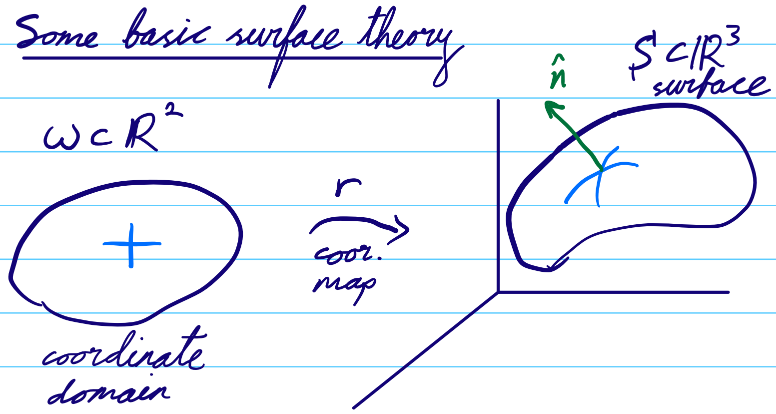
III Thin non-Euclidean elastic bodies

Some basic surface theory



$\partial_i r \cdot \partial_j r \;=\; \bar{g}_{ij}$ (1^{st} fund. form - distance on S)

$-\partial_i r \cdot \partial_j n \;=\; \bar{II}_{ij}$ (2^{nd} fund. form - change of normal on S)

$\bar{g}$ & $\bar{II}$ satisfy the Gauss-Codazzi equations:

Gauss: $\bar{K} = \frac{\det \bar{II}}{\det \bar{g}}$, $\bar{K}$ = second order in $\bar{g}$

Codazzi: $\nabla_{[i} \bar{II}_{j]k} = 0$, i.e. $\partial_2 \bar{II}_{1k} - \partial_1 \bar{II}_{2k} = \bar{II}_{11} \Gamma^1_{k2} + \bar{II}_{12} (\Gamma^2_{k2} - \Gamma^1_{1k}) - N \Gamma^2_{1k}$

$\bar{g}$ & $\bar{II}$ characterize S up to rigid motion:

Thm: If $\bar{g}$ & $\bar{II}$ satisfy GC and ω is simply connected, then there exists a unique immersed surface (up to rigid motion) S with forms $\bar{g}, \bar{II}$.

Definition: $\bar{g}$ and $\bar{II}$ are symmetric $(2,0)$ tensors on TS.

The shape operator $\bar{S}$ is a $(1,1)$ tensor associated w. $\bar{II}$, i.e.

$\bar{g}(\bar{S}(v), w) = \bar{II}(v, w)$, or $\bar{g}_{ik} S^k_j = \bar{II}_{ij}$.

It can be easily verified that $\bar{S}(v) = -\nabla_v \hat{n}$.

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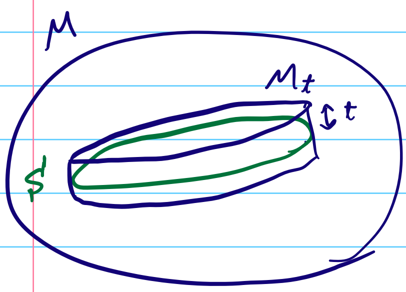
GC and Riemann's thm.

Remark: Define $\Omega = \omega \times (-\varepsilon_0, \varepsilon_0)$, and a metric $\bar{G}$ on Ω by

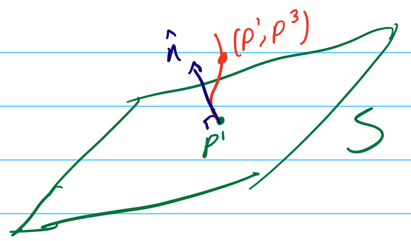
$$\bar{G}(x', x_3) = \left(\begin{array}{c|c} \bar{g}(x') - 2x_3 \bar{\mathrm{II}}(x') + x_3^2 \bar{S}^T(x') \bar{g}(x') \bar{S}(x') & \\ \hline & 1 \end{array} \right)$$

then $\bar{g}$ & $\bar{\mathrm{II}}$ satisfy GC iff $\bar{G}$ is flat, i.e. if $(\Omega, \bar{G})$ can be (locally) isometrically-immersed in $\mathbb{R}^3$.

In the other direction, given a 3-dim. manifold M, and a subman. S



the t-tubular neigh. of S, M_t, is obtained by going via normal geodesics up to time t:



The metric $\bar{G}$, restricted to $\bar{M}_t$, satisfies the following expansion:

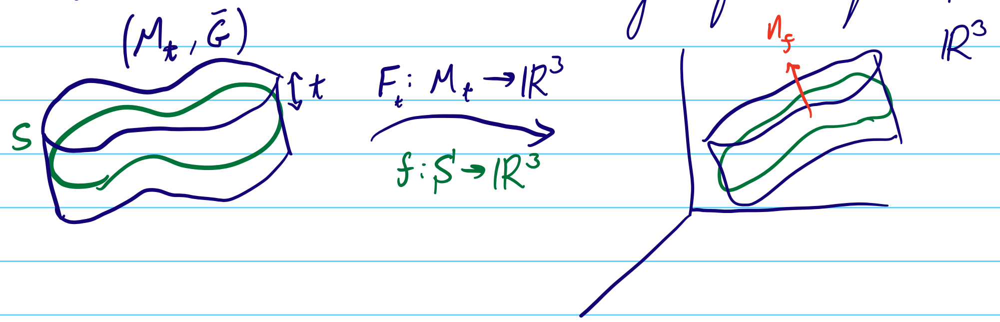
$$\bar{G}(p', p^3) = \left(\begin{array}{c|c} \bar{g}(p') - 2p^3 \bar{\mathrm{II}}(p') + O(|p^3|^2) & \\ \hline & 1 \end{array} \right)$$

where $\bar{g}$ is the induced metric on S, and $\bar{\mathrm{II}} = \mathrm{II}_{S,M}$ is the second form of S in M.

$\bar{g}$ & $\bar{\mathrm{II}}$ satisfy Gauss-Codazzi iff $R^{\bar{G}}(X,Y) = 0 \ \forall X, Y \in TS$, i.e., if $R^{\bar{G}}_{12ij}\big|_S \equiv 0$.

Elastic energy – formal asymptotics

M_t – thin 3-dim. man., t-tub. neigh. of a surface S



Assume $F_t(p', p^3) = f(p') + p^3 n_f(p')$, and that $W(A) = \mathrm{dist}^2(A, SO(3))$

Then, formally,

$$E_{M_t}(F_t) = \underbrace{\underbrace{\int_S \mathrm{dist}^2\big(Df \circ \bar g^{-\frac12}, \overset{\{A:\mathbb{R}^2\to\mathbb{R}^3 \,|\, A^TA = I_{2\times 2}\}}{\overset{\parallel}{O(2,3)}}\big)\, dV_{\bar g}}_{\text{stretching } E_S^{str}(f)} + \underbrace{\frac{t^2}{3}\int_S |\nabla n_f + Df \circ \bar S|^2\, dV_{\bar g}}_{\text{bending } t^2 E_S^{bnd}(f)}}_{E_S(f) \quad \text{Kirchhoff shell energy}} + O(t^3)$$

Remark: An alternative bending energy, which differs from the above by higher order terms, is

$$\frac{t^2}{3}\int_S |\mathrm{II}_f - \bar{\mathrm{II}}|^2\, dV_{\bar g}$$

where $\mathrm{II}_f = Df^T \nabla n_f$.
While more common in the physics community, it is worse from analytic point of view, because of the product structure of II_f.

