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Biased Loss Prediction Caused by Aggregation

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Abstract

In the present paper we consider two subportfolios of risks as well as the union of these portfolios and for the chain–ladder method and the additive method we study the problem of whether or not the predictors based on the subportfolios are consistent with those based on the full portfolio.
In the case of the chain–ladder method we extend the results of Ajne [1994] and Klemmt [2005], using the duality of the chain–ladder method applied to incremental losses; we also give a short proof for this duality, which was first observed by Barnett, Zehnwirth and Dubossarky [2005].
Apparently, in the case of the additive method the aggregation problem has not been considered before.

1 Introduction

To model a portfolio of risks, we consider a family of strictly positive random variables

$$\{Z_{i,k}\}_{i,k\in\{0,1,\dots,n\}}$$

We interpret the random variable $Z_{i,k}$ as the loss of *accident year* i which is settled with a delay of k years and hence in *development year* k and in *calendar year* $i+k$ and refer to $Z_{i,k}$ as the *incremental loss* of accident year i and development year k. We assume that the incremental losses $Z_{i,k}$ are *observable* for calender years $i+k \le n$ and that they are *non–observable* for calender years $i+k > n$. The observable incremental losses are represented in a *run–off triangle* and the non–observable ones are to be predicted.

Consider now two subportfolios of risks with incremental losses $\bar{Z}_{i,k}$ and $\tilde{Z}_{i,k}$, respectively, as well as the union of these portfolios with incremental losses

$$\breve{Z}_{i,k} := \bar{Z}_{i,k} + \tilde{Z}_{i,k}$$

For a given loss reserving method the problem arises whether or not the sum of the predictors based on the subportfolios equals the corresponding predictor based on the aggregated portfolio. This problem is called the *aggregation problem.*

In the present paper we study the aggregation problem for the chain–ladder method (Section 2) and for the additive method (Section 3). For the chain–ladder method we extend results of Ajne [1994] and Klemmt [2005]; for the additive method, it appears that the aggregation problem has not yet been considered in literature.

2 Chain-ladder method and aggregation problem

In the present section we study the aggregation problem for the chain–ladder method. To this end, it is convenient to consider not only the chain–ladder method but also its dual version, which is the usual chain–ladder method applied to the transposed run–off triangle where accident and development years are interchanged. This allows for a prediction of the non–observable incremental losses also in terms of the dual version.

The uncommon approach of using chain–ladder method together with its dual version is twofold: First, the use of usual and dual chain–ladder factors facilitates the presentation of our results. Second, the proofs benefit from the fact that the chain–ladder predictors of the non–observable incremental losses in terms of the usual chain–ladder method coincide with those of the dual chain–ladder method which we call *duality* of the chain–ladder method. This result is due to Barnett, Zehnwirth and Dubossarsky [2005] and we include a proof of it which is particularly short.

The results on the aggregation problem presented in this section extend those of Ajne [1994] and Klemmt [2005].

2.1 Chain–ladder method and dual chain–ladder method

We refer to

$$S_{i,k} := \sum_{l=0}^{k} Z_{i,l} \qquad \text{and} \qquad T_{i,k} := \sum_{j=0}^{i} Z_{j,k}$$

as the *primal cumulative loss* and the *dual cumulative loss* of accident year i and development year k, respectively. Obviously, the cumulative losses $S_{i,k}$ and $T_{i,k}$ are observable for calendar years $i+k \leq n$ and they are non–observable for calendar years $i+k > n$. Moreover, the incremental losses can be recovered from both primal and dual cumulative losses by letting

$$Z_{i,k} = \begin{cases} S_{i,0} & \text{if } k=0 \\ S_{i,k} - S_{i,k-1} & \text{else} \end{cases} \qquad \text{and} \qquad Z_{i,k} = \begin{cases} T_{0,k} & \text{if } i=0 \\ T_{i,k} - T_{i-1,k} & \text{else} \end{cases}$$

This fact motivates the prediction of the non–observable incremental losses along both development and accident years and hence to distinguish between primal (usual) and dual chain–ladder method.

The *primal chain–ladder method* is based on the *primal chain–ladder factors*

$$\varphi_k^{\text{CL}} := \frac{\sum_{i=0}^{n-k} S_{i,k}}{\sum_{i=0}^{n-k} S_{i,k-1}}$$

of development year k, $k \in \{1, ..., n\}$, and consists in the prediction of the primal cumulative losses $S_{i,k}$, $i+k > n$, by the *chain–ladder predictors*

$$S_{i,k}^{\text{CL}} := S_{i,n-i} \prod_{l=n-i+1}^{k} \varphi_l^{\text{CL}}$$

with $i+k > n$. Analogously, the *dual chain–ladder method* is based on the *dual chain–ladder factors*

$$\psi_i^{\text{DCL}} := \frac{\sum_{l=0}^{n-i} T_{i,l}}{\sum_{l=0}^{n-i} T_{i-1,l}}$$

of accident year i, $i \in \{1, ..., n\}$, and consists in the prediction of the dual cumulative losses $T_{i,k}$, $i+k > n$, by the *chain–ladder predictors*

$$T_{i,k}^{\text{DCL}} := T_{n-k,k} \prod_{j=n-k+1}^{i} \psi_j^{\text{DCL}}$$

with $i+k > n$. Thereby, the terms primal and dual are justified by the identities

$$\varphi_k^{\text{CL}} = \frac{\sum_{j=0}^{n-k} \sum_{l=0}^{k} Z_{j,l}}{\sum_{j=0}^{n-k} \sum_{l=0}^{k-1} Z_{j,l}} \qquad \text{and} \qquad \psi_i^{\text{DCL}} = \frac{\sum_{l=0}^{n-i} \sum_{j=0}^{i} Z_{j,l}}{\sum_{l=0}^{n-i} \sum_{j=0}^{i-1} Z_{j,l}}$$

The dual chain–ladder factors are the primal chain–ladder factors applied to the transposed run–off triangle where accident and development years are interchanged, i.e. in the dual chain–ladder method prediction runs along accident years instead of development years.

For the remainder of this paper, we set $S^{\text{CL}}_{i,k} := S_{i,k}$ and $T^{\text{DCL}}_{i,k} := T_{i,k}$ for all $i+k \leq n$.

Using the primal respectively the dual chain–ladder method, the non–observable incremental losses are predicted by the *chain–ladder predictors*

$$Z^{\text{CL}}_{i,k} := S^{\text{CL}}_{i,k} - S^{\text{CL}}_{i,k-1} = S_{i,n-i} \left(\prod_{l=n-i+1}^{k-1} \varphi^{\text{CL}}_l \right) \left(\varphi^{\text{CL}}_k - 1\right) = S^{\text{CL}}_{i,k-1} \left(\varphi^{\text{CL}}_k - 1\right)$$

respectively

$$Z^{\text{DCL}}_{i,k} := T^{\text{DCL}}_{i,k} - T^{\text{DCL}}_{i-1,k} = T_{n-k,k} \left(\prod_{j=n-k+1}^{i-1} \psi^{\text{DCL}}_j \right) \left(\psi^{\text{DCL}}_i - 1\right) = T^{\text{DCL}}_{i-1,k} \left(\psi^{\text{DCL}}_i - 1\right)$$

with $i+k > n$.

The following result is due to Barnett, Zehnwirth and Dubossarsky [2005].

2.1.1 Theorem. *The identity*

$$Z^{\text{CL}}_{i,k} = Z^{\text{DCL}}_{i,k}$$

holds for all $i+k > n$.

In order to give a proof of this duality we introduce the random variables

$$A_{i,k} := \sum_{j=0}^{i} \sum_{l=0}^{k} Z_{j,l}$$

for all $i,k \in \{0,1,...,n\}$, which will turn out to be a useful technical device. Then, the chain–ladder factors satisfy

$$\varphi^{\text{CL}}_k = \frac{A_{n-k,k}}{A_{n-k,k-1}} \qquad \text{and} \qquad \psi^{\text{DCL}}_i = \frac{A_{i,n-i}}{A_{i-1,n-i}}$$

for all $i,k \in \{1,...,n\}$ and the cumulative losses satisfy

$$\begin{aligned} S_{i,k} &= A_{i,k} - A_{i-1,k} \qquad && i \in \{1,...,n\}, k \in \{0,1,...,n\},\ i+k \leq n \\ T_{i,k} &= A_{i,k} - A_{i,k-1} \qquad && k \in \{1,...,n\}, i \in \{0,1,...,n\},\ i+k \leq n \end{aligned}$$

The use of the rectangles $A_{i,k}$ allows for a particularly short proof of Theorem 2.1.1.

