Notes for the course

Nonlinear Elasticity

(preliminary lecture notes)[1]

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This is a short summary of the topics discussed in the lectures. It cannot replace a careful study of the literature. Relevant references include

- L.C. Evans, Partial Differential Equations, AMS Graduate Studies in Mathematics
- E. Gurtin, An introduction to continuum mechanics, Academic Press, 1981
- P. Ciarlet, Mathematical Elasticity Vol I, North-Holland
- C. Eck, H. Garcke, P. Knabner, Mathematische Modellierung, Springer

These notes are based on the books mentioned above and lecture notes by S. Müller, M. Brokate and other sources, which sometimes are not mentioned explicitly.

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This text is only intended for students of the course
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0 Content of the lecture and Motivation

For motivation we discussed

- What is continuum mechanics about?
- Elastic solids
- Equilibrium Equation for a harmonic spring. 3 parts of modelling: kinematics, constitutive relation (material law), equilibrium equation
- Cauchy stress principle and the equilibrium equations for elastic bodies
- Equations of linear elasticity (linear 2nd order elliptic system, divergence form, solution theory Lax-Milgram)
- In general: material law is not linear, even not monotone. fully non-linear 2nd order elliptic system, divergence form, no general existence theory
- The framework of Nonlinear Elasticity (Hyperelasticity). Transformation to a minimization problem. Theory based on the Calculus of Variations

Content

- (continuum-mechanical) modelling: Derivation of the equilibrium equations for elastic solids; Deformation, Cauchy stress principle, objectivity, material symmetries, hyperelasticity
- analysis: Elements of the Calculus of Variations, notions of convexity, existence theorem by J.Ball
- asymptotic analysis: Derive linear elasticity from nonlinear elasticity; derive nonlinear elasticity from discrete models; derive lower dimension theories (rods, plates) from nonlinear elasticity

Required preliminaries

- basic analysis: notion of submannifold in $\mathbb{R}^d$, surface measure, integral transformation
- basic functional analysis, in particular weak and weak* convergence, Theorem of Banach-Alaoglu
- basic PDE theory: Sobolev spaces, notion of weak solution, Lax-Milgram

1 Introduction

1.1 Analytical Toolbox and notational convention – Part I

1.1.1 Linear and differentiable maps

Let X, Y be normed vector spaces. We denote by $\mathcal{L}(X;Y)$ the vector space of all bounded linear mappings $T : X \to Y$. It is a normed vector space with the operator norm

$$\|T\| := \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Ax\|}{\|x\|};$$

it is *complete* if Y is complete.

Fréchet derivative. A mapping $f : \Omega \subset X \to Y$ (with Ω open) is called **differentiable at** $x_0 \in \Omega$, if there exists $Df(x_0) \in \mathcal{L}(X;Y)$ – the **Fréchet derivative** of f at x_0 – such that

$$f(x_0 + h) = f(x_0) + Df(x_0)h + o(h) \qquad \text{for all } h \in X,$$

where $o(h) := \|h\|O(h)$ and $O(h)$ satisfies $\limsup_{h\to 0} O(h) = 0$.
f is called **differentiable**, if it is differentiable at x_0 for all $x_0 \in \Omega$; it is called continuously differentiable if it is differentiable and $\Omega \ni x_0 \mapsto Df(x_0) \in \mathcal{L}(X;Y)$ is continuous. We write $C^1(X;Y)$ for the space of all continuously differentiable functions from X to Y.

Chain rule. Suppose $f : X \to Y$ and $g : Y \to Y$ are continuously differentiable, then $(g \circ f) : X \to Z$ is differentiable and for all $x_0 \in X$ we have $D(g\circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$, where $\circ$ denotes the concatenation of mappings.

The case of a Euclidean space. We write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{R}^d$. We denote by $x_1, \ldots, x_n$ the coordinates of $x = \sum_{i=1}^n x_i x_i \in \mathbb{R}^d$, and write

$$a \cdot b := \sum_{i=1}^n a_i b_i, \qquad |a| := \sqrt{a \cdot a}$$

for the Euclidean scalar product and norm on $\mathbb{R}^d$. For $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^d$ we introduce the linear map

$$a \otimes b : \mathbb{R}^d \to \mathbb{R}^m, \qquad (a \otimes b)x := a(b \cdot x).$$

Exercise 1.1. *Convince yourself that* $\big\{e_i \otimes e_j \,:\, i = 1, \ldots, m, j = 1, \ldots, d\big\}$ *– the standard basis of* $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)$ *– is indeed a basis.*

We denote by A_{ij} the coordinates of $A = \sum_{i=1}^m \sum_{j=1}^d A_{ij}(e_i \otimes e_j) \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$. We tacitly identify a linear map $A : \mathbb{R}^d \to \mathbb{R}^m$ with the associated coordinate matrix $(A_{ij}) \in \mathbb{R}^{m\times d}$. The Euclidean scalar product and norm on $\mathbb{R}^{m\times n}$ (resp. $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m)$) are denoted by

$$A \cdot B := A_{ij}B_{ij} = \sum_{i=1}^m \sum_{j=1}^n A_{ij}B_{ij}, \qquad |A| := \sqrt{A \cdot A},$$

where we use Einstein's summation convention to sum over repeating indices.

