}\|_L = \|\boldsymbol{\gamma}\|_L = 1$ and hence $\alpha_1+\alpha_2 = \beta_1+\beta_2 = \gamma_1+\gamma_2+\gamma_3 = 1$. Under this condition, the marginal–sum equations become

$$\mu\alpha_1 = 6/(30\beta_1\gamma_1+1) \tag{1}$$
$$\mu(1-\alpha_1) = 6/(30(1-\beta_1)\gamma_2+1) \tag{2}$$
$$\mu\beta_1 = 6/(30\alpha_1\gamma_1+1) \tag{3}$$

$$\mu(1-\beta_1) = 6/(30(1-\alpha_1)\gamma_2+1) \tag{4}$$
$$\mu\gamma_1 = 4/(30\alpha_1\beta_1+1) \tag{5}$$
$$\mu\gamma_2 = 4/(30(1-\alpha_1)(1-\beta_1)+1) \tag{6}$$
$$\mu(1-\gamma_1-\gamma_2) = 4 \tag{7}$$

Multiplication of (1) and (3) with the denominators and taking the difference yields

$$\alpha_1 \;=\; \beta_1$$

and inserting this identity into (1), (5), (6) and using (7) then gives

$$\alpha_1(30\alpha_1\gamma_1+1) = 6/\mu = 3(1-\gamma_1-\gamma_2)/2 \tag{8}$$
$$\gamma_1(30{\alpha_1}^2+1) = 4/\mu = 1-\gamma_1-\gamma_2 \tag{9}$$
$$\gamma_2(30(1-\alpha_1)^2+1) = 4/\mu = 1-\gamma_1-\gamma_2 \tag{10}$$

Taking the difference of (8) and (9) yields

$$\gamma_2 \;=\; 1+\gamma_1-2\alpha_1$$

and inserting this identity into (9) and (10) then gives the identities

$$\gamma_1-\alpha_1 \;=\; \frac{2\alpha_1}{30{\alpha_1}^2+3}-\alpha_1$$
$$\gamma_1-\alpha_1 \;=\; \frac{2(1-\alpha_1)}{30(1-\alpha_1)^2+3}-(1-\alpha_1)$$

Therefore, the function $f$ given by

$$f(x) \;:=\; \frac{2x}{30x^2+3}-x$$

satisfies $f(\alpha_1) = \gamma_1-\alpha_1 = f(1-\alpha_1)$. Since $f$ is strictly decreasing this identity yields $\alpha_1 = 1-\alpha_1$ and hence $\alpha_1 = 1/2, \beta_1 = 1/2, \gamma_1 = 2/21, \gamma_2 = 2/21$.

*Claim 2.* *The map $\bar{\Xi}$ has a unique fixed–point $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ and the fixed–point satisfies*

$$(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*) = \left(\frac{\boldsymbol{\alpha}^\bullet}{\|\boldsymbol{\alpha}^\bullet\|}, \frac{\boldsymbol{\beta}^\bullet}{\|\boldsymbol{\beta}^\bullet\|}, \frac{\boldsymbol{\gamma}^\bullet}{\|\boldsymbol{\gamma}^\bullet\|}\right)$$

*with*

$$\boldsymbol{\alpha}^\bullet = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \boldsymbol{\beta}^\bullet = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \boldsymbol{\gamma}^\bullet = \left(\frac{2}{21}, \frac{2}{21}, \frac{17}{21}\right)$$

This follows from Claim 1 and Theorem 3.6.

*Claim 3.* *The vectors $(\boldsymbol{\alpha}^+, \boldsymbol{\beta}^+, \boldsymbol{\gamma}^+)$ and $(\boldsymbol{\alpha}^-, \boldsymbol{\beta}^-, \boldsymbol{\gamma}^-)$ given by*

$$\boldsymbol{\alpha}^\pm \;:=\; \left(\frac{15 \pm \sqrt{-285+30\sqrt{226}}}{30}, \frac{15 \mp \sqrt{-285+30\sqrt{226}}}{30}\right)$$

$$\boldsymbol{\beta}^{\pm} := \left(\frac{15 \pm \sqrt{-285+30\sqrt{226}}}{30}, \frac{15 \mp \sqrt{-285+30\sqrt{226}}}{30}\right)$$

$$\boldsymbol{\gamma}^{\pm} := \left(\frac{-1+\sqrt{226} \pm \sqrt{-285+30\sqrt{226}}}{510-30\sqrt{226}}, \frac{-1+\sqrt{226} \mp \sqrt{-285+30\sqrt{226}}}{510-30\sqrt{226}}, \frac{512-32\sqrt{226}}{510-30\sqrt{226}}\right)$$

*satisfy*

$$\begin{aligned} \bar{\Xi}_L(\boldsymbol{\alpha}^+, \boldsymbol{\beta}^+, \boldsymbol{\gamma}^+) &= (\boldsymbol{\alpha}^-, \boldsymbol{\beta}^-, \boldsymbol{\gamma}^-) \\ \bar{\Xi}_L(\boldsymbol{\alpha}^-, \boldsymbol{\beta}^-, \boldsymbol{\gamma}^-) &= (\boldsymbol{\alpha}^+, \boldsymbol{\beta}^+, \boldsymbol{\gamma}^+) \end{aligned}$$

*such that none of the sequences* $\{\bar{\Xi}_L^n(\boldsymbol{\alpha}^+, \boldsymbol{\beta}^+, \boldsymbol{\gamma}^+)\}_{n\in\mathbb{N}}$ *and* $\{\bar{\Xi}_L^n(\boldsymbol{\alpha}^-, \boldsymbol{\beta}^-, \boldsymbol{\gamma}^-)\}_{n\in\mathbb{N}}$ *converges to the unique fixed–point of* $\bar{\Xi}_L$*.*

The verification of Claim 3 is tedious but straightforward.