Proof. (Proof of Theorem 2.1.1) We obtain

$$
\begin{aligned}
Z_{i,k}^{\text{CL}} &= \big(A_{i,n-i} - A_{i-1,n-i}\big)\left(\prod_{l=n-i+1}^{k-1} \frac{A_{n-l,l}}{A_{n-l,l-1}}\right)\left(\frac{A_{n-k,k}}{A_{n-k,k-1}} - 1\right) \\
&= \big(A_{i,n-i} - A_{i-1,n-i}\big)\,\frac{\prod_{l=n-i+1}^{k-1} A_{n-l,l}}{\prod_{l=n-i+1}^{k} A_{n-l,l-1}}\,\big(A_{n-k,k} - A_{n-k,k-1}\big) \\
&= \big(A_{n-k,k} - A_{n-k,k-1}\big)\,\frac{\prod_{j=i-1}^{n-k+1} A_{j,n-j}}{\prod_{j=i}^{n-k+1} A_{j-1,n-j}}\,\big(A_{i,n-i} - A_{i-1,n-i}\big) \\
&= \big(A_{n-k,k} - A_{n-k,k-1}\big)\left(\prod_{j=n-k+1}^{i-1} \frac{A_{j,n-j}}{A_{j-1,n-j}}\right)\left(\frac{A_{i,n-i}}{A_{i-1,n-i}} - 1\right) \\
&= Z_{i,k}^{\text{DCL}}
\end{aligned}
$$

for all $i+k>n$. □

2.2 Aggregation problem

For the cumulative losses, the chain–ladder factors, the chain–ladder predictors and the rectangles we take over the notation used for the incremental losses.

As a first result, we obtain that the primal respectively dual chain–ladder factor based on the aggregated portfolio lies in the convex hull of the primal respectively dual chain–ladder factors based on the subportfolios.

2.2.1 Lemma. *The identities*

$$
\breve{\varphi}_k^{\text{CL}} = \frac{\bar{A}_{n-k,k-1}}{\breve{A}_{n-k,k-1}}\,\bar{\varphi}_k^{\text{CL}} + \frac{\tilde{A}_{n-k,k-1}}{\breve{A}_{n-k,k-1}}\,\tilde{\varphi}_k^{\text{CL}}
$$

and

$$
\breve{\psi}_i^{\text{DCL}} = \frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\,\bar{\psi}_i^{\text{DCL}} + \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\,\tilde{\psi}_i^{\text{DCL}}
$$

hold for all $i,k \in \{1,...,n\}$.

First of all, we want to study the aggregation problem on the basis of incremental losses. We have the following result:

2.2.2 Theorem. *Consider $i, k \in \{1, ..., n\}$ with $i + k > n$.*

(1) *Assume that*

$$\left(\left(\prod_{l=n-j+1}^{k-1} \bar{\varphi}_l^{\mathrm{CL}}\right)(\bar{\varphi}_k^{\mathrm{CL}} - 1) - \left(\prod_{l=n-j+1}^{k-1} \tilde{\varphi}_l^{\mathrm{CL}}\right)(\tilde{\varphi}_k^{\mathrm{CL}} - 1)\right)\left(\bar{\psi}_j^{\mathrm{DCL}} - \tilde{\psi}_j^{\mathrm{DCL}}\right) > 0$$

holds for all $j \in \{n-k+1, ..., i\}$. Then

$$\bar{Z}_{j,k}^{\mathrm{CL}} + \tilde{Z}_{j,k}^{\mathrm{CL}} > \breve{Z}_{j,k}^{\mathrm{CL}}$$

for all $j \in \{n-k+1, ..., i\}$.

(2) *Assume that*

$$\left(\left(\prod_{l=n-j+1}^{k-1} \bar{\varphi}_l^{\mathrm{CL}}\right)(\bar{\varphi}_k^{\mathrm{CL}} - 1) - \left(\prod_{l=n-j+1}^{k-1} \tilde{\varphi}_l^{\mathrm{CL}}\right)(\tilde{\varphi}_k^{\mathrm{CL}} - 1)\right)\left(\bar{\psi}_j^{\mathrm{DCL}} - \tilde{\psi}_j^{\mathrm{DCL}}\right) < 0$$

holds for all $j \in \{n-k+1, ..., i\}$. Then

$$\bar{Z}_{j,k}^{\mathrm{CL}} + \tilde{Z}_{j,k}^{\mathrm{CL}} < \breve{Z}_{j,k}^{\mathrm{CL}}$$

for all $j \in \{n-k+1, ..., i\}$.

(3) *The following are equivalent:*

(a) *The identity*

$$\left(\left(\prod_{l=n-j+1}^{k-1} \bar{\varphi}_l^{\mathrm{CL}}\right)(\bar{\varphi}_k^{\mathrm{CL}} - 1) - \left(\prod_{l=n-j+1}^{k-1} \tilde{\varphi}_l^{\mathrm{CL}}\right)(\tilde{\varphi}_k^{\mathrm{CL}} - 1)\right)\left(\bar{\psi}_j^{\mathrm{DCL}} - \tilde{\psi}_j^{\mathrm{DCL}}\right) = 0$$

holds for all $j \in \{n-k+1, ..., i\}$.

(b) *The identity*

$$\bar{Z}_{j,k}^{\mathrm{CL}} + \tilde{Z}_{j,k}^{\mathrm{CL}} = \breve{Z}_{j,k}^{\mathrm{CL}}$$

holds for all $j \in \{n-k+1, ..., i\}$.

Proof. Assume that the condition of (1) is fulfilled. Initially, we prove the assertion for $j = n-k+1$. Using Theorem 2.1.1 and Lemma A.3 we obtain

$$\begin{aligned}
&\bar{Z}_{n-k+1,k}^{\mathrm{CL}} + \tilde{Z}_{n-k+1,k}^{\mathrm{CL}} - \breve{Z}_{n-k+1,k}^{\mathrm{CL}} \\
&= \bar{Z}_{n-k+1,k}^{\mathrm{DCL}} + \tilde{Z}_{n-k+1,k}^{\mathrm{DCL}} - \breve{Z}_{n-k+1,k}^{\mathrm{DCL}} \\
&= \bar{T}_{n-k,k}^{\mathrm{DCL}}\left(\bar{\psi}_{n-k+1}^{\mathrm{DCL}} - 1\right) + \tilde{T}_{n-k,k}^{\mathrm{DCL}}\left(\tilde{\psi}_{n-k+1}^{\mathrm{DCL}} - 1\right) - \breve{T}_{n-k,k}^{\mathrm{DCL}}\left(\breve{\psi}_{n-k+1}^{\mathrm{DCL}} - 1\right) \\
&= \bar{T}_{n-k,k}^{\mathrm{DCL}}\left(\bar{\psi}_{n-k+1}^{\mathrm{DCL}} - 1\right) + \tilde{T}_{n-k,k}^{\mathrm{DCL}}\left(\tilde{\psi}_{n-k+1}^{\mathrm{DCL}} - 1\right) - \left(\bar{T}_{n-k,k}^{\mathrm{DCL}} + \tilde{T}_{n-k,k}^{\mathrm{DCL}}\right)\left(\breve{\psi}_{n-k+1}^{\mathrm{DCL}} - 1\right) \\
&= \bar{T}_{n-k,k}^{\mathrm{DCL}}\left(\bar{\psi}_{n-k+1}^{\mathrm{DCL}} - \breve{\psi}_{n-k+1}^{\mathrm{DCL}}\right) + \tilde{T}_{n-k,k}^{\mathrm{DCL}}\left(\tilde{\psi}_{n-k+1}^{\mathrm{DCL}} - \breve{\psi}_{n-k+1}^{\mathrm{DCL}}\right) \\
&= \gamma_{n-k+1}\left(\bar{\varphi}_k^{\mathrm{CL}} - \tilde{\varphi}_k^{\mathrm{CL}}\right)\left(\bar{\psi}_{n-k+1}^{\mathrm{DCL}} - \tilde{\psi}_{n-k+1}^{\mathrm{DCL}}\right) \\
&> 0
\end{aligned}$$

where $\gamma_{n-k+1} := (\bar{A}_{n-k,k-1}\, \tilde{A}_{n-k,k-1})/\breve{A}_{n-k,k-1}$. Now, consider $j \in \{n-k+2, ..., i\}$.

We assume that the assertion holds for $j-1$ and we prove the requested inequalitiy for j. Using Theorem 2.1.1 and Lemma A.3 we obtain

$$
\begin{aligned}
& \bar{Z}^{\mathrm{CL}}_{j,k} + \tilde{Z}^{\mathrm{CL}}_{j,k} - \breve{Z}^{\mathrm{CL}}_{j,k} \\
&= \bar{Z}^{\mathrm{DCL}}_{j,k} + \tilde{Z}^{\mathrm{DCL}}_{j,k} - \breve{Z}^{\mathrm{DCL}}_{j,k} \\
&= \bar{T}^{\mathrm{DCL}}_{j-1,k}\left(\bar{\psi}^{\mathrm{DCL}}_j - 1\right) + \tilde{T}^{\mathrm{DCL}}_{j-1,k}\left(\tilde{\psi}^{\mathrm{DCL}}_j - 1\right) - \breve{T}^{\mathrm{DCL}}_{j-1,k}\left(\breve{\psi}^{\mathrm{DCL}}_j - 1\right) \\
&> \bar{T}^{\mathrm{DCL}}_{j-1,k}\left(\bar{\psi}^{\mathrm{DCL}}_j - 1\right) + \tilde{T}^{\mathrm{DCL}}_{j-1,k}\left(\tilde{\psi}^{\mathrm{DCL}}_j - 1\right) - \left(\bar{T}^{\mathrm{DCL}}_{j-1,k} + \tilde{T}^{\mathrm{DCL}}_{j-1,k}\right)\left(\breve{\psi}^{\mathrm{DCL}}_j - 1\right) \\
&= \bar{T}^{\mathrm{DCL}}_{j-1,k}\left(\bar{\psi}^{\mathrm{DCL}}_j - \breve{\psi}^{\mathrm{DCL}}_j\right) + \tilde{T}^{\mathrm{DCL}}_{j-1,k}\left(\tilde{\psi}^{\mathrm{DCL}}_j - \breve{\psi}^{\mathrm{DCL}}_j\right) \\
&= \gamma_j \left(\left(\prod_{l=n-j+1}^{k-1} \bar{\varphi}^{\mathrm{CL}}_l\right)(\bar{\varphi}^{\mathrm{CL}}_k - 1) - \left(\prod_{l=n-j+1}^{k-1} \tilde{\varphi}^{\mathrm{CL}}_l\right)(\tilde{\varphi}^{\mathrm{CL}}_k - 1)\right)\left(\bar{\psi}^{\mathrm{DCL}}_j - \tilde{\psi}^{\mathrm{DCL}}_j\right) \\
&> 0
\end{aligned}
$$

where $\gamma_j := (\bar{A}_{j-1,n-j}\, \tilde{A}_{j-1,n-j})/\breve{A}_{j-1,n-j}$. Cases (2) and (3) can be proved analogously. □

The duality of the chain–ladder method allows for a dual version of Theorem 2.2.2 where accident and development years are interchanged.

We illustrate the results of Theorem 2.2.2 by the following example:

2.2.3 Example. Incremental losses and predictors of the incremental losses of portfolio I:

accident year i	**development year** k 0	1	2	$\bar{\psi}^{\mathrm{DCL}}_i$
0	150	270	240	
1	420	200	*354*	*2.48*
2	400	*330*	*417*	*1.70*
$\bar{\varphi}^{\mathrm{CL}}_k$		*1.82*	*1.57*	

Incremental losses and predictors of the incremental losses of portfolio II:

accident year i	**development year** k 0	1	2	$\tilde{\psi}^{\mathrm{DCL}}_i$
0	200	200	200	
1	200	200	*200*	*2.00*
2	200	*200*	*200*	*1.50*
$\tilde{\varphi}^{\mathrm{CL}}_k$		*2.00*	*1.50*	

Sum of the predictors of portfolios I & II:

accident year i	**development year** k 0	1	2	
0				
1			*554*	
2		*530*	*617*	

Incremental losses and predictors of the incremental losses of the aggregated portfolio:

accident year i	**development year** k 0	1	2	
0	350	470	440	
1	620	400	*547*	
2	600	*538*	*611*	
$\breve{\varphi}_k^{\text{CL}}$		*1.90*	*1.54*	

The conditions of Theorem 2.2.2 (1) are fulfilled for $(i,k) = (1,2)$ as well as $(i,k) = (2,2)$ and those of Theorem 2.2.2 (2) for $(i,k) = (2,1)$. So, we have

$$\begin{aligned} \bar{Z}_{1,2}^{\text{CL}} + \tilde{Z}_{1,2}^{\text{CL}} = 554 &\quad > \quad 547 = \breve{Z}_{1,2}^{\text{CL}} \\ \bar{Z}_{2,1}^{\text{CL}} + \tilde{Z}_{2,1}^{\text{CL}} = 530 &\quad < \quad 538 = \breve{Z}_{2,1}^{\text{CL}} \\ \bar{Z}_{2,2}^{\text{CL}} + \tilde{Z}_{2,2}^{\text{CL}} = 617 &\quad > \quad 611 = \breve{Z}_{2,2}^{\text{CL}} \end{aligned}$$

However, for the primal cumulative losses we have

$$\bar{S}_{2,2}^{\text{CL}} + \tilde{S}_{2,2}^{\text{CL}} = 1747 \ngtr 1749 = \breve{S}_{2,2}^{\text{CL}}$$

Now, we want to study the aggregation problem on the basis of cumulative losses. For that purpose, we need to strengthen the conditions given in Theorem 2.2.2.

2.2.4 Lemma. *Consider $i, k \in \{1, ..., n\}$ with $i + k > n$. Assume that*

$$\prod_{l=n-j+1}^{k} \bar{\varphi}_l^{\text{CL}} > \prod_{l=n-j+1}^{k} \tilde{\varphi}_l^{\text{CL}}$$

holds for all $j \in \{n-k+1, ..., i\}$. Then

$$\left(\prod_{l=n-j+1}^{k-1} \bar{\varphi}_l^{\text{CL}} \right) (\bar{\varphi}_k^{\text{CL}} - 1) > \left(\prod_{l=n-j+1}^{k-1} \tilde{\varphi}_l^{\text{CL}} \right) (\tilde{\varphi}_k^{\text{CL}} - 1)$$

for all $j \in \{n-k+1, ..., i\}$.

Proof. By assumption, we have $\bar{\varphi}_k^{\text{CL}} > \tilde{\varphi}_k^{\text{CL}}$, hence $(\bar{\varphi}_k^{\text{CL}} - 1)/\bar{\varphi}_k^{\text{CL}} > (\tilde{\varphi}_k^{\text{CL}} - 1)\tilde{\varphi}_k^{\text{CL}}$ and thus

$$\begin{aligned} \left(\prod_{l=n-j+1}^{k-1} \bar{\varphi}_l^{\text{CL}} \right) (\bar{\varphi}_k^{\text{CL}} - 1) &= \left(\prod_{l=n-j+1}^{k} \bar{\varphi}_l^{\text{CL}} \right) \frac{\bar{\varphi}_k^{\text{CL}} - 1}{\bar{\varphi}_k^{\text{CL}}} \\ &> \left(\prod_{l=n-j+1}^{k} \tilde{\varphi}_l^{\text{CL}} \right) \frac{\tilde{\varphi}_k^{\text{CL}} - 1}{\tilde{\varphi}_k^{\text{CL}}} \\ &= \left(\prod_{l=n-j+1}^{k-1} \tilde{\varphi}_l^{\text{CL}} \right) (\tilde{\varphi}_k^{\text{CL}} - 1) \end{aligned}$$

for all $j \in \{n-k+1, ..., i\}$. $\square$

As can be seen from Example 2.2.3, the converse implication is in general not true.

For the cumulative losses we have the following result:

2.2.5 Theorem. *Consider $i, k \in \{1, ..., n\}$ with $i + k > n$.*

(1) *Assume that*

$$\left(\prod_{l=n-j+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-j+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right) \left(\bar{\psi}_j^{\mathrm{DCL}} - \tilde{\psi}_j^{\mathrm{DCL}} \right) > 0$$

holds for all $j \in \{n-k+1, ..., i\}$. Then

$$\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}} > \breve{S}_{j,k}^{\mathrm{CL}}$$

for all $j \in \{n-k+1, ..., i\}$.

(2) *Assume that*

$$\left(\prod_{l=n-j+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-j+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right) \left(\bar{\psi}_j^{\mathrm{DCL}} - \tilde{\psi}_j^{\mathrm{DCL}} \right) < 0$$

holds for all $j \in \{n-k+1, ..., i\}$. Then

$$\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}} < \breve{S}_{j,k}^{\mathrm{CL}}$$

for all $j \in \{n-k+1, ..., i\}$.