Note that

$$\{f : S \to \mathbb{R}^3 \,|\, E_S(f) < \infty\} = \{f \in W^{1,2}(S;\mathbb{R}^3) \,|\, \mathrm{rnk}\, Df = 2 \text{ a.e.}, \overset{\text{implies existence}}{n_f \in W^{1,2}(S;\mathbb{R}^3)}\} =: \mathrm{Imm}^2(S;\mathbb{R}^3)$$

Interesting "critical" Sobolev space, similar to $\{f \in W^{1,2}(S;\mathbb{R}^2) \,|\, \det Df > 0\}$

Open question: $\mathrm{Imm}^2(S;\mathbb{R}^3) \subset C(S;\mathbb{R}^3)$? (This is true if $\nabla n_f = 0$ Goldstein, Šverák, ...)
Does Kirchhoff energy allow for cavitation?

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<u>Thm</u> (AKM'22): $\inf E_{\bar S} = 0 \iff \exists f \in C^\infty(S, \mathbb{R}^3)$ iso. imm w. forms $\bar g, \bar{\mathrm{II}}$.
In particular, $\bar g$ & $\bar{\mathrm{II}}$ satisfy GC.
If S is simply-connected, this also implies $\inf E_S = 0$.

<u>Pf</u> (sketch): "undo the dimension reduction".

Define a manifold $M = S^1 \times (-\varepsilon_0, \varepsilon_0)$ w. metric

$$\bar G_{(x^1; x_3)}((v,s),(w,r)) = \bar g_{x^1}(v - s\bar S(v), w - r\bar S(w)) + sr.$$

This only works for const. curv. target space. For general targets a direct approach is needed (AKM'23)

Given $f \in \mathrm{Imm}^2(S^1; \mathbb{R}^3)$, define $F \in W^{1,2}(M; \mathbb{R}^3)$ by

$$F(x^1, x^3) = f(x^1) + x^3 n_f(x^1)$$

Then $\int_M \mathrm{dist}^2(DF \circ \bar G^{-\frac12}, SO(3)) \leq C E_S(f)$.

Thus, $\inf E_S = 0$ implies $\inf E_M = 0$, and thus $\exists$ smooth iso. imm.
$F: M \to \mathbb{R}^3$. $F|_{S^1 \times \{0\}}$ is the wanted map. $\square$

Non-Euclidean shells – rigorous limits

As $t \to 0$ we can see that stretching becomes infinitely costly compared to bending, thus we expect

$$\lim_{t\to 0} \frac{1}{t^2} E_{M_t} = \begin{cases} E_S^{bnd} & E_S^{str} = 0 \\ +\infty & E_S^{str} \neq 0 \end{cases}$$

Note that if $E_S^{str}(f) = 0$ then $Df^T Df = \bar{g}$ a.e., and thus

1) $f \in Imm^2(S; \mathbb{R}^3) \Rightarrow f \in W^{2,2}(S; \mathbb{R}^3)$, since $\partial_{ij} f = \bar{\Gamma}_{ij}^k \partial_k f - \bar{\Gamma}_{il}^k \bar{g}_{kj} \bar{g}^{lm} n_m$

We denote this space by $W^{2,2}_{\bar{g}}(S; \mathbb{R}^3)$.

2) $E_S^{bnd}(f) = \frac{1}{3} \int_S |\mathrm{II}_f - \bar{\mathrm{II}}|^2 dV_{\bar{g}}$

<u>Thm</u> (Sch'07, LP'11, BLS'16, KS'14): ← generalizations of FJM'02, FJMM'03

- $E_{M_t}(f_t) \leq C t^2 \Rightarrow f_t \xrightarrow{W^{1,2}} f \in W^{2,2}_{\bar{g}}(S; \mathbb{R}^3)$ (after rescaling ...)
- $\frac{1}{t^2} E_{M_t} \xrightarrow{\Gamma} \begin{cases} E_S^{bnd}(f) & f \in W^{2,2}_{\bar{g}} \\ +\infty & \text{else} \end{cases}$

Corollary:
(MS'19)

- $\inf E_{M_t} = O(t^2) \iff W^{2,2}_{\bar g}(S;\mathbb{R}^3) \neq \emptyset$

$\Leftarrow$ S simply-connected

- $\inf E_{M_t} = o(t^2) \iff \exists f \in W^{2,2}_{\bar g}(S;\mathbb{R}^3)\cap C^\infty,\ \mathrm{II}_f = \bar{\mathrm{II}} \Rightarrow \bar g, \bar{\mathrm{II}}$ satisfy GC

$$\iff \inf E_{M_t} = O(t^4)$$

- $\inf E_{M_t} = o(t^4) \Rightarrow R^{\bar G}|_S \equiv 0$ (follows from local estimate)

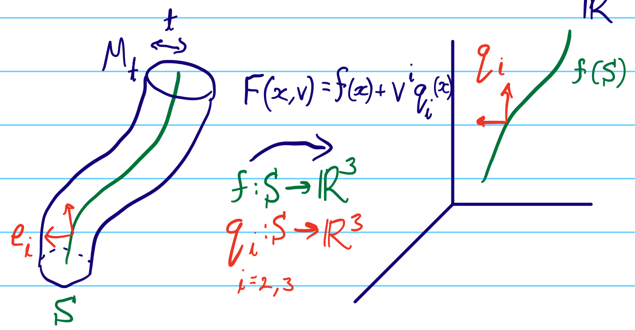
$\Leftarrow$ simply conn.

Higher order scaling in Lewicka '20.

If time permits

Rods

For rods we have a different behavior



Thm (KS'14, Aharoni et al.'12):

- $E_{M_t}(f_t) \le C t^2 \Rightarrow f_t \xrightarrow{W^{1,2}} f \in W^{2,2}(S;\mathbb{R}^3)$, $Df_t \xrightarrow{L^2} \overbrace{(Df|q_2|q_3)}^{q} \in W^{1,2}(S;SO(\bar G))$

- $\frac{1}{t^2}E_{M_t} \xrightarrow{\Gamma} \begin{cases} C\int |\underbrace{q_2'\cdot q_1 - \bar\Gamma^1_{12}}_{\kappa_1}|^2 + |\underbrace{q_3'\cdot q_1 - \bar\Gamma^1_{13}}_{\kappa_2}|^2 + |\underbrace{q_3'\cdot q_2 - \bar\Gamma^2_{13}}_{\tau}|^2 & q \in SO(\bar G) \\ +\infty \end{cases}$

For $W(\cdot) = \mathrm{dist}^2(\cdot, SO(3))$, round cross-section.
Otherwise an equivalent energy

Since we can always solve $q' = \begin{pmatrix} 0 & -\bar\Gamma^1_{12} & -\bar\Gamma^1_{13} \\ \bar\Gamma^1_{12} & 0 & -\bar\Gamma^2_{13} \\ \bar\Gamma^1_{13} & \bar\Gamma^2_{13} & 0 \end{pmatrix} q$ (no compatibility conditions)

Cor (MS'19):
- $\inf E_{M_t} = O(t^4)$
- $\inf E_{M_t} = o(t^4) \Rightarrow R^{\bar G}|_S \equiv 0$.

Summary:

	Shells	Rods
Generic scaling of E_{M_t}:	t^2	t^4
Behavior of min:	$\min \int \lvert \mathrm{II}_f - \bar{\mathrm{II}} \rvert^2$, II_f subject to GC	" $\mathrm{II}_f = \bar{\mathrm{II}}$ " (satisfy curvature, torsion)

If time permits

Comment: In KS'14 dimension reduction at the t^2 scaling is done

for any $\underbrace{\dim S}_{m} < \underbrace{\dim M}_{n} \hookrightarrow \mathbb{R}^n \qquad k = n-m$

We can choose a orthonormal frame $e_1, \dots, e_k$ of NS, which induce coordinates on M s.t.