*Claim 4.* *For any norm* $\|\,.\,\|$*, each of the sequences* $\{\bar{\Xi}^n(\boldsymbol{\alpha}^+, \boldsymbol{\beta}^+, \boldsymbol{\gamma}^+)\}_{n\in\mathbb{N}}$ *and* $\{\bar{\Xi}^n(\boldsymbol{\alpha}^-, \boldsymbol{\beta}^-, \boldsymbol{\gamma}^-)\}_{n\in\mathbb{N}}$ *is divergent.*

This is a consequence of the following general theorem and Claim 3.

**6.2 Theorem.** *Consider* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ *and define*

$$\begin{aligned} (\boldsymbol{\alpha}^{(n)}, \boldsymbol{\beta}^{(n)}, \boldsymbol{\gamma}^{(n)}) &:= \bar{\Xi}^n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \\ (\boldsymbol{\alpha}_L^{(n)}, \boldsymbol{\beta}_L^{(n)}, \boldsymbol{\gamma}_L^{(n)}) &:= \bar{\Xi}_L^n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \end{aligned}$$

*(where* $\Xi$ *refers to an arbitrary norm). Then the sequence* $\{(\boldsymbol{\alpha}^{(n)}, \boldsymbol{\beta}^{(n)}, \boldsymbol{\gamma}^{(n)})\}_{n\in\mathbb{N}}$ *converges if and only if the sequence* $\{(\boldsymbol{\alpha}_L^{(n)}, \boldsymbol{\beta}_L^{(n)}, \boldsymbol{\gamma}_L^{(n)})\}_{n\in\mathbb{N}}$ *converges.*

**Proof.** Assume first that the sequence $\{(\boldsymbol{\alpha}^{(n)}, \boldsymbol{\beta}^{(n)}, \boldsymbol{\gamma}^{(n)})\}_{n\in\mathbb{N}}$ converges. Because of the properties of $\mathbf{H}, \mathbf{G}, \mathbf{F}$ established in Lemma 2.2, we have

$$\begin{aligned} \boldsymbol{\alpha}_L^{(n)} &= \boldsymbol{\alpha}^{(n)} \frac{\|\mathbf{H}(\boldsymbol{\beta}^{(n-1)}, \boldsymbol{\gamma}^{(n-1)})\|}{\|\mathbf{H}(\boldsymbol{\beta}^{(n-1)}, \boldsymbol{\gamma}^{(n-1)})\|_L} \\ \boldsymbol{\beta}_L^{(n)} &= \boldsymbol{\beta}^{(n)} \frac{\|\mathbf{G}(\boldsymbol{\alpha}^{(n-1)}, \boldsymbol{\gamma}^{(n-1)})\|}{\|\mathbf{G}(\boldsymbol{\alpha}^{(n-1)}, \boldsymbol{\gamma}^{(n-1)})\|_L} \\ \boldsymbol{\gamma}_L^{(n)} &= \boldsymbol{\gamma}^{(n)} \frac{\|\mathbf{F}(\boldsymbol{\alpha}^{(n-1)}, \boldsymbol{\beta}^{(n-1)})\|}{\|\mathbf{F}(\boldsymbol{\alpha}^{(n-1)}, \boldsymbol{\beta}^{(n-1)})\|_L} \end{aligned}$$

and we also see that the right hand side of these equations converges. This proves one of the implications, and the other one is obtained by interchanging the roles of the norms. $\square$

We note that the previous result and its proof can be extended to arbitrary order $p \geq 2$.

**6.3 Remark.** We have computed the fixed–point iteration for $\bar{\Xi}_L$ for various other choices of the initial values, including initial values in a small neighbourhood of the fixed–point. The numerical results indicate that for every choice of the initial values either the even iterates converge to $(\boldsymbol{\alpha}^+, \boldsymbol{\beta}^+, \boldsymbol{\gamma}^+)$ and the odd iterates converge to $(\boldsymbol{\alpha}^-, \boldsymbol{\beta}^-, \boldsymbol{\gamma}^-)$, or vice versa.

## 7 Further Remarks

In the case $p \geq 3$ no general results are known on radial uniqueness and on the convergence of fixed–point iterations. The proofs for the case $p = 2$ given in Section 5 cannot be extended to the case $p \geq 3$. We also note that extensive numerical computations in the case $p \geq 3$ have provided the following observations:

- In all examples the marginal–sum equation has a radially unique solution.
- In all examples and for all choices of the initial values the fixed–point iteration for the map $\mathbf{\Phi}$ appears to converge to a fixed–point; see Theorem 5.1 for the case $p = 2$.
- In all examples and for all choices of the initial values the fixed–point iteration for the map $\mathbf{\Xi}$ appears to possess the accumulation points $\mathbf{0}$ and $+\boldsymbol{\infty}$. This differs from the case $p = 2$; see Corollary 5.4.
- Except for very few examples the fixed–point iteration for the map $\bar{\mathbf{\Xi}}_L$ appears to be convergent; see Section 6 for such an example.

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Siegfried Dietze
Thomas Riedrich
Klaus D. Schmidt
Fachrichtung Mathematik
Technische Universitt Dresden
D–01062 Dresden

e–mail: `klaus.d.schmidt@tu-dresden.de`