(3) *The following are equivalent:*

(a) *The identity*

$$\left(\prod_{l=n-j+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-j+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right) \left(\bar{\psi}_j^{\mathrm{DCL}} - \tilde{\psi}_j^{\mathrm{DCL}} \right) = 0$$

holds for all $j \in \{n-k+1, ..., i\}$.

(b) *The identity*

$$\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}} = \breve{S}_{j,k}^{\mathrm{CL}}$$

holds for all $j \in \{n-k+1, ..., i\}$.

Proof. Assume that the condition of (1) is fulfilled. Initially, we prove the assertion for $j = n-k+1$. Using Lemma A.4 for cell $(n-k+1, k)$ we obtain

$$\begin{aligned}
&\bar{S}_{n-k+1,k}^{\mathrm{CL}} + \tilde{S}_{n-k+1,k}^{\mathrm{CL}} - \breve{S}_{n-k+1,k}^{\mathrm{CL}} \\
&= \bar{S}_{n-k+1,k-1}^{\mathrm{CL}} \bar{\varphi}_k^{\mathrm{CL}} + \tilde{S}_{n-k+1,k-1}^{\mathrm{CL}} \tilde{\varphi}_k^{\mathrm{CL}} - \breve{S}_{n-k+1,k-1}^{\mathrm{CL}} \breve{\varphi}_k^{\mathrm{CL}} \\
&= \bar{S}_{n-k+1,k-1}^{\mathrm{CL}} \bar{\varphi}_k^{\mathrm{CL}} + \tilde{S}_{n-k+1,k-1}^{\mathrm{CL}} \tilde{\varphi}_k^{\mathrm{CL}} - \left(\bar{S}_{n-k+1,k-1}^{\mathrm{CL}} + \tilde{S}_{n-k+1,k-1}^{\mathrm{CL}} \right) \breve{\varphi}_k^{\mathrm{CL}} \\
&= \bar{S}_{n-k+1,k-1}^{\mathrm{CL}} (\bar{\varphi}_k^{\mathrm{CL}} - \breve{\varphi}_k^{\mathrm{CL}}) + \tilde{S}_{n-k+1,k-1}^{\mathrm{CL}} (\tilde{\varphi}_k^{\mathrm{CL}} - \breve{\varphi}_k^{\mathrm{CL}}) \\
&= \gamma_{n-k+1} \left(\bar{\varphi}_k^{\mathrm{CL}} - \tilde{\varphi}_k^{\mathrm{CL}} \right) \left(\bar{\psi}_{n-k+1}^{\mathrm{DCL}} - \tilde{\psi}_{n-k+1}^{\mathrm{DCL}} \right) \\
&> 0
\end{aligned}$$

where $\gamma_{n-k+1} := (\bar{A}_{n-k,k-1} \, \tilde{A}_{n-k,k-1}) / \breve{A}_{n-k,k-1}$. Now, consider $j \in \{n-k+2, ..., i\}$.

We assume that the assertion holds for $j-1$ and we prove the requested inequality for j. Using Lemmas A.1 and A.4 we obtain

$$\begin{aligned}
&\bar{S}^{\mathrm{CL}}_{j,k} + \tilde{S}^{\mathrm{CL}}_{j,k} - \breve{S}^{\mathrm{CL}}_{j,k} \\
&= \bar{S}_{j,n-j} \prod_{l=n-j+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l + \tilde{S}_{j,n-j} \prod_{l=n-j+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l - \breve{S}_{j,n-j} \prod_{l=n-j+1}^{k} \breve{\varphi}^{\mathrm{CL}}_l \\
&= \bar{S}_{j,n-j} \prod_{l=n-j+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l + \tilde{S}_{j,n-j} \prod_{l=n-j+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l - (\bar{S}_{j,n-j} + \tilde{S}_{j,n-j}) \prod_{l=n-j+1}^{k} \breve{\varphi}^{\mathrm{CL}}_l \\
&= \bar{S}_{j,n-j} \left(\prod_{l=n-j+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l - \prod_{l=n-j+1}^{k} \breve{\varphi}^{\mathrm{CL}}_l \right) + \tilde{S}_{j,n-j} \left(\prod_{l=n-j+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l - \prod_{l=n-j+1}^{k} \breve{\varphi}^{\mathrm{CL}}_l \right) \\
&= \bar{S}_{j,n-j} \left(\prod_{l=n-j+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l - \frac{\sum_{h=0}^{j-1} \breve{S}^{\mathrm{CL}}_{h,k}}{\breve{A}_{j-1,n-j}} \right) + \tilde{S}_{j,n-j} \left(\prod_{l=n-j+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l - \frac{\sum_{h=0}^{j-1} \breve{S}^{\mathrm{CL}}_{h,k}}{\breve{A}_{j-1,n-j}} \right) \\
&> \bar{S}_{j,n-j} \left(\prod_{l=n-j+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l - \frac{\sum_{h=0}^{j-1} (\bar{S}^{\mathrm{CL}}_{h,k} + \tilde{S}^{\mathrm{CL}}_{h,k})}{\breve{A}_{j-1,n-j}} \right) \\
&\quad + \tilde{S}_{j,n-j} \left(\prod_{l=n-j+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l - \frac{\sum_{h=0}^{j-1} (\bar{S}^{\mathrm{CL}}_{h,k} + \tilde{S}^{\mathrm{CL}}_{h,k})}{\breve{A}_{j-1,n-j}} \right) \\
&= \gamma_j \left(\prod_{l=n-j+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l - \prod_{l=n-j+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l \right) \left(\bar{\psi}^{\mathrm{DCL}}_j - \tilde{\psi}^{\mathrm{DCL}}_j \right) \\
&> 0
\end{aligned}$$

where $\gamma_j := (\bar{A}_{j-1,n-j}\, \tilde{A}_{j-1,n-j})/\breve{A}_{j-1,n-j}$. Cases (2) and (3) can be proved analogously. □

For the case $i = k = n$ Theorem 2.2.5 is due to Ajne [1994].

Due to Lemma 2.2.4 the results of Theorem 2.2.5 are also valid for the incremental losses. As can be seen from Example 2.2.3, the converse implication is in general not true. Additionally, the duality of the chain–ladder method allows for a dual version of Theorem 2.2.5.

2.2.6 Example. Incremental losses and predictors of the incremental losses of portfolio I:

accident year i	**development year** k 0	1	2	$\bar{\psi}^{\mathrm{DCL}}_i$
0	100	100	300	
1	300	100	*600*	*3.00*
2	400	*200*	*900*	*2.00*
$\bar{\varphi}^{\mathrm{CL}}_k$		*1.50*	*2.50*	

Incremental losses and predictors of the incremental losses of portfolio II:

accident year i	**development year** k 0	1	2	$\tilde{\psi}_i^{\mathrm{DCL}}$
0	200	200	200	
1	200	200	*200*	*2.00*
2	200	*200*	*200*	*1.50*
$\tilde{\varphi}_k^{\mathrm{CL}}$		*2.00*	*1.50*	

Sum of the predictors of portfolios I & II:

accident year i	**development year** k 0	1	2	
0				
1			*800*	
2		*400*	*1100*	

Incremental losses and predictors of the incremental losses of the aggregated portfolio:

accident year i	**development year** k 0	1	2	
0	300	300	500	
1	500	300	*667*	
2	600	*450*	*875*	
$\breve{\varphi}_k^{\mathrm{CL}}$		*1.75*	*1.83*	

The conditions of Theorem 2.2.5 (1) are fulfilled for $(i,k) = (1,2)$ as well as $(i,k) = (2,2)$ and those of Theorem 2.2.5 (2) for $(i,k) = (2,1)$. So, we have

$$\begin{aligned}
\bar{S}_{1,2}^{\mathrm{CL}} + \tilde{S}_{1,2}^{\mathrm{CL}} = 1600 \quad &> \quad 1467 = \breve{S}_{1,2}^{\mathrm{CL}} \\
\bar{S}_{2,1}^{\mathrm{CL}} + \tilde{S}_{2,1}^{\mathrm{CL}} = 1000 \quad &< \quad 1050 = \breve{S}_{2,1}^{\mathrm{CL}} \\
\bar{S}_{2,2}^{\mathrm{CL}} + \tilde{S}_{2,2}^{\mathrm{CL}} = 2100 \quad &> \quad 1925 = \breve{S}_{2,2}^{\mathrm{CL}}
\end{aligned}$$

The same inequalities are valid for the incremental losses.

In contrast to Theorems 2.2.2 and 2.2.5 which contain local results for the aggregation problem, i.e results for the chain–ladder predictors of a single cell, Corollary 2.2.8 finally provides a global result. The following lemma is obvious.

2.2.7 Lemma. *Assume that* $\bar{\varphi}_l^{\mathrm{CL}} > \tilde{\varphi}_l^{\mathrm{CL}}$ *holds for all* $l \in \{1, ..., n\}$. *Then*

$$\prod_{l=n-j+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} > \prod_{l=n-j+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}}$$

for all $i, k \in \{1, ..., n\}$ *such that* $i + k > n$ *and* $j \in \{n-k+1, ..., i\}$.

As can be seen from Example 2.2.6, the converse implication is in general not true.

If both chain–ladder factors, primal as well as dual, of one subportfolio are dominated by both factors of the second subportfolio, then the sum of the chain–ladder predictors based on the subportfolios exceeds the corresponding chain–ladder predictor based on the aggregated portfolio. The following corollary describes this result and includes analogous statements for the converse inequality and identity. Corollary 2.2.8 is an immediate consequence of Theorem 2.2.2 and Lemmas 2.2.4 and 2.2.7.

2.2.8 Corollary.

(1) *Assume that* $\bar{\varphi}_k^{\mathrm{CL}} > \tilde{\varphi}_k^{\mathrm{CL}}$ *holds for all* $k \in \{1, ..., n\}$ *and* $\bar{\psi}_i^{\mathrm{DCL}} > \tilde{\psi}_i^{\mathrm{DCL}}$ *holds for all* $i \in \{1, ..., n\}$. *Then*

$$\bar{Z}_{i,k}^{\mathrm{CL}} + \tilde{Z}_{i,k}^{\mathrm{CL}} > \breve{Z}_{i,k}^{\mathrm{CL}} \qquad \text{and} \qquad \bar{S}_{i,k}^{\mathrm{CL}} + \tilde{S}_{i,k}^{\mathrm{CL}} > \breve{S}_{i,k}^{\mathrm{CL}}$$

for all $i + k > n$.

(2) *Assume that* $\bar{\varphi}_k^{\mathrm{CL}} > \tilde{\varphi}_k^{\mathrm{CL}}$ *holds for all* $k \in \{1, ..., n\}$ *and* $\bar{\psi}_i^{\mathrm{DCL}} < \tilde{\psi}_i^{\mathrm{DCL}}$ *holds for all* $i \in \{1, ..., n\}$. *Then*

$$\bar{Z}_{i,k}^{\mathrm{CL}} + \tilde{Z}_{i,k}^{\mathrm{CL}} < \breve{Z}_{i,k}^{\mathrm{CL}} \qquad \text{and} \qquad \bar{S}_{i,k}^{\mathrm{CL}} + \tilde{S}_{i,k}^{\mathrm{CL}} < \breve{S}_{i,k}^{\mathrm{CL}}$$

for all $i + k > n$.

(3) *Assume that* $\bar{\varphi}_k^{\mathrm{CL}} = \tilde{\varphi}_k^{\mathrm{CL}}$ *holds for all* $k \in \{1, ..., n\}$ *or* $\bar{\psi}_i^{\mathrm{DCL}} = \tilde{\psi}_i^{\mathrm{DCL}}$ *holds for all* $i \in \{1, ..., n\}$. *Then*

$$\bar{Z}_{i,k}^{\mathrm{CL}} + \tilde{Z}_{i,k}^{\mathrm{CL}} = \breve{Z}_{i,k}^{\mathrm{CL}} \qquad \text{and} \qquad \bar{S}_{i,k}^{\mathrm{CL}} + \tilde{S}_{i,k}^{\mathrm{CL}} = \breve{S}_{i,k}^{\mathrm{CL}}$$

for all $i + k > n$.

Corollary 2.2.8 extends a result of Klemmt [2005].

2.2.9 Example. Incremental losses and predictors of the incremental losses of portfolio I:

accident year i	**development year** k: 0	1	2	3	$\bar{\psi}_i^{\mathrm{DCL}}$
0	230	110	60	20	
1	240	120	80	*22*	*2.10*
2	230	120	*70*	*21*	*1.50*
3	210	*105*	*63*	*19*	*1.30*
$\bar{\varphi}_k^{\mathrm{CL}}$		*1.50*	*1.20*	*1.05*	

Incremental losses and predictors of the incremental losses of portfolio II:

accident year i	**development year** k: 0	1	2	3	$\tilde{\psi}_i^{\mathrm{DCL}}$
0	780	140	80	10	
1	760	120	100	*10*	*1.98*
2	410	130	*54*	*6*	*1.30*
3	390	*78*	*47*	*5*	*1.20*
$\tilde{\varphi}_k^{\mathrm{CL}}$		*1.20*	*1.10*	*1.01*	

Sum of the predictors of portfolios I & II:

accident year i	**development year** k 0	1	2	3	
0					
1				*32*	
2			*124*	*27*	
3		*183*	*110*	*24*	

Incremental losses and predictors of the incremental losses of the aggregated portfolio:

accident year i	**development year** k 0	1	2	3	
0	1010	250	140	30	
1	1000	240	180	*30*	
2	640	250	*114*	*22*	
3	600	*168*	*98*	*19*	
$\breve{\varphi}_k^{\text{CL}}$		*1.28*	*1.13*	*1.02*	

These results are in accordance with Corollary 2.2.8 (1).

2.3 Remarks

The results given in Theorem 2.2.2 (1), (2) and Theorem 2.2.5 (1), (2) for a given cell remain valid under weaker conditions. If we replace the given conditions up to a certain index by non–strict inequalities, we also obtain non–strict inequalities for the chain–ladder predictors up to this index.

The chain–ladder method is based on the assumption that there exists a development pattern for factors.

– If we assume that there exists a development pattern for factors for each subportfolio, then we have parameters $\bar{\varphi}_k$ and $\tilde{\varphi}_k$ with

$$\begin{aligned} E[\bar{S}_{i,k}] &= E[\bar{S}_{i,k-1}]\,\bar{\varphi}_k \\ E[\tilde{S}_{i,k}] &= E[\tilde{S}_{i,k-1}]\,\tilde{\varphi}_k \end{aligned}$$

for all $k \in \{1, ..., n\}$ and $i \in \{0, 1, ..., n\}$.

– If we assume that there exists a development pattern for factors for the full portfolio, then we have parameters $\breve{\varphi}_k$ with

$$E[\breve{S}_{i,k}] = E[\breve{S}_{i,k-1}]\,\breve{\varphi}_k$$

for all $k \in \{1, ..., n\}$ and $i \in \{0, 1, ..., n\}$.

Hence, there exists both a development pattern for factors for each subportfolio and a development pattern for factors for the full portfolio, if and only if, for every $k \in \{1, ..., n\}$ there exists some $c_{k-1} \in (0, \infty)$ with

$$\frac{E[\bar{S}_{i,k-1}]}{E[\tilde{S}_{i,k-1}]} = c_{k-1}$$

for all $i \in \{0, 1, ..., n\}$.

For a consistent modelling of the subportfolios and the full portfolio one can use the multivariate chain–ladder model which is the basis of the multivariate chain–ladder method; see Pröhl and Schmidt [2005]. The multivariate chain–ladder model describes not only the subportfolios but also the relation between them.

3 Additive method and aggregation problem

In the present section we study the aggregation problem for the additive method. Because of the volume measures used in the additive method, the problem of whether or not the sum of the predictors based on the subportfolios equals the corresponding predictor based on the aggregated portfolio can be decided from a single and simple equation (Lemma 3.2.2).

Apparently, the aggregation problem for the additive method has not yet been considered in literature.

3.1 Additive method

The *additive method* is based on known *volume measures* v_i of accident year i, $i \in \{0, 1, ..., n\}$ and on the *additive incremental loss ratios*

$$\zeta_k^{\mathrm{AD}} := \frac{\sum_{i=0}^{n-k} Z_{i,k}}{\sum_{i=0}^{n-k} v_i}$$

of development year k, $k \in \{0, 1, ..., n\}$, and consists in the prediction of the incremental losses $Z_{i,k}$, $i + k > n$, by the *additive predictors*

$$Z_{i,k}^{\mathrm{AD}} := v_i \, \zeta_k^{\mathrm{AD}}$$

with $i + k > n$. For the prediction of the cumulative losses we use the *additive predictors*

$$S_{i,k}^{\mathrm{AD}} := S_{i,n-i} + v_i \sum_{l=n-i+1}^{k} \zeta_l^{\mathrm{AD}}$$

with $i + k > n$.

3.2 Aggregation problem

For the cumulative losses, the incremental loss ratios, the volume measures and the additive predictors we take over the notation used for the incremental losses.

Since the additive predictors of the incremental losses are invariant with respect to the multiplication of the volume measures with a positive scalar, we set

$$\breve{v}_i := \alpha \, (\bar{v}_i + \tilde{v}_i)$$

for all $i \in \{0, 1, ..., n\}$ and some $\alpha \in (0, \infty)$.

A similar result to Lemma 2.2.1 holds for the additive incremental loss ratios.

3.2.1 Lemma. *The identity*

$$\breve{\zeta}_k^{\mathrm{AD}} = \frac{\sum_{i=0}^{n-k} \bar{v}_i}{\sum_{i=0}^{n-k} \breve{v}_i} \, \bar{\zeta}_k^{\mathrm{AD}} + \frac{\sum_{i=0}^{n-k} \tilde{v}_i}{\sum_{i=0}^{n-k} \breve{v}_i} \, \tilde{\zeta}_k^{\mathrm{AD}}$$

holds for all $k \in \{0, 1, ..., n\}$.

First of all, we want to study the aggregation problem on the basis of incremental losses. We have the following result:

3.2.2 Lemma. *For each* $i, k \in \{1, ..., n\}$ *with* $i + k > n$ *there exists some* $c_k \in (0, \infty)$ *with*

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} - \breve{Z}_{i,k}^{\mathrm{AD}} = c_k \left(\frac{\bar{v}_i}{\sum_{j=0}^{n-k} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-k} \tilde{v}_j} \right) \left(\bar{\zeta}_k^{\mathrm{AD}} - \tilde{\zeta}_k^{\mathrm{AD}} \right)$$

Proof. Using Lemma 3.2.1 we obtain

$$\begin{aligned}
&\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} - \breve{Z}_{i,k}^{\mathrm{AD}} \\
&= \bar{v}_i \, \bar{\zeta}_k^{\mathrm{AD}} + \tilde{v}_i \, \tilde{\zeta}_k^{\mathrm{AD}} - \breve{v}_i \, \breve{\zeta}_k^{\mathrm{AD}} \\
&= \bar{v}_i \, \bar{\zeta}_k^{\mathrm{AD}} + \tilde{v}_i \, \tilde{\zeta}_k^{\mathrm{AD}} - \breve{v}_i \left(\frac{\sum_{j=0}^{n-k} \bar{v}_j}{\sum_{j=0}^{n-k} \breve{v}_j} \, \bar{\zeta}_k^{\mathrm{AD}} + \frac{\sum_{j=0}^{n-k} \tilde{v}_j}{\sum_{j=0}^{n-k} \breve{v}_j} \, \tilde{\zeta}_k^{\mathrm{AD}} \right) \\
&= \left(\bar{v}_i - \breve{v}_i \, \frac{\sum_{j=0}^{n-k} \bar{v}_j}{\sum_{j=0}^{n-k} \breve{v}_j} \right) \bar{\zeta}_k^{\mathrm{AD}} + \left(\tilde{v}_i - \breve{v}_i \, \frac{\sum_{j=0}^{n-k} \tilde{v}_j}{\sum_{j=0}^{n-k} \breve{v}_j} \right) \tilde{\zeta}_k^{\mathrm{AD}} \\
&= \left(\frac{\bar{v}_i \sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)}{\sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)} - \frac{\alpha \, (\bar{v}_i + \tilde{v}_i) \, \sum_{j=0}^{n-k} \bar{v}_j}{\sum_{j=0}^{n-k} \alpha \, (\bar{v}_j + \tilde{v}_j)} \right) \bar{\zeta}_k^{\mathrm{AD}} \\
&\quad + \left(\frac{\tilde{v}_i \sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)}{\sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)} - \frac{\alpha \, (\bar{v}_i + \tilde{v}_i) \, \sum_{j=0}^{n-k} \tilde{v}_j}{\sum_{j=0}^{n-k} \alpha \, (\bar{v}_j + \tilde{v}_j)} \right) \tilde{\zeta}_k^{\mathrm{AD}} \\
&= \left(\frac{\bar{v}_i \sum_{j=0}^{n-k} \tilde{v}_j - \tilde{v}_i \, \sum_{j=0}^{n-k} \bar{v}_j}{\sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)} \right) \bar{\zeta}_k^{\mathrm{AD}} + \left(\frac{\tilde{v}_i \sum_{j=0}^{n-k} \bar{v}_j - \bar{v}_i \, \sum_{j=0}^{n-k} \tilde{v}_j}{\sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)} \right) \tilde{\zeta}_k^{\mathrm{AD}} \\
&= \left(\frac{\bar{v}_i \sum_{j=0}^{n-k} \tilde{v}_j - \tilde{v}_i \, \sum_{j=0}^{n-k} \bar{v}_j}{\sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)} \right) \left(\bar{\zeta}_k^{\mathrm{AD}} - \tilde{\zeta}_k^{\mathrm{AD}} \right) \\
&= \frac{(\sum_{j=0}^{n-k} \bar{v}_j)(\sum_{j=0}^{n-k} \tilde{v}_j)}{\sum_{j=0}^{n-k} (\bar{v}_j + \tilde{v}_j)} \left(\frac{\bar{v}_i}{\sum_{j=0}^{n-k} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-k} \tilde{v}_j} \right) \left(\bar{\zeta}_k^{\mathrm{AD}} - \tilde{\zeta}_k^{\mathrm{AD}} \right)
\end{aligned}$$

□

The previous lemma completely solves the aggregation problem for the incremental losses. The following theorem takes up this result.

3.2.3 Theorem. *Consider $i, k \in \{1, ..., n\}$ with $i + k > n$ and $\bowtie \in \{>, <, =\}$. Then the following are equivalent:*
(a)

$$\left(\frac{\bar{v}_i}{\sum_{j=0}^{n-k} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-k} \tilde{v}_j}\right)\left(\bar{\zeta}_k^{\mathrm{AD}} - \tilde{\zeta}_k^{\mathrm{AD}}\right) \bowtie 0$$

(b)

$$\bar{Z}_{i,k}^{\mathrm{AD}} + \tilde{Z}_{i,k}^{\mathrm{AD}} \bowtie \breve{Z}_{i,k}^{\mathrm{AD}}$$

3.2.4 Example. Incremental losses and predictors of the incremental losses of portfolio I:

accident year i	**development year** k 0	1	2	$\bar{v}_i$
0	260	120	70	*100*
1	205	180	*105*	*150*
2	300	*240*	*140*	*200*
$\bar{\zeta}_k^{\mathrm{AD}}$	*1,70*	*1,20*	*0,70*	

Incremental losses and predictors of the incremental losses of portfolio II:

accident year i	**development year** k 0	1	2	$\tilde{v}_i$
0	300	200	160	*200*
1	260	250	*200*	*250*
2	340	*300*	*240*	*300*
$\tilde{\zeta}_k^{\mathrm{AD}}$	*1,20*	*1,00*	*0,80*	

Sum of the predictors of portfolios I & II:

accident year i	**development year** k 0	1	2	
0				
1			*305*	
2		*540*	*380*	

Incremental losses and predictors of the incremental losses of the aggregated portfolio:

accident year i	**development year** k 0	1	2	$\breve{v}_i$
0	560	320	230	*300*
1	465	430	*307*	*400*
2	640	*536*	*383*	*500*
$\breve{\zeta}_k^{\mathrm{AD}}$	*1,39*	*1,07*	*0,77*	

In the case of incremental losses, Lemma 3.2.2 demonstrates completely the relation between the sum of the additive predictors based on the subportfolios and the corresponding additive predictor based on the aggregated portfolio. So, we have

$$\begin{aligned} \bar{Z}_{1,2}^{\mathrm{AD}} + \tilde{Z}_{1,2}^{\mathrm{AD}} = 305 \quad &< \quad 307 = \breve{Z}_{1,2}^{\mathrm{AD}} \\ \bar{Z}_{2,1}^{\mathrm{AD}} + \tilde{Z}_{2,1}^{\mathrm{AD}} = 540 \quad &> \quad 536 = \breve{Z}_{2,1}^{\mathrm{AD}} \\ \bar{Z}_{2,2}^{\mathrm{AD}} + \tilde{Z}_{2,2}^{\mathrm{AD}} = 380 \quad &< \quad 383 = \breve{Z}_{2,2}^{\mathrm{AD}} \end{aligned}$$

However, for the cumulative losses we have

$$\bar{S}_{2,2}^{\mathrm{AD}} + \tilde{S}_{2,2}^{\mathrm{AD}} = 1560 \not< 1559 = \breve{S}_{2,2}^{\mathrm{AD}}$$

As can be seen from Example 3.2.4, the aggregation problem on the basis of cumulative losses shall be considered separately. The following result is an immediate consequence of Lemma 3.2.2.

3.2.5 Corollary. *Consider $i, k \in \{1, ..., n\}$ with $i + k > n$.*

(1) *Assume that*

$$\left(\frac{\bar{v}_i}{\sum_{j=0}^{n-l} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-l} \tilde{v}_j}\right)\left(\bar{\zeta}_l^{\mathrm{AD}} - \tilde{\zeta}_l^{\mathrm{AD}}\right) > 0$$

holds for all $l \in \{n-i+1, ..., k\}$. Then

$$\bar{S}_{i,l}^{\mathrm{AD}} + \tilde{S}_{i,l}^{\mathrm{AD}} > \breve{S}_{i,l}^{\mathrm{AD}}$$

for all $l \in \{n-i+1, ..., k\}$.

(2) *Assume that*

$$\left(\frac{\bar{v}_i}{\sum_{j=0}^{n-l} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-l} \tilde{v}_j}\right)\left(\bar{\zeta}_l^{\mathrm{AD}} - \tilde{\zeta}_l^{\mathrm{AD}}\right) < 0$$

holds for all $l \in \{n-i+1, ..., k\}$. Then

$$\bar{S}_{i,l}^{\mathrm{AD}} + \tilde{S}_{i,l}^{\mathrm{AD}} < \breve{S}_{i,l}^{\mathrm{AD}}$$

for all $l \in \{n-i+1, ..., k\}$.

(3) *The following are equivalent:*

(a) *The identity*

$$\left(\frac{\bar{v}_i}{\sum_{j=0}^{n-l} \bar{v}_j} - \frac{\tilde{v}_i}{\sum_{j=0}^{n-l} \tilde{v}_j}\right)\left(\bar{\zeta}_l^{\mathrm{AD}} - \tilde{\zeta}_l^{\mathrm{AD}}\right) = 0$$

holds for all $l \in \{n-i+1, ..., k\}$.

(b) *The identity*

$$\bar{S}_{i,l}^{\mathrm{AD}} + \tilde{S}_{i,l}^{\mathrm{AD}} = \breve{S}_{i,l}^{\mathrm{AD}}$$

holds for all $l \in \{n-i+1, ..., k\}$.

The concluding corollary is obvious.

3.2.6 Corollary. *Assume that there exists some $c \in (0, \infty)$ with $\bar{v}_i/\tilde{v}_i = c$ for all $i \in \{0, 1, ..., n\}$. Then*

$$\bar{Z}^{\mathrm{AD}}_{i,k} + \tilde{Z}^{\mathrm{AD}}_{i,k} = \breve{Z}^{\mathrm{AD}}_{i,k} \qquad \textit{and} \qquad \bar{S}^{\mathrm{AD}}_{i,k} + \tilde{S}^{\mathrm{AD}}_{i,k} = \breve{S}^{\mathrm{AD}}_{i,k}$$

for all $i, k \in \{1, ..., n\}$. In this case the additive method is additive.

3.3 Remarks

Unlike the chain–ladder method, the additive method uses not only the information contained in the run–off triangle but also certain known volume measures, and the choice of the volume measures may considerably effect the aggregation problem. A typical example of such a situation is the classification of a given portfolio of risks into large and basic losses.

– Using the number of policies as volume measures for both subportfolios, then $\bar{v}_i/\tilde{v}_i = 1$ and the additivity of the additive method follows from Corollary 3.2.6. In this case there is no reason to worry about the aggregation problem.
– Otherwise, if we choose for both subportfolios the expected number of claims as volume measures, then these volume measures will in general fail to be proportional and the additive method may fail to be additive.

Thus, the choice of the underlying volume measures may have an effect on the additivity or non–additivity of the additive method.

The additive method is based on the assumption that there exists a development pattern for cumulative quotas.

– If we assume that there exists a development pattern for cumulative quotas for each subportfolio, then we have parameters $\bar{\zeta}_k$ and $\tilde{\zeta}_k$ with

$$\begin{aligned} E[\bar{Z}_{i,k}] &= \bar{v}_i \, \bar{\zeta}_k \\ E[\tilde{Z}_{i,k}] &= \tilde{v}_i \, \tilde{\zeta}_k \end{aligned}$$

for all $k \in \{0, 1, ..., n\}$ and $i \in \{0, 1, ..., n\}$.

– If we assume that there exists a development pattern for cumulative quotas for the full portfolio, then we have parameters $\breve{\zeta}_k$ with

$$E[\breve{Z}_{i,k}] = \breve{v}_i \, \breve{\zeta}_k$$

for all $k \in \{0, 1, ..., n\}$ and $i \in \{0, 1, ..., n\}$.

Hence, there exists both a development pattern for cumulative quotas for each subportfolio and a development pattern for cumulative quotas for the full portfolio, if and only if, there exists some $c \in (0, \infty)$ with

$$\frac{\bar{v}_i}{\tilde{v}_i} = c$$

for all $i \in \{0, 1, ..., n\}$.

For a consistent modelling of the subportfolios and the full portfolio one can use the multivariate additive model which is the basis of the multivariate additive method; see Hess, Schmidt and Zocher [2006]. The multivariate additive model describes not only the subportfolios but also the relation between them.

A Appendix

In this section we prove some technical Lemmas. Recall that $S^{\mathrm{CL}}_{i,k} := S_{i,k}$ and $T^{\mathrm{DCL}}_{i,k} := T_{i,k}$ for all $i+k \leq n$.

A.1 Lemma. *The identities*

$$\frac{\sum_{j=0}^{i} S^{\mathrm{CL}}_{j,k}}{A_{i,n-i}} = \prod_{l=n-i+1}^{k} \varphi^{\mathrm{CL}}_l = \frac{\sum_{j=0}^{i-1} S^{\mathrm{CL}}_{j,k}}{A_{i-1,n-i}}$$

hold for all $i \in \{1, ..., n\}$, $k \in \{0, 1, ..., n\}$ such that $i+k \geq n$.

Proof. The first identity is a generalization of Lemma 13.5.1 in Schmidt [2009]. There, the proof is given for $k = n$; the general case can be proved quite similar. The second identity is an immediate consequence of the first identity and the definition of the chain–ladder predictors of the primal cumulative losses $S^{\mathrm{CL}}_{i,k}$. □

A.2 Lemma. *The identities*

$$\frac{S^{\mathrm{CL}}_{i,k-1}}{A_{n-k,k-1}} = \left(\prod_{j=n-k+1}^{i-1} \psi^{\mathrm{DCL}}_j\right)(\psi^{\mathrm{DCL}}_i - 1)$$

and

$$\frac{T^{\mathrm{DCL}}_{i-1,k}}{A_{i-1,n-i}} = \left(\prod_{l=n-i+1}^{k-1} \varphi^{\mathrm{CL}}_l\right)(\varphi^{\mathrm{CL}}_k - 1)$$

hold for all $i+k > n$.

Proof. First of all, we obtain

$$\frac{1}{A_{n-k,k-1}} \prod_{l=n-i+1}^{k-1} \varphi^{\mathrm{CL}}_l = \frac{\prod_{l=n-i+1}^{k-1} A_{n-l,l}}{\prod_{l=n-i+1}^{k} A_{n-l,l-1}} = \frac{\prod_{j=n-k+1}^{i-1} A_{j,n-j}}{\prod_{j=n-k+1}^{i} A_{j-1,n-j}} = \frac{1}{A_{i-1,n-i}} \prod_{j=n-k+1}^{i-1} \psi^{\mathrm{DCL}}_j$$

for all $i+k > n$. Now, using this result we also obtain

$$\begin{aligned}
\frac{S^{\mathrm{CL}}_{i,k-1}}{A_{n-k,k-1}} &= \frac{1}{A_{n-k,k-1}} \left(A_{i,n-i} - A_{i-1,n-i}\right) \prod_{l=n-i+1}^{k-1} \varphi^{\mathrm{CL}}_l \\
&= \frac{1}{A_{i-1,n-i}} \left(A_{i,n-i} - A_{i-1,n-i}\right) \prod_{j=n-k+1}^{i-1} \psi^{\mathrm{DCL}}_j \\
&= (\psi^{\mathrm{DCL}}_i - 1) \prod_{j=n-k+1}^{i-1} \psi^{\mathrm{DCL}}_j
\end{aligned}$$

for all $i+k > n$. The dual part can be proved analogously. □

A.3 Lemma. *The identity*

$$\begin{aligned}
&\bar{T}^{\mathrm{DCL}}_{i-1,k}\left(\bar{\psi}^{\mathrm{DCL}}_i - \breve{\psi}^{\mathrm{DCL}}_i\right) + \tilde{T}^{\mathrm{DCL}}_{i-1,k}\left(\tilde{\psi}^{\mathrm{DCL}}_i - \breve{\psi}^{\mathrm{DCL}}_i\right)\\
&= \gamma_i \left(\left(\prod_{l=n-i+1}^{k-1} \bar{\varphi}^{\mathrm{CL}}_l\right)(\bar{\varphi}^{\mathrm{CL}}_k - 1) - \left(\prod_{l=n-i+1}^{k-1} \tilde{\varphi}^{\mathrm{CL}}_l\right)(\tilde{\varphi}^{\mathrm{CL}}_k - 1)\right)\left(\bar{\psi}^{\mathrm{DCL}}_i - \tilde{\psi}^{\mathrm{DCL}}_i\right)
\end{aligned}$$

holds for all $i+k>n$ where $\gamma_i := (\bar{A}_{i-1,n-i}\,\tilde{A}_{i-1,n-i})/\breve{A}_{i-1,n-i}$.

Proof. First of all, using Lemma 2.2.1 we obtain

$$\begin{aligned}
\bar{\psi}^{\mathrm{DCL}}_i - \breve{\psi}^{\mathrm{DCL}}_i &= \bar{\psi}^{\mathrm{DCL}}_i - \frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\,\bar{\psi}^{\mathrm{DCL}}_i - \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\,\tilde{\psi}^{\mathrm{DCL}}_i\\
&= \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\,\bar{\psi}^{\mathrm{DCL}}_i - \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\,\tilde{\psi}^{\mathrm{DCL}}_i\\
&= \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\left(\bar{\psi}^{\mathrm{DCL}}_i - \tilde{\psi}^{\mathrm{DCL}}_i\right)
\end{aligned}$$

for all $i \in \{1, ..., n\}$ and analogously

$$\tilde{\psi}^{\mathrm{DCL}}_i - \breve{\psi}^{\mathrm{DCL}}_i = \frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\left(\tilde{\psi}^{\mathrm{DCL}}_i - \bar{\psi}^{\mathrm{DCL}}_i\right)$$

for all $i \in \{1, ..., n\}$. Using this result and Lemma A.2 we obtain

$$\begin{aligned}
&\bar{T}^{\mathrm{DCL}}_{i-1,k}\left(\bar{\psi}^{\mathrm{DCL}}_i - \breve{\psi}^{\mathrm{DCL}}_i\right) + \tilde{T}^{\mathrm{DCL}}_{i-1,k}\left(\tilde{\psi}^{\mathrm{DCL}}_i - \breve{\psi}^{\mathrm{DCL}}_i\right)\\
&= \bar{T}^{\mathrm{DCL}}_{i-1,k}\,\frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\left(\bar{\psi}^{\mathrm{DCL}}_i - \tilde{\psi}^{\mathrm{DCL}}_i\right) + \tilde{T}^{\mathrm{DCL}}_{i-1,k}\,\frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\left(\tilde{\psi}^{\mathrm{DCL}}_i - \bar{\psi}^{\mathrm{DCL}}_i\right)\\
&= \frac{\bar{A}_{i-1,n-i}\,\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}}\left(\frac{\bar{T}^{\mathrm{DCL}}_{i-1,k}}{\bar{A}_{i-1,n-i}} - \frac{\tilde{T}^{\mathrm{DCL}}_{i-1,k}}{\tilde{A}_{i-1,n-i}}\right)\left(\bar{\psi}^{\mathrm{DCL}}_i - \tilde{\psi}^{\mathrm{DCL}}_i\right)\\
&= \gamma_i \left(\left(\prod_{l=n-i+1}^{k-1} \bar{\varphi}^{\mathrm{CL}}_l\right)(\bar{\varphi}^{\mathrm{CL}}_k - 1) - \left(\prod_{l=n-i+1}^{k-1} \tilde{\varphi}^{\mathrm{CL}}_l\right)(\tilde{\varphi}^{\mathrm{CL}}_k - 1)\right)\left(\bar{\psi}^{\mathrm{DCL}}_i - \tilde{\psi}^{\mathrm{DCL}}_i\right)
\end{aligned}$$

for all $i+k>n$. □

A.4 Lemma. *The identity*

$$\begin{aligned}
&\bar{S}_{i,n-i}\left(\prod_{l=n-i+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l - \frac{\sum_{j=0}^{i-1}(\bar{S}^{\mathrm{CL}}_{j,k} + \tilde{S}^{\mathrm{CL}}_{j,k})}{\breve{A}_{i-1,n-i}}\right) + \tilde{S}_{i,n-i}\left(\prod_{l=n-i+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l - \frac{\sum_{j=0}^{i-1}(\bar{S}^{\mathrm{CL}}_{j,k} + \tilde{S}^{\mathrm{CL}}_{j,k})}{\breve{A}_{i-1,n-i}}\right)\\
&= \gamma_i\left(\prod_{l=n-i+1}^{k} \bar{\varphi}^{\mathrm{CL}}_l - \prod_{l=n-i+1}^{k} \tilde{\varphi}^{\mathrm{CL}}_l\right)\left(\bar{\psi}^{\mathrm{DCL}}_i - \tilde{\psi}^{\mathrm{DCL}}_i\right)
\end{aligned}$$

holds for all $i+k>n$ where $\gamma_i := (\bar{A}_{i-1,n-i}\,\tilde{A}_{i-1,n-i})/\breve{A}_{i-1,n-i}$.

Proof. First, using Lemma A.1 we obtain

$$
\begin{aligned}
&\prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \frac{\sum_{j=0}^{i-1}(\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}})}{\breve{A}_{i-1,n-i}} \\
&= \prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \frac{\sum_{j=0}^{i-1} \bar{S}_{j,k}^{\mathrm{CL}}}{\breve{A}_{i-1,n-i}} - \frac{\sum_{j=0}^{i-1} \tilde{S}_{j,k}^{\mathrm{CL}}}{\breve{A}_{i-1,n-i}} \\
&= \prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \\
&= \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \left(\prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right)
\end{aligned}
$$

for all $i + k > n$ and analogously

$$
\prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} - \frac{\sum_{j=0}^{i-1}(\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}})}{\breve{A}_{i-1,n-i}} = \frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \left(\prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} \right)
$$

for all $i + k > n$. Moreover, applying Lemma A.2 for $k = n - i + 1$ yields

$$
\frac{S_{i,n-i}}{A_{i-1,n-i}} = \psi_i^{\mathrm{DCL}} - 1
$$

for all $i \in \{1, ..., n\}$. Thus, using the previous results we obtain

$$
\begin{aligned}
&\bar{S}_{i,n-i} \left(\prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \frac{\sum_{j=0}^{i-1}(\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}})}{\breve{A}_{i-1,n-i}} \right) + \tilde{S}_{i,n-i} \left(\prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} - \frac{\sum_{j=0}^{i-1}(\bar{S}_{j,k}^{\mathrm{CL}} + \tilde{S}_{j,k}^{\mathrm{CL}})}{\breve{A}_{i-1,n-i}} \right) \\
&= \bar{S}_{i,n-i} \frac{\tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \left(\prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right) \\
&\quad + \tilde{S}_{i,n-i} \frac{\bar{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \left(\prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} \right) \\
&= \frac{\bar{A}_{i-1,n-i}\, \tilde{A}_{i-1,n-i}}{\breve{A}_{i-1,n-i}} \left(\frac{\bar{S}_{i,n-i}}{\bar{A}_{i-1,n-i}} - \frac{\tilde{S}_{i,n-i}}{\tilde{A}_{i-1,n-i}} \right) \left(\prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right) \\
&= \gamma_i \left(\bar{\psi}_i^{\mathrm{DCL}} - \tilde{\psi}_i^{\mathrm{DCL}} \right) \left(\prod_{l=n-i+1}^{k} \bar{\varphi}_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \tilde{\varphi}_l^{\mathrm{CL}} \right)
\end{aligned}
$$

for all $i + k > n$. □

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