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Solvency Capital Requirement und die Wurzelformel

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Introduction

Risk measures

Risk measures

Let $L^1 = L^1(\Omega, \mathcal{F}, P)$ denote the set of all integrable real-valued random variables (risks).

Definition

A *risk measure* R is a map $R : L^1 \to \mathbb{R}$ with the property

$$P_X = P_Y \quad \Longrightarrow \quad R[X] = R[Y]$$

Definition

A risk measure $R : L^1 \to \mathbb{R}$ is called

- *positively homogeneous* if for all $X \in L^1$ and every $c \in \mathbb{R}_+$

$$R[cX] = cR[X]$$

- *translative* if for all $X \in L^1$ and every $c \in \mathbb{R}$

$$R[X + c] = R[X] + c$$

Value at Risk

Definition

Let $\alpha \in (0, 1)$. Then the map $VaR_\alpha : L^1 \to \mathbb{R}$ given by

$$VaR_\alpha[X] := \inf\left\{ x \in \mathbb{R} \;\middle|\; P[X \leq x] \geq \alpha \right\}$$

is called *Value at Risk* with respect to α.

Lemma

Value at Risk is a positively homogeneous and translative risk measure.

Expected Shortfall

Definition

Let $\alpha \in (0, 1)$. Then the map $ES_\alpha : L^1 \rightarrow \mathbb{R}$ given by

$$ES_\alpha[X] := \frac{1}{1-\alpha} \int_{(\alpha,1)} VaR_\beta[X] \,\mathrm{d}\lambda(\beta)$$

is called *Expected Shortfall* with respect to α.

Lemma

Expected Shortfall is a positively homogeneous and translative risk measure.

Tail Value at Risk

Definition

Let $\alpha \in (0,1)$. Then the map $\boldsymbol{TVaR_\alpha} : L^1 \to \mathbb{R}$ given by

$$TVaR_\alpha[X] := E\Big[X \,\Big|\, X \geq VaR_\alpha[X]\Big]$$

is called *Tail Value at Risk* with respect to α.

Lemma

Tail Value at Risk is a positively homogeneous and translative risk measure.

Solvency Capital Requirement

Solvency Capital Requirement

Definition

Let R be a risk measure. Then the map $SCR_R : L^1 \to \mathbb{R}_+$ given by

$$SCR_R[X] := R[X] - E[X]$$

is called *Solvency Capital Requirement* with respect to R.

Let $\mathbf{X}$ be a square integrable random vector with coordinates X_i and define

$$\rho_{ij} := \frac{\operatorname{cov}[X_i, X_j]}{\sqrt{\operatorname{var}[X_i]}\sqrt{\operatorname{var}[X_j]}}$$

Definition

For a risk measure R and a square integrable random vector $\mathbf{X}$ define

$$\widehat{SCR_R}[\mathbf{1}'\mathbf{X}] \quad := \quad \sqrt{\sum_{i=1}^{d} \sum_{j=1}^{d} \rho_{ij} SCR_R[X_i] SCR_R[X_j]}$$

(*"square root formula"*)

Problem

Find conditions which guarantee that

$$\widehat{SCR_R}[\mathbf{1}'\mathbf{X}] = SCR_R[\mathbf{1}'\mathbf{X}]$$

Theorem

Let $\boldsymbol{R}$ be a positively homogeneous and translative risk measure.
If $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$, then

$$\widehat{SCR_R}[\mathbf{1}'\mathbf{X}] = SCR_R[\mathbf{1}'\mathbf{X}]$$

Theorem

Let $\boldsymbol{R}$ be a positively homogeneous and translative risk measure,
let $\mathbf{X}$ be a square integrable random vector and define

$$Z_i := \frac{X_i - E[X_i]}{\sqrt{\mathrm{var}[X_i]}} \qquad \text{and} \qquad Z := \frac{\mathbf{1}'\mathbf{X} - E[\mathbf{1}'\mathbf{X}]}{\sqrt{\mathrm{var}[\mathbf{1}'\mathbf{X}]}}$$

If $P_{Z_i} = P_Z$ holds for all $i \in \{1, \ldots, d\}$, then

$$\widehat{SCR_R}[\mathbf{1}'\mathbf{X}] = SCR_R[\mathbf{1}'\mathbf{X}]$$

Spherical and elliptical distributions

Spherical distributions

Spherical distributions can be seen as an extension of the multivariate standard normal distribution $\boldsymbol{N_d}(\mathbf{0}, \boldsymbol{I})$.

Theorem

Let $\mathbf{Y}$ *be a random vector. Then the following are equivalent:*

(i) *The identity*

$$P_{U\mathbf{Y}} = P_{\mathbf{Y}}$$

holds for every orthogonal matrix $\boldsymbol{U} \in \mathbb{R}^{d\times d}$.

(ii) *There exists a function* $\vartheta : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *such that*

$$\phi_{\mathbf{Y}}(\mathbf{t}) = \vartheta(\|\mathbf{t}\|^2)$$

(iii) *The identity*

$$P_{\mathbf{a}'\mathbf{Y}} = P_{\|\mathbf{a}\| Y_1}$$

holds for every vector $\mathbf{a} \in \mathbb{R}^d$.

Definition

A random vector $\mathbf{Y}$ has a *spherical distribution* if it fulfils one and therefore all properties of the theorem. In this case we write $\mathbf{Y} \sim S_d(\vartheta)$.
The function ϑ is called the *characteristic generator*.

Lemma

Let $\mathbf{Y} \sim S_d(\vartheta)$. *Then:*

- *The coordinates of* $\mathbf{Y}$ *are identically distributed.*
- *If* $\mathbf{Y}$ *is integrable, then* $E[\mathbf{Y}] = \mathbf{0}$.
- *If* $\mathbf{Y}$ *is square integrable, then* $\mathrm{var}[\mathbf{Y}] = -2\vartheta'(0) \cdot I$.

Example

(Multivariate standard normal)

Let $\mathbf{Y} \sim N_d(\mathbf{0}, I)$. Then its characteristic function satisfies

$$\phi_{\mathbf{Y}}(\mathbf{t}) = \exp\left(-\frac{1}{2}\|\mathbf{t}\|^2\right)$$

such that $\mathbf{Y} \sim S_d(\vartheta)$ with $\vartheta(z) = \exp(-\frac{1}{2}z)$.

□

Theorem

Let $\mathbf{Y}$ *be a random vector with density* $f_{\mathbf{Y}}$. *Then the following are equivalent:*

(i) $\mathbf{Y}$ *has a spherical distribution.*

(ii) *There exists a measurable function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

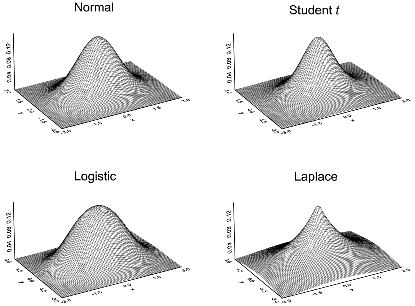
$$f_{\mathbf{Y}}(\mathbf{y}) = g(\|\mathbf{y}\|^2)$$

The function g is called the *density generator*.

Some families of spherical distributions and their density generators:

Normal	$g(z) \sim \exp(-\frac{z}{2})$	
Exponential power	$g(z) \sim \exp(-rz^s)$,	$r, s \in (0, \infty)$
Students t	$g(z) \sim \left(1 + \frac{z}{m}\right)^{-(d+m)/2}$,	$m \in \mathbb{N}$
Laplace	$g(z) \sim \exp(-\sqrt{z})$	
Logistic	$g(z) \sim \frac{\exp(-\sqrt{z})}{(1+\exp(-\sqrt{z}))^2}$	

Densities of some bivariate elliptical distributions:



Elliptical distributions

Elliptical distributions can be seen as an extension of the multivariate normal distribution $N_d(\boldsymbol{\mu}, \Sigma)$.
They are obtained by affine transformations of spherical distributions.

Definition

A random vector $\mathbf{X}$ has an *elliptical distribution* if there exist a random vector $\mathbf{Y} \sim S_k(\vartheta)$, a matrix $A \in \mathbb{R}^{d \times k}$ and a vector $\boldsymbol{\mu} \in \mathbb{R}^d$ satisfying

$$\mathbf{X} = \boldsymbol{\mu} + A\mathbf{Y}$$

In this case, we write

$$\mathbf{X} \sim E_d(\boldsymbol{\mu}, AA', \vartheta)$$

Lemma

(affine transformations)

Assume that $\mathbf{X}$ *has an elliptical distribution. Then:*

- *Every affine transformation of* $\mathbf{X}$ *has an elliptical distribution.*
- *Every coordinate of* $\mathbf{X}$ *has an elliptical distribution.*
- *The sum* $\mathbf{1}'\mathbf{X}$ *of the coordinates of* $\mathbf{X}$ *has an elliptical distribution.*

Lemma

(moments)

Let $\mathbf{X} \sim E_d(\boldsymbol{\mu}, AA', \vartheta)$.

- *If* $\mathbf{X}$ *is integrable, then*

$$E[\mathbf{X}] = \boldsymbol{\mu}$$

- *If* $\mathbf{X}$ *is square integrable, then*

$$\mathrm{var}[\mathbf{X}] = -2\vartheta'(0) \cdot AA'$$

Solvency Capital Requirement

Theorem

Let $\boldsymbol{R}$ *be a positively homogeneous and translative risk measure and let* $\mathbf{X}$ *be a square integrable random vector.*
If $\mathbf{X}$ *has an elliptical distribution, then*

$$\widehat{SCR_R}[\mathbf{1}'\mathbf{X}] = SCR_R[\mathbf{1}'\mathbf{X}]$$

Lemma

Assume that $\mathbf{Y}$ *has a spherical distribution with density generator* $\boldsymbol{g}$*.*
Consider $\mu \in \mathbb{R}^{d}$ *and an invertible matrix* $\boldsymbol{A} \in \mathbb{R}^{d \times d}$*.*
Then $\mathbf{X} = \boldsymbol{\mu} + A\mathbf{Y}$ *has a density* $f_{\mathbf{X}}$ *satisfying*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{|\det(A)|} \cdot g\left(\|A^{-1}(\mathbf{x} - \boldsymbol{\mu})\|^2\right)$$