Exercise 1.2. *Convince yourself that*

- $\{e_i \otimes e_j \,:\, i = 1, \dots, m, j = 1, \dots, n\}$ *is an orthormal basis of* $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$*;*
- *for any* $A \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$*, we have*
$$A_{ij} = A \cdot (e_i \otimes e_j);$$
- *the scalar product are independent of the choice of the orthonormal basis of* $\mathbb{R}^d$ *(resp.* $\mathbb{R}^m$*): Let* $\bar{e}_1, \dots, \bar{e}_n$ *and* $\tilde{e}_1, \dots, \tilde{e}_m$ *denote orthonormal bases of* $\mathbb{R}^d$ *and* $\mathbb{R}^m$ *respectively; consider coordinate matrices* $A' = (A'_{ij}), B' = (B'_{ij}) \in \mathbb{R}^{m\times n}$ *and the linear mappings*
$$A := \sum_{ij} A'_{ij}(\bar{e}_i \otimes \tilde{e}_j), \qquad B := \sum_{ij} B'_{ij}(\bar{e}_i \otimes \tilde{e}_j);$$
note that in general $A_{ij} \neq A'_{ij}$ *and show that* $A \cdot B = \sum_{ij} A'_{ij}B'_{ij}$*.*

Let $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}^m$ (with $\Omega \subset \mathbb{R}^d$ open) be differentiable at $x_0 \in \Omega$. We identify $Df(x_0)$ with a matrix in $\mathbb{R}^{m\times d}$ – which in applications to elasticity is called the **deformation gradient**.

Next, we consider the scalar case, i.e. codimension $m = 1$. $Df(x_0) \in \mathcal{L}(\mathbb{R}^d; \mathbb{R})$ is a linear functional. The **gradient** of f at x_0 – denoted by $\nabla f(x_0)$ – is defined as the unique vector in $\mathbb{R}^d$ such that
$$Df(x_0)h = \nabla f(x_0) \cdot h \qquad \text{for all } h \in \mathbb{R}^d.$$
We write $\partial_i f(x_0) := \nabla f(x_0) \cdot e_i$ for the partial derivative of f in direction e_i. Note that $\nabla f(x_0) \in \mathbb{R}^d$, while we identify $Df(x_0)$ with a matrix in $\mathbb{R}^{1\times n}$.

More generally, if H is a Hilbert space with scalar product $(\cdot, \cdot)$, and $f : \Omega \subset H \to \mathbb{R}$ is differentiable at $x_0 \in \Omega$, we define the gradient of f at x_0 as the unique $\nabla f(x_0) \in H$ satisfying $Df(x_0)h = \nabla f(x_0) \cdot h$ for all $h \in H$.

Exercise 1.3.

- *Consider* $f : \mathbb{R}^{n\times m} \to \mathbb{R}$*,* $f(A) := A_{11}^3 + A_{nm}^2$*. Compute* $\nabla f(A)$*.*
- *Consider* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$*,* $f(A) := \det A$*. Compute* $\nabla f(Id)$*, where* Id *denotes the identity matrix.*
- *Let* $g_1 : \mathbb{R}^d \to \mathbb{R}^m, g_2 : \mathbb{R}^d \to \mathbb{R}^m$ *be differentiable, set* $f(x) := g_1(x) \cdot g_2(x)$ *and compute* $\nabla f(x)$*.*
- *Let* $g_1 : \mathbb{R}^d \to \mathbb{R}^m$*, and* $g_2 : \mathbb{R}^m \to \mathbb{R}$ *be continuously differentiable. Define* $f(x) := g_2(g_1(x))$ *and express* $Df(x)$ *(resp.* $\nabla f(x)$*) in terms* Dg_1*,* Dg_2 *(resp.* ∇g_2*) in a coordinate wise and coordinate-free format.*
$$\begin{aligned} Df(x)h &= Dg_2(g_1(x))[Dg_1(x)[h]] = \nabla g_2(g_1(x)) \cdot (Dg_1(x)h) \\ &= \partial_i g_2(g_1(x))(Dg_1(x))_{ij}h_j \\ \nabla f(x) &= Dg_1(x)^t \nabla g_2(g_1(x)). \end{aligned}$$
Plausibility test: $g_1(x) = Ax$ *with* $A \in \mathbb{R}^{m\times d}$*,* $g_2(y) = a \cdot y$ *with* $a \in \mathbb{R}^m$*,* $f(x) = a \cdot Ax = A^t a \cdot x$*, and thus* $\nabla f(x) = A^t a = (Dg_1)^t \nabla g_2$*.*

Notation for classical function spaces. Let $\Omega \subset \mathbb{R}^d$ be open.

- $C(\Omega; \mathbb{R}^m) := C^0(\Omega; \mathbb{R}^m)$ – space of continuous functions on Ω
- $C(\bar{\Omega}; \mathbb{R}^m)$ – space of continuous functions on $\overline{\Omega}$ $(= \{f_{|\Omega} \,:\, f \in C(\mathbb{R}^d; \mathbb{R}^m)\})$.
- $C^k(\Omega; \mathbb{R}^m)$ – space of k-times continuously differentiable functions on Ω.
- $C_c^\infty(\Omega; \mathbb{R}^m)$ – space of infinitely continuously differentiable functions on Ω with compact support $\overline{\{f \neq 0\}}$ contained in Ω.

1.1.2 Volume, Area and Length

In this section we discuss transformation formulas for volume-, area- and length integrals under a C^1-diffeomorphism $\varphi : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$.

Definition 1.4 (right Cauchy-Green strain tensor)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and $\varphi : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ a C^1-diffeomorphism, i.e. $\varphi : \Omega \to \varphi(\Omega)$ is bijective, and f, f^{-1} are continuously differentiable. The map $x \mapsto C(x) := D\varphi(x)^t D\varphi(x)$ is called the right Cauchy-Green strain tensor (associated with φ).*

The Cauchy-Green strain tensor captures changes in length and volume: We recall that the length of a C^1-curve $\gamma : [0,1] \to \mathbb{R}^d$ is given by

$$L(\gamma) := \int_0^1 |\gamma'(t)| \, dt = \int_0^1 \sqrt{\gamma'(t) \cdot \gamma'(t)} \, dt.$$

Lemma 1.5 (Transformation rule for length integrals)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $\varphi : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ be a C^1-diffeomorphism. Let $\gamma : [0,1] \to \Omega$ be a C^1-curve. Then $\varphi(\gamma) : [0,1] \ni s \mapsto \varphi(\gamma(s))$ defines a C^1-curve, and*

$$L(\varphi(\gamma)) = \int_0^1 \sqrt{\gamma'(t) \cdot C(\gamma(t))\gamma'(t)} \, dt.$$

Beweis. By the chain rule we have $(\varphi(\gamma))'(t) = D\varphi(\gamma(t))\gamma'(t)$. □

Theorem 1.6 (Transformation rule for volume integrals)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $\varphi : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ be a C^1-diffeomorphism. Then $f \in L^1(\varphi(\Omega)) \;\Leftrightarrow\; f(\varphi(\cdot))|\det D\varphi(\cdot)| \in L^1(\Omega)$, and*

$$\int_{\varphi(\Omega)} f(y) \, dy = \int_\Omega f(\varphi(x))|\det D\varphi(x)| \, dx = \int_\Omega f(\varphi(x))\sqrt{\det C(x)} \, dx$$

(The function $\Omega \ni x \mapsto |\det D\varphi(x)|$ is called the Jacobian of φ).

For a proof see Königsberger: Analysis 2.

Example 1.7 (Affine case)**.** *If $\varphi(x) = Ax + c$ with $A \in GL(n)$ and $c \in \mathbb{R}^d$, then $|\det D\varphi(x)| = |\det A|$, and thus for any $Q \subset \mathbb{R}^d$ open, $Q' := \varphi(Q)$,*

$$|Q'| := \mathrm{Vol}(Q') := \int_{Q'} dx = |\det A| \, \mathrm{Vol}(Q).$$

In particular,

$$A = \lambda_i(e_i \otimes e_i), \qquad \frac{|Q'|}{|Q|} = \prod_{i=1}^{n} |\lambda_i| \qquad \textit{(dilation)}$$
$$A \in O(n), \qquad \frac{|Q'|}{|Q|} = 1 \qquad \textit{(rotation / reflection)}$$
$$A = Id + \lambda(e_1 \otimes e_2), \qquad \frac{|Q'|}{|Q|} = 1 \qquad \textit{(shear).}$$

In the following we discuss how surface integrals and fields transform under deformation. To that end we recall the definitions of a C^1-domain and of the surface integral.

Definition 1.8. *(a) We call $\Omega \subset \mathbb{R}^d$ **domain**, if it is open, bounded and connected.*

*(b) A C^1**-domain** is a domain $\Omega \subset \mathbb{R}^d$ such that $\partial\Omega$ is a $(d-1)$-dimensional C^1-manifold and Ω locally lies only on one side of $\partial\Omega$, i.e.:*
for all $x \in \partial\Omega$ there exists an open neighborhood $U \subset \mathbb{R}^d$ and a function $f \in C^1(U)$ such that

- $\nabla f \neq 0$ *in* U,
- $\partial\Omega \cap U = \{ f = 0 \}$,
- $\Omega \cap U = \{ f < 0 \}$.

(Implicit characterization of the manifold).

(c) Let $\Omega \subset \mathbb{R}^d$ be a domain with C^1-boundary. Let $x \in \partial\Omega$.

- *A vector $v \in \mathbb{R}^d$ is called tangent vector to $\partial\Omega$ at x, if there exists a C^1-curve $\gamma : (-1,1) \to \mathbb{R}^d$ with $\gamma((-1,1)) \subset \partial\Omega$, $\gamma(0) = x$ and $\gamma'(0) = v$. We denote by $T_x(\partial\Omega) := \{ v \in \mathbb{R}^d \;\; v$ is tangent vector to $\partial\Omega$ at $x \}$ the tangent space to $\partial\Omega$ at x.*
- *A vector $\nu \in \mathbb{R}^d$ with $|\nu| = 1$ is called a normal to $\partial\Omega$ at x, if $\nu \cdot v = 0$ for all $v \in T_x(\partial\Omega)$. It is called outer unit normal, if additionally $x + t\nu \notin \Omega$ for all $t > 0$ sufficiently small.*

(d) A pair (ϕ, V) is called a parametrization for $\partial\Omega$ at x, if $V \subset \mathbb{R}^{d-1}$ open, $\phi \in C^1(V; \mathbb{R}^d)$ and

- $\phi(V) \subset \partial\Omega$,
- $\phi : V \mapsto \phi(V)$ *is a bijection with continuous inverse,*
- $\operatorname{rank}(D\phi) = d - 1$.

Lemma 1.9 (Existence of a local parametrization)**.** *Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain, $x \in \partial\Omega$. Then there exists a parametrization (ϕ, V) for $\partial\Omega$ at x.*

Beweis. Application of the implicit function theorem. □

Exercise 1.10 (Outer unit normal)**.** *Let Ω be C^1-domain, $x \in \partial\Omega$. Let (f, U) be as in Definition 1.8 (b), and (ϕ, V) a parametrization at x. Show that the outer unit normal $\nu(x)$ at x is unique and given by $\nu(x) = \frac{\nabla f(x)}{|\nabla f(x)|}$. Show that* $\mathrm{span}(\nu(x)) = \{ v \in \mathbb{R}^d : D\phi(y)^t v = 0 \}$ *where $y = \phi^{-1}(x)$.*

If Ω is a C^1-domain, then we can define the surface integral

$$\int_{\partial\Omega} f(x) \mathrm{d}\mathcal{H}^{d-1}(x). \tag{1.1}$$

Here $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure.[3] For C^1-domains, the surface integral (1.1) can be expressed in terms of the $(d-1)$-dimensional Lebesgue measure. Indeed, if (ϕ, V) is a parametrization for $\partial\Omega$ and $\mathrm{supp}(f) \cap \partial\Omega \subset \phi(V)$, then

$$\int_{\partial\Omega} f(x)\mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\phi(V)} f(x)\mathrm{d}\mathcal{H}^{d-1}(x) = \int_V f(\phi(x))\sqrt{\det D\phi(x)^t D\phi(x)}\mathrm{d}x, \tag{1.2}$$

If $\mathrm{supp}(f)$ cannot be parametrized by a single parametrization, then the surface integral can be expressed similarly by appealing to a partition of unity (i.e. to an atlas). For details on the Hausdorff measure see the textbook *Measure theory and fine properties of function. Evans & Gariepy.* For our purpose we can take the right-hand side in (1.2) as a definition.

Exercise 1.11 (Divergence theorem)**.** *Let $V \subset \mathbb{R}^{d-1}$ be open and bounded, $h \in C^1(\overline{V})$. Consider $\Omega := \{x = (x', x_d) : x' \in V,\, x_d < h(x')\}$ and set $\Gamma := \mathrm{graph}(h) \subset \partial\Omega$.*

(a) Let $u : \Gamma \to \mathbb{R}$ be continuous and bounded. Show that

$$\int_\Gamma u \, d\mathcal{H}^{d-1} = \int_V u(x', h(x'))\rho(x')\, dx',$$

for some $\rho : V \to \mathbb{R}$.

(b) Compute the outer unit normal to $\Gamma \subset \partial\Omega$ for $x \in \Gamma$.

(c) Let $u \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$ with $\mathrm{supp}(u) \Subset V \times \mathbb{R}$. Prove that

$$\int_\Omega \mathrm{div}\, u \, dx = \int_\Gamma u \cdot \nu \, d\mathcal{H}^{d-1}.$$

[3] The *Hausdorff Measure* is a (geometrically intuitive way) to introduce a measure for lower dimensional subsets $A \subset \mathbb{R}^d$. The construction is based on covering A by balls:

- For $0 \le s < \infty$ and $\varepsilon > 0$ one defines the quantity

$$\mathcal{H}^s_\varepsilon(A) := \inf \Big\{ \sum_{i=1}^\infty \alpha(s)(\frac{1}{2}\, diam(C_i))^s \,:\, A \subset \cup_{i=1}^\infty C_i,\, C_i \subset \mathbb{R}^d \text{ open with } diam\, C_i \le \varepsilon \Big\}$$

  where $\alpha(s) := \frac{\pi^{s/e}}{\Gamma(\frac{s}{2}+1)}$; note for $s \in \mathbb{N}$, $\alpha(s)(\frac{1}{2}\ell)^s$ is equal to the s-dimensional volume of a ball diameter ℓ.

- Given $A \subset \mathbb{R}^d$, the s-dimensional Hausdorff measure of A is defined as $\mathcal{H}^s(A) := \lim_{\varepsilon \downarrow 0} \mathcal{H}^s_\varepsilon(A)$ and is a (Borel-regular) measure. The *Hausdorff dimension* of A is the defined as $\inf\{0 \le s < \infty : \mathcal{H}^s(A) = 0\}$.

The assumption of C^1-regularity in Definition 1.8 can be relaxed. E.g. we want to consider polygonal domains Ω, which are special cases of Lipschitz domains.

Definition 1.12. *A domain $\Omega \subset \mathbb{R}^d$ is called (uniformly) Lipschitz, if there exists $C > 0$ and for all $x \in \partial\Omega$ there exists an open neighbourhood $U \subset \mathbb{R}^d$ and a bijection $\varphi : B(0;1) \to U$ such that*

$$\begin{aligned} \varphi, \varphi^{-1} &\text{ are Lipschitz (with Lipschitz constant at most } C\text{)},\\ \varphi^{-1}(U \cap \partial\Omega) &= B(0;1) \cap \{x_d = 0\},\\ \varphi^{-1}(U \cap \Omega) &= B(0;1) \cap \{x_d > 0\}. \end{aligned}$$

Note that C^1-domains are Lipschitz. One can show that $\mathcal{H}^{d-1}$ turns $\partial\Omega$ into a measure space, and for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ an outer normal $\nu(x)$ to $\partial\Omega$ at x can be defined. The divergence theorem extends to Lipschitz domains:

Theorem 1.13 (Divergence Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $f \in C^1(\bar{\Omega}, \mathbb{R}^d)$ (i.e. f admits a C^1-extension to an open neighborhood of $\bar{\Omega}$). Then*

$$\int_\Omega \operatorname{div} f(x)\, dx = \int_{\partial\Omega} f(x) \cdot \nu(x)\, d\mathcal{H}^{d-1}(x).$$

For details see the textbook *Measure theory and fine properties of function. Evans & Gariepy*

Later we discuss transformation rules for surface integrals.

2 Elements of continuum mechanics

We discuss a subarea of continuum mechanics, namely the basic problem of *elastostatics*: Imagine a material body that at a initial time occupies some domain $\Omega \subset \mathbb{R}^d$. *External forces* are acting on the body (bodyforces as gravity, or surface forces acting on the boudary of the body). As a result the body deforms. If the forces are constant in time, then often (e.g. under presence of friction) the body will relax (as time evolves) into a static *equilibrium configuration*. In that state the *external forces* and the *internal forces* (the so called stresses) are in *equilibrium*. We are interested to find, describe, and analyze this equilibrium state.

Typical questions are:

- Is there a unique equilibrium state? (Existence theory for PDEs and variational problems)
- Is it stable? (Bifurcation theory)
- What is the shape of the equilibrium state and how can we compute it? (numerics, FEM)
- What are the extremal stresses? (Regularity theory)
- Can we find simpler models with good predictivity? (Asymptotic Analysis, Homogenization)

The model of elastostatics (as all continuum mechanical models) is based on three parts:ingredient

- *kinematics* describing the deformation (or the motion in the case of time-dependent problems) of the body;
- *balance laws* coming from physics;
- *constitutive laws* describing specific properties of the material.

2.1 Kinematics: Deformation

The notion of deformation.

Definition 2.1 (Deformation). *Let $\Omega \subset \mathbb{R}^d$ be a domain, let $k \geq 1$. A C^k-deformation is a map $\varphi \in C^k(\Omega; \mathbb{R}^d) \cap C(\bar{\Omega}; \mathbb{R}^d)$ that is*

(i) φ is orientation preserving, i.e. $\det D\varphi > 0$ in Ω,

(ii) φ is injective on Ω,

(iii) $D\varphi \in C(\bar{\Omega}; \mathbb{R}^{d\times d})$.

Moreover,

- *The set Ω is called the reference domain of a body, and $x \in \Omega$ are called material points.*
- *$D\varphi$ is called deformation gradient.*
- *The map $u : \Omega \to \mathbb{R}^d$ defined by $\varphi = \mathrm{id} + u$ is called displacement, and Du displacement gradient.*

Definition 2.2 (Standard terminology). *It is a convention in mechanics to use specific letters and symbols to denote strain and stress tensors associated:*

- *deformationt gradient: $F := D\varphi$.*
- *right Cauchy-Green strain tensor: $C := F^t F$.*
- *Green-Lagrange strain tensor: $E := \frac{1}{2}(F^t F - I) = \frac{1}{2}(C - I)$.*
- *Infinitesimal strain tensor: $\boldsymbol{\varepsilon} := \frac{1}{2}\,\mathrm{sym}\,\nabla u$, where $\nabla u = \nabla\varphi - I$ denotes the displacement gradient.*
  Note: $E = \frac{1}{2}((I + \nabla u)^t(I + \nabla u) - I) = \mathrm{sym}\,\nabla u + \frac{1}{2}\nabla u^t \nabla u = \varepsilon + O(|\nabla u|^2)$.
  Warning: In asymptotic analysis ε is usually used to denote a small scaling parameter; in this in mathematical texts the infinitesimal strain is often denoted by "e" or "e(u)".

Example 2.3 (affine deformation). *An affine map $\varphi(x) = Fx + b$ with $F \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$ is a deformation if and only if* $\det F > 0$. *Such deformations are called affine deformations. Note that any deformation can be locally approximated by an affine map, that is,*

$$\varphi(z+x) = \underbrace{\varphi(z) + D\varphi(z)x}_{\text{affine in } x} + o(|x|).$$

Thanks to this property, for many questions we can concentrate on infinitesimally parts of the body where the deformation can be considered to be affine.

Remark 2.4 (orientation-preserving, non-interpenetration, selfcontact).

- *To illustrate condition (i) consider the action of an affine map on the square $\Omega = (0,1)^2$ in Figure 1. In case (a) the square is slightly compressed and the determinant of the deformation gradient may have the value* $1/2$, *in case (b) the square is rotated and the determinant of the deformation gradient equals* 1. *In the case (c) the square is reflected and the determinant of the gradient is* -1. *Such a state cannot be reached from the reference state by a continuous process without going through a state with* $\det D\varphi(x) = 0$ *in at least one material point x. Hence, such deformations are deemed un-physical and are not allowed.*

- *The injectivity-property excludes that under deformation two distinct material points are mapped to the same position.*

- *Typically the reference configuration is chosen such that there is no self-contact, i.e.*

$$\Omega = \operatorname{int}(\Omega), \qquad \partial\Omega = \partial\bar{\Omega}. \tag{2.1}$$

  (This is fulfilled if Ω is a Lipschitz domain)

- *A deformation may map boundary points to the same position (self-contact under deformation), i.e. $\varphi(\Omega)$ might not satisfy* (2.1).
  E.g. consider

$$\Omega = \{x \in \mathbb{R}^d : \frac{1}{2} < x_1 < 1, -1 < x_2 < 1\}, \qquad \varphi(x) = (x_1\cos(\theta x_2), x_1\sin(\theta x_2)).$$

  Then for $\theta = \pi$, $\varphi(\Omega)$ is an annulus $\big(B(0;1)\setminus\overline{B(0;\frac{1}{2})}\big)$ without the linesegment from $(-1,0)$ to $(-\frac{1}{2},0)$.

Remark 2.5 (Variants of the definition). *A variant of the definition of a deformation is as follows:*

- *(ii) is replaced by the stronger assumption: φ is injective on $\overline{\Omega}$.*
  This excludes self-contact under deformation.

- *(iii) is dropped.*
  It is technical assumption to assure that certain quantities that emerge in the transformation of surface integrals are continuous and bounded.

We discuss the change of topological properties under deformation:

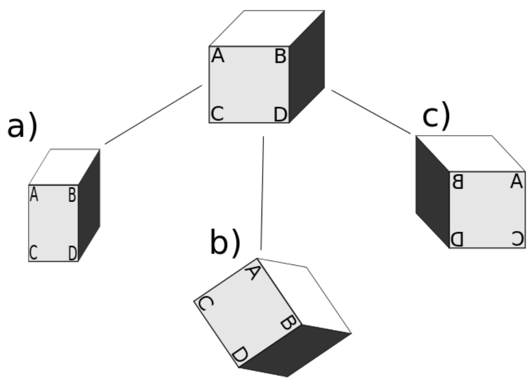

Abbildung 1: A cube deformed by an affine map.

Theorem 2.6. *(a)* $\Omega \subset \mathbb{R}^d$ *open,* $\varphi \in C(\Omega; \mathbb{R}^d)$ *injective* $\Rightarrow \varphi(\Omega)$ *open.*

(b) $\Omega \subset \mathbb{R}^d$ *domain,* $\varphi : \bar{\Omega} \to \mathbb{R}^d$ *deformation* $\Rightarrow \varphi(\Omega)$ *is domain and*

$$\varphi(\bar{\Omega}) = \overline{\varphi(\Omega)}. \tag{2.2}$$

(c) $\Omega \subset \mathbb{R}^d$ *domain with* $\Omega = \mathrm{int}(\bar{\Omega})$*,* $\varphi : \bar{\Omega} \to \mathbb{R}^d$ *deformation injective on* $\bar{\Omega} \Rightarrow \varphi(\Omega)$ *is domain and*

$$\varphi(\Omega) = \mathrm{int}(\varphi(\bar{\Omega})), \qquad \varphi(\partial\Omega) = \partial\varphi(\Omega) = \partial\varphi(\bar{\Omega}). \tag{2.3}$$

Beweis. (a) follows from a Theorem in Topology, e.g. see Zeidler, Vol I, Chap 16.

(b) $\varphi(\Omega)$ is a domain, i.e. open, bounded and connected:
$\varphi(\Omega)$ open follows from (a); $\varphi(\Omega)$ is bounded, since $\varphi(\Omega) \subset \varphi(\overline{\Omega})$ and by assumption φ is continuous on (the compact) set $\overline{\Omega}$, and thus uniformly continuous.
Argument for $\varphi(\Omega)$ connected. Suppose the opposite, i.e. $\varphi(\Omega) = U_1 \cup U_2$ with U_1, U_2 non-empty, open, disjoint. Then $\Omega_i := \varphi^{-1}(U_i)$, $i = 1, 2$, are open, non-empty, and disjoint, since φ is continuous and injective. Moreover, $\Omega = \Omega_1 \cup \Omega_2$. This contradicts that Ω is connected.
Argument for (2.2):

$$\varphi(\bar{\Omega}) \overset{\varphi \in C(\bar{\Omega})}{\subset} \overline{\varphi(\Omega)} \subset \overline{\varphi(\bar{\Omega})} \subset \varphi(\bar{\Omega}),$$

where the argument for the last inclusion is as follows: For $y \in \overline{\varphi(\bar{\Omega})}$, $\exists (x_i) \subset \bar{\Omega}$ s.t. $\varphi(x_i) \to y$. Since $\bar{\Omega}$ is compact, we may pass to a subsequence s.t. $x_i \to x$ in $\bar{\Omega}$ (not relabeled). φ is continuous on $\bar{\Omega}$, and thus $y \leftarrow y_i = \varphi(x_i) \to \varphi(x)$, i.e. $y = \varphi(x)$ and thus $y \in \varphi(\bar{\Omega})$.

(c) Argument for $\varphi(\Omega) = \mathrm{int}\,\varphi(\bar{\Omega})$: "$\subset$" follows from (a). We prove the other inclusion:

- From $\varphi : \bar{\Omega} \to \varphi(\bar{\Omega})$ continuous bijection, and $\bar{\Omega}$ compact, we conclude that φ^{-1} is continuous.

– From (a) we learn that $\varphi^{-1}(\int \varphi(\bar\Omega))$ is open, and thus

$$\varphi^{-1}(\operatorname{int}\varphi(\bar\Omega)) = \operatorname{int}\varphi^{-1}(\operatorname{int}\varphi(\bar\Omega)) \subset \operatorname{int}\varphi^{-1}(\varphi(\bar\Omega)) = \operatorname{int}\bar\Omega = \Omega.$$

$\operatorname{int}\varphi(\bar\Omega) \subset \varphi(\Omega)$ then follows by applying φ.

Argument for $\varphi(\partial\Omega) = \partial\varphi(\Omega) = \partial\varphi(\bar\Omega)$:

– Ω open $\Rightarrow \partial\Omega = \bar\Omega \setminus \Omega$
– $\varphi(\Omega)$ open $\Rightarrow \partial\varphi(\Omega) = \bar\varphi(\Omega) \setminus \varphi(\Omega)$

Since $\varphi(\bar\Omega) = \overline{\varphi(\Omega)}$,

$$\partial\varphi(\Omega) = \varphi(\bar\Omega) \setminus \varphi(\Omega) \overset{\text{bijection}}{=} \varphi(\bar\Omega \setminus \Omega) = \varphi(\partial\Omega).$$

□

Exercise 2.7 (Orientation preserving implies local injectivity)**.** *Let $\Omega \subset \mathbb{R}^d$ open, $\varphi \in C^1(\Omega;\mathbb{R}^d)$.*

- *Show: If $\det D\varphi(x) \neq 0$ for some $x \in \Omega$, then there exists an open neighbourhood $U \subset \Omega$ of x and $\varphi : U \to \varphi(U)$ is a C^1-diffeomorphism.*
- *Find an example for a non-injective $\varphi \in C^1(\Omega;\mathbb{R}^d)$ satisfying $\det D\varphi > 0$ in Ω.*

Notation: We define

$$\begin{aligned} O(d) &:= \{\, F \in \mathbb{R}^{d\times d} \;:\; F^tF = \mathrm{Id} \,\} \\ SO(d) &:= \{\, F \in O(d) \;:\; \det F = 1 \,\}. \end{aligned}$$

Note that $F \in \mathbb{R}^{d\times d}$ as a linear map on $\mathbb{R}^d$ is an isometry, i.e. $Fx \cdot Fy = x \cdot y$ for all $x, y \in \mathbb{R}^d$, if and only if $F \in O(d)$. The sets $O(d)$ and $SO(d)$ are groups under matrix multiplication.

Theorem 2.8 (Liouville's Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and connected, $\varphi \in C^1(\Omega,\mathbb{R}^d)$. TFAE:*

(i) $D\varphi(x) \in O(d)$ *for all* $x \in \Omega$,

(ii) $\forall x \in \Omega\ \exists U$ *open neighbourhood of* x, *s.t.* $\forall y, z \in U$: $|\varphi(y) - \varphi(z)| = |y - z|$

(iii) $\exists Q \in O(d), c \in \mathbb{R}^d$, *s.t.* $\varphi(x) = Qx + c$.

Beweis. • $(iii) \Rightarrow (i)$ is trivial.

- Argument for $(i) \Rightarrow (ii)$. Let $x \in \Omega$. Since $\det D\varphi(x) = 1 \neq 0$, by the inverse function theorem there exists $r > 0$ such that with $U := B(x;r)$ and $V := \varphi(U)$, $\varphi : U \to V$ is a C^1-diffeomorphism, and $D\varphi^{-1}(\varphi(y)) = D\varphi(y)^{-1} \in O(d)$ for all $y \in U$. Let $y, z \in U$. Since U is convex, the line segment $\gamma : [0,1] \to \mathbb{R}^d$, $\gamma(t) := \varphi(y + t(z-y))$ lies in U, and

$$\begin{aligned} |\varphi(z) - \varphi(y)| &= |\gamma(1) - \gamma(0)| = |\int_0^1 \gamma'(s)\,ds| = |\int_0^1 D\varphi(\gamma(s))(z-y)\,ds| \\ &\leq \int_0^1 |D\varphi(\gamma(s))(z-y)|\,ds \\ &= \int_0^1 |z-y|\,ds = |z-y|. \end{aligned}$$

  Same argument applied to φ^{-1} yields

$$|z-y| = |\varphi^{-1}(\varphi(z)) - \varphi^{-1}(\varphi(y))| \leq |\varphi(z) - \varphi(y)|.$$

- Argument for $(ii) \Rightarrow (iii)$. Let $B \subset \Omega$ be an arbitrary ball. By (ii) have

$$|y-z|^2 = |\varphi(y) - \varphi(z)|^2 \qquad \text{for all } y, z \in B.$$

  Differentiation w.r.t. y yields

$$2(y-z) = 2D\varphi(y)^t(\varphi(y) - \varphi(z))$$

  Differentiation w.r.t. z yields

$$-2Id = -2D\varphi(y)^t D\varphi(z).$$

  By continuity of $D\varphi(\cdot)$, we deduce that $D\varphi(y) \in O(d)$ for all $y \in B$, and thus multiplikation with $D\varphi(y)$ yields

$$Q := D\varphi(y) = D\varphi(z) \qquad \text{for all } y, z \in B.$$

  Since $B \subset \Omega$ is arbitrary, we conclude that the set $\{x \in \Omega : D\varphi(x) = Q\}$ is open and that same time closed (since $D\varphi$ is continuous). Since Ω is connected, we deduce that $\{x \in \Omega : D\varphi(x) = Q\} = \Omega$, and thus $\varphi(x) = Qx + a$ for some $a \in \mathbb{R}^d$.

□

Definition 2.9 (Rigid deformation)**.** *A deformation of the form $\varphi(x) = Rx + a$ with $R \in SO(d)$ and $c \in \mathbb{R}^d$ is called rigid.*

Since the superposition with a rigid deformation corresponds to an orientation preserving change of the coordinate frame, we are interested in properties of $D\varphi$ that are invariant under such changes.

Lemma 2.10 (Polar decomposition)**.** *Let $A \in \mathbb{R}^{d\times d}$, $\det A > 0$. There exist unique pairs $(U,R), (V,P)$ such that $U, V \in \mathbb{R}^{d\times d}$ are symmetric and positive definite, $R, P \in SO(d)$, and $A = RU = VP$. If $\det A = 0$ then we have a similar decomposition, but in this case U and V are only positive semidefinite and R and P are not longer uniquely determined.*

For the proof we recall from linear algebra the definition of the square root of a symmetric matrix:

Lemma 2.11 (Square root of a symmetric matrix)**.** *Let $W \in \mathbb{R}^{d\times d}$ be a symmetric and positive semidefinite matrix. Then there exists exactly one symmetric and positive semidefinite matrix U such that*

$$U^2 = W.$$

If W is positive definite so is U.

Notation: $U = \sqrt{W} = W^{\frac{1}{2}}$.

Proof of Lemma 2.10. We only discuss the decomposition $A = RU$. The other decomposition can be reduced to that case by considering A^t. We start with the case $\det A > 0$. