The admissible-strain model

$$E_M(f) = \int_M W(Df \circ P^{-1})\, d\mathrm{Vol}_{\bar G} \quad , \quad P = I + \frac{v}{2\pi} d\varphi$$

Assume: • Small dislocation $P \sim I$
• small energy $E_M(f) \ll 1 \Rightarrow Df \approx U \in SO(2)$

Then: $U^T Df \circ P^{-1} \simeq U^T Df + (I - P) \equiv \beta$

Def: Admissible strains: $\{ \beta \in L^2(M; \mathbb{R}^{2\times 2}) \mid \operatorname{curl} \beta = 0 ,\ \oint_\gamma \beta = -\underline{v} \}$

$$E^{as}(\beta) = \begin{cases} \int_M W(\beta)\, dx & \text{SZ12, MSZ14, ...} \\ \int_M Q(\beta - I)\, dx \quad , \quad Q(A) = \frac{1}{2} D_I^2 W(A,A) & \text{GLP10, ...} \end{cases}$$

Homogenization of dislocations

Single dislocation of size ε: $E \sim \varepsilon^2 \log \frac{1}{\varepsilon}$

n dislocations of size ε: $E_{n,\varepsilon} \sim \underbrace{n \varepsilon^2 \log \frac{1}{\varepsilon}}_{\text{self energy}} + \underbrace{\overbrace{n^2}^{\#\text{pairs}} \varepsilon^2}_{\text{interaction energy}} \sim$ interaction between pair of disloc.

We would like to understand homogenization limits

$$\Gamma\text{-}\lim_{\substack{\varepsilon \to 0 \\ n_\varepsilon \to \infty}} \frac{1}{h_\varepsilon^2} E_{n,\varepsilon} \quad , \quad h_\varepsilon^2 = \max \{ n \varepsilon^2 \log \frac{1}{\varepsilon}, n^2 \varepsilon^2 \}$$

Results:

- LGP'10, DGP'12,

  $\Gamma\text{-}\lim \frac{1}{h_\varepsilon^2} E_{n,\varepsilon}^{as,lin}$, $\log n_\varepsilon \ll \log \frac{1}{\varepsilon}$ $(h_\varepsilon \ll 1)$

- SZ'12, MSZ'14,'15 , Gin'19 , ...

  $\Gamma\text{-}\lim \frac{1}{h_\varepsilon^2} E_{n,\varepsilon}^{lin}$, $n_\varepsilon = \text{const}$ (SZ'12), $n_\varepsilon \sim \log \frac{1}{\varepsilon}$ (MSZ'14...)

- $\underbrace{\text{CGO'15}}_{\text{linear}}$, $\underbrace{\text{GMS'21, CGM'22}}_{\text{non-linear}}$ 3D admissible strains, $\underbrace{\text{"}n_\varepsilon < C\text{"}}_{\text{line-tension models}}$

- KM'16, EKM'20

  $\Gamma\text{-}\lim E_{n,\varepsilon}$, $n_\varepsilon \sim \frac{1}{\varepsilon}$ $(h_\varepsilon \sim 1)$

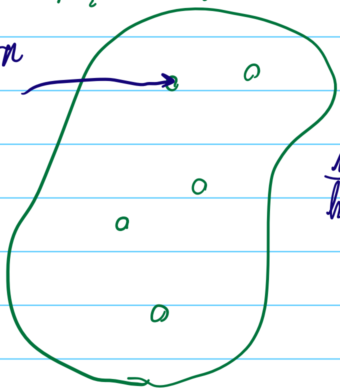
- KM'23 Our focus today

  $\Gamma\text{-}\lim E_{n,\varepsilon}$, $\log n_\varepsilon \ll \log \frac{1}{\varepsilon}$ $(h_\varepsilon \ll 1)$

The limit in the admissible strain model (GLP'10, ...)

$M_\varepsilon = \Omega \setminus B_\varepsilon^i$

dislocation cores B_ε^i



$$AS_\varepsilon = \{ \beta \in L^2(M_\varepsilon; \mathbb{R}^{2\times 2}) \mid \operatorname{curl}\beta = 0,\ \oint_{\partial B_\varepsilon^i} \beta = -\varepsilon v_\varepsilon^i \}$$

$$\frac{1}{h_\varepsilon^2} E_\varepsilon^{as}(\beta_\varepsilon) = \frac{1}{h_\varepsilon^2} \int_{M_\varepsilon} Q(\beta_\varepsilon - I)\, dx$$

$$\Big\downarrow \Gamma \qquad \underbrace{\frac{1}{h_\varepsilon}(\beta_\varepsilon - I)}_{\text{"}\frac{1}{\varepsilon}\operatorname{curl}\beta\text{"}} \xrightarrow{L^2} J, \quad \frac{1}{n_\varepsilon} \sum v_\varepsilon^i \delta_{x_\varepsilon^i} \xrightarrow{*} \mu$$

$$E_0(J,\mu) = \underbrace{\int_\Omega Q(J)\, dx}_{\text{linear elastic energy}} + \underbrace{\int \Sigma\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|}_{\text{self-energy}}$$

strain (J), dislocation density (μ)

self-energy ($n_\varepsilon \lesssim \log \frac{1}{\varepsilon}$)

$$\operatorname{curl} J = \begin{cases} 0 & n_\varepsilon \ll \log\frac{1}{\varepsilon} \quad \text{elastic part is ind. of } \mu \\ -\mu & n_\varepsilon \gtrsim \log \frac{1}{\varepsilon} \end{cases}$$

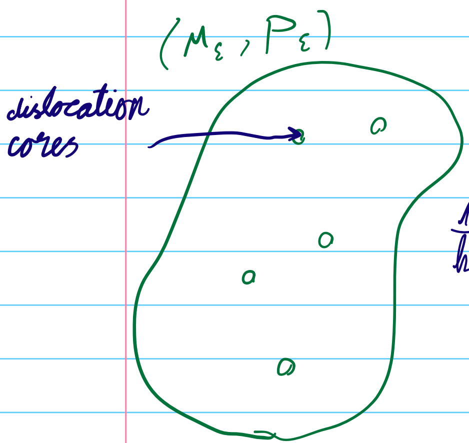
Models of the type

$$E(u, \beta_p) = \int Q(\underbrace{\nabla u - \beta_p}_{\text{elastic strain}}) + \int \varphi(\operatorname{curl}\beta_p)$$

β_p: plastic strain (additive decomposition)

are called "strain-gradient" models (Fleck & Hutchinson '93, Gurtin '00, ...)

The limit in the geometric model (KM'23)



$$f_\varepsilon \in H^1(M_\varepsilon; \mathbb{R}^2)$$

$$\frac{1}{h_\varepsilon^2} E_\varepsilon(f_\varepsilon) = \frac{1}{h_\varepsilon^2} \int_{M_\varepsilon} W(Df_\varepsilon \circ P_\varepsilon^{-1})\, dV_{G_\varepsilon}$$

$$\Big\downarrow \; \Gamma \quad \begin{array}{ll} (M_\varepsilon, P_\varepsilon) \xrightarrow{n_\varepsilon} (\Omega, \mu) & ① \\ \frac{1}{h_\varepsilon}(R_\varepsilon^T Df_\varepsilon - P_\varepsilon) \xrightarrow{L^2} J & ② \end{array}$$

$$E_0(\underbrace{J}_{\text{strain}}, \underbrace{\mu}_{\text{dislocation density}}) = \underbrace{\int_\Omega Q(J)\,dx}_{\text{linear elastic energy}} + \underbrace{\int \Sigma\left(\frac{d\mu}{d|\mu|}\right) d|\mu|}_{\text{self-energy } (n_\varepsilon \lesssim \log\frac{1}{\varepsilon})}$$

$$\operatorname{curl} J = \begin{cases} 0 & n_\varepsilon \ll \log\frac{1}{\varepsilon} \\ -\mu & n_\varepsilon \gtrsim \log\frac{1}{\varepsilon} \end{cases}$$

① <u>Manifold convergence</u> $(M_\varepsilon, P_\varepsilon) \to (\Omega, \mu)$ w.r.t. parameter n_ε if

$\exists Z_\varepsilon : M_\varepsilon \hookrightarrow \Omega$ uniformly bilipschitz s.t.