$\langle \nabla^{\bar{G}}_{\partial_i} \partial_j, e_\alpha \rangle = (\mathrm{II}_\alpha)_{ij}$ second forms (sym. mat.)

$\langle \nabla^{\bar{G}}_{\partial_i} e_\alpha, e_\beta \rangle = (\tau_i)_{\alpha\beta}$ is the twist matrices (skew)

Then we have

- $E_{M_t}(f_t) \le C t^2 \Rightarrow f_t \xrightarrow{W^{1,2}} f \in W^{2,2}_{g}(S; \mathbb{R}^n)$, $Df \xrightarrow{L^2} (Df \mid \underbrace{q^\perp}_{q^\perp_1, \dots, q^\perp_k}) \in W^{1,2}(S, SO(\bar{G}))$

- $\Gamma\text{-}\lim \frac{1}{t^2} E_{M_t} \overset{\text{if not too}}{=} C(m,k) \int_S \sum_\alpha 2 \lvert \underbrace{P^{\shortparallel}_F}_{\text{proj. on } T(F(S))} \circ \nabla_\cdot q^\perp_\alpha + DF \circ \underbrace{S^{(\cdot)}_\alpha}_{\text{shape operator associated w. } \mathrm{II}_\alpha} \rvert^2 + \lvert \underbrace{P^\perp_F}_{\text{" " } N(F(S))} \circ \nabla_\cdot q^\perp_\alpha - \sum_\beta (\tau_\cdot)_{\alpha\beta} q^\perp_\beta \rvert^2$

for $W(\cdot) = \mathrm{dist}^2(\cdot, SO(n))$ and circular cross-section

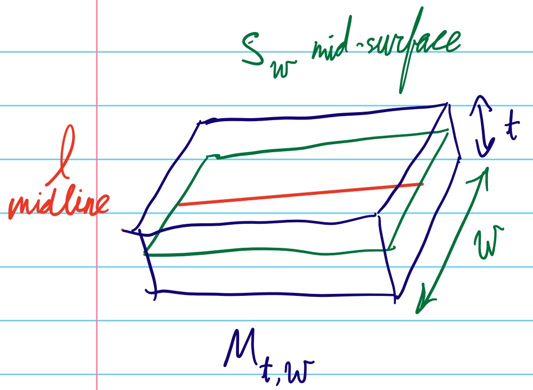
Ribbons

ribbons interpolate between these behaviors

Summary: Shells $\longleftrightarrow$ Rods

	Shells	Rods
Generic scaling of E_{M_t}:	t^2	t^4
Behavior of min:	$\min \int \lvert \mathrm{II}_f - \bar{\mathrm{II}} \rvert^2$, II_f subject to GC	"$\mathrm{II}_f = \bar{\mathrm{II}}$" (satisfy curvature, torsion)

Ribbons see the difference between Gauss & Codazzi



$t \ll w \ll 1$

$\boxed{\text{Show examples}}$

$$E_{t,w}(f) = \int_{M_{t,w}} \mathrm{dist}^2(Df \circ \bar{G}^{-1/2}, SO(3))\, dV_{\bar{G}}$$

$$E^{2D}_{t,w}(f) = \int_{S_w} \mathrm{dist}^2(Df \circ \bar{g}^{-1/2}, SO(2,3))\, dV_{\bar{g}} + t^2 \int_{S_w} \lvert Dn_f + Df \circ \bar{S} \rvert^2 dV_{\bar{g}}$$

$$\approx \int_{S_w} \lvert \underbrace{Df^T Df}_{g_f} - \bar{g} \rvert^2 dV_{\bar{g}} + t^2 \int_{S_w} \lvert \underbrace{\mathrm{II}_f}_{-Df^T Dn_f} - \bar{\mathrm{II}} \rvert^2 dV_{\bar{g}}$$

Example 1: $S^w = (0,\ell) \times (-\frac{w}{2}, \frac{w}{2})$, $\bar{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{\mathrm{II}} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$

$\times$ **Gauss:** $\det \bar{\mathrm{II}} = -\lambda^2$, $K_{\bar{g}} \det \bar{g} = 0$

$\checkmark$ **Codazzi:** $\partial_i \mathrm{II}_{jk} = 0 = \partial_j \mathrm{II}_{ik}$ since $\nabla_i^{\bar{g}} = \partial_i$

$$\bar{g}_f = \begin{pmatrix} 1 - K_0(x_1) x_2 + O(x_2^2) & 0 \\ 0 & 1 \end{pmatrix}$$

$K_0(x_1)$: Gaussian curvature of $f(S_w)$ along the midline.

Thus, formally,

$$E^{2D}_{t,w} = w^4 \int_L K_0^2(x_1)\, dx_1 + t^2 \int_L |\mathrm{II}_f(x_1,0) - \bar{\mathrm{II}}|^2 dx_1 + h.o.t$$

leading order constraint: $K_0(x_1) = \det \mathrm{II}_f(x_1,0)$

Cor: Wide ribbon ($t \ll w^2$) $K_0 \equiv 0$ $\quad E^{2D}_{t,w} \sim t^2 \int_L |\underbrace{\mathrm{II}_f(x_1,0)}_{\det \mathrm{II}_f = 0} - \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}|^2 dx_1 \sim t^2 \lambda^2$

dev. surface

Narrow ribbon ($w^2 \ll t$) $\mathrm{II}_f(x_1,0) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \Rightarrow K_0 = -\lambda^2$ $\quad E^{2D}_{t,w} \sim w^4 \lambda^4$

pure twist

$\Rightarrow$ transition @ $t \sim w^2$

Example 2: $S^w = (0,\ell) \times (-\frac{w}{2}, \frac{w}{2})$, $\bar{g} = \begin{pmatrix} (1-\kappa x_2)^2 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{\mathrm{II}} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$

✓ Gauss: $\det \bar{\mathrm{II}} = 0 = K_{\bar{g}}$

× Codazzi: $\nabla_2 \bar{\mathrm{II}}_{11} - \nabla_1 \bar{\mathrm{II}}_{21} = -\lambda \kappa (1-\kappa x_2)^2 \neq 0$

$K_0(x) = K_{g_f}(x_1,0)$, $K_1(x_1) = \partial_2 K_{g_f}(x_1,0)$

It is always energetically favorable to respect the reference geodesic curvature of the mid-line

$$g_f = \begin{pmatrix} (1-\kappa x_2)^2 - K_0 x_2^2 + \frac{1}{3}(4\kappa K_0 - K_1) x_2^3 + \dots & 0 \\ 0 & 1 \end{pmatrix}$$

having $\mathrm{II}_f(x_1,0) = \bar{\mathrm{II}}$ is consistent with $K_0 = \bar{K} = 0$, but then

by Codazzi we get $\partial_2 \mathrm{II}_f(x_1,0) = -\lambda \kappa$ hence $\mathrm{II}_f = \begin{pmatrix} -\lambda \kappa x_2 & 0 \\ 0 & \lambda \end{pmatrix} + h.o.t.$

hence if $\mathrm{II}_f(x_1,0) = \bar{\mathrm{II}}$ we get $K_1 = -\lambda^2 \kappa \neq 0$, i.e. effectively we have

$$E^{2D}_{t,w} = \underbrace{w^6 \int_L K_1^2(x_1)\, dx_1 + t^2 \int_L |\mathrm{II}_f(x_1,0) - \bar{\mathrm{II}}|^2 dx_1}_{\text{cannot be simultaneously zero}} + h.o.t.$$

Thus we expect transition between $\mathrm{II}_f = \bar{\mathrm{II}}$ for narrow ribbons $t \gg w^3$ and $K_f = O(x_2^2)$ for wide ribbons $t \ll w^3$.

However, it can be shown that the transition can be moved at least to $t \sim w^{8/3}$