(i) $|\Omega \setminus M_\varepsilon| \to 0$ (ii) $\|P_\varepsilon - DZ_\varepsilon\|_{L^2} = O(h_\varepsilon)$

(iii) $|P_\varepsilon - DZ_\varepsilon| \lesssim \underbrace{\frac{\varepsilon}{r} + o(1)}$ → distance to closest dislocation

(iv) $\frac{1}{n_\varepsilon \varepsilon} \int_{\partial M_\varepsilon} (\psi \circ Z_\varepsilon) \cdot P_\varepsilon \longrightarrow \int_\Omega \psi \cdot d\mu \quad \forall \psi \in C_c(\Omega)$ ("$\operatorname{curl} P_\varepsilon \xrightarrow{*} \mu$")

② <u>Convergence of strains</u>

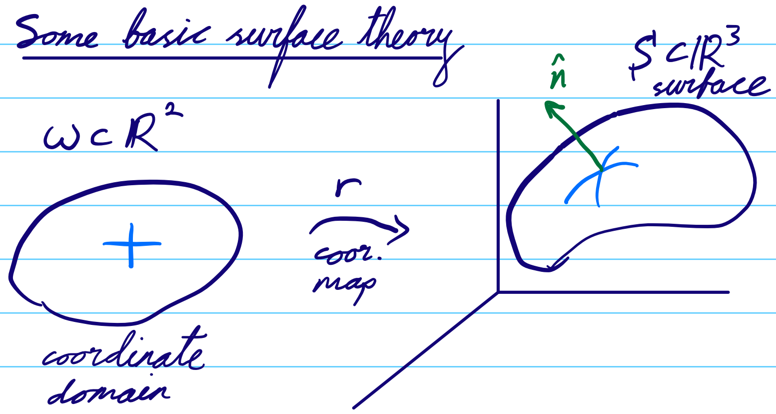
<u>Thm</u> ("FJM"): If $(M_\varepsilon, P_\varepsilon) \xrightarrow{n_\varepsilon} (\Omega, \mu)$, then $\forall f_\varepsilon \in H^1(M_\varepsilon; \mathbb{R}^2)$ $\exists R_\varepsilon \in SO(2)$

$$\int_{M_\varepsilon} |Df_\varepsilon - R_\varepsilon P_\varepsilon|^2 \le C\left(\int_{M_\varepsilon} \operatorname{dist}^2(Df_\varepsilon \circ P_\varepsilon^{-1}, SO(2)) + h_\varepsilon^2\right)$$

R_ε in ② is as in this thm.

II Thin non-Euclidean elastic bodies

Some basic surface theory



$\partial_i r \cdot \partial_j r \quad = \bar{g}_{ij}$ (1^{st} fund. form - distance on S)

$-\partial_i r \cdot \partial_j n \quad = \bar{\mathrm{II}}_{ij}$ (2^{nd} fund. form - change of normal on S)

$\bar{g}$ & $\bar{\mathrm{II}}$ satisfy the Gauss-Codazzi equations:

Gauss: $\bar{K} = \dfrac{\det \bar{\mathrm{II}}}{\det \bar{g}}$, $\bar{K}$ = second order in $\bar{g}$

Codazzi: $\nabla_{[i} \bar{\mathrm{II}}_{j]k} = 0$, i.e. $\partial_2 \bar{\mathrm{II}}_{1k} - \partial_1 \bar{\mathrm{II}}_{2k} = \bar{\mathrm{II}}_{11} \Gamma^1_{k2} + \bar{\mathrm{II}}_{12} (\Gamma^2_{k2} - \Gamma^1_{1k}) - N \Gamma^2_{1k}$

$\bar{g}$ & $\bar{\mathrm{II}}$ charcterize S up to rigid motion:

Thm: If $\bar{g}$ & $\bar{\mathrm{II}}$ satisfy GC and ω is simply connected, then there exists a unique immersed surface (up to rigid motion) S with forms $\bar{g}, \bar{\mathrm{II}}$.

Definition: $\bar{g}$ and $\bar{\mathrm{II}}$ are symmetric (2,0) tensors on TS.

The shape operator $\bar{S}$ is a (1,1) tensor associated w. $\bar{\mathrm{II}}$, i.e.

$\bar{g}(\bar{S}(v), w) = \bar{\mathrm{II}}(v, w)$, or $\bar{g}_{ik} S^k_j = \bar{\mathrm{II}}_{ij}$.

It can be easily verified that $\bar{S}(v) = -D_v \hat{n}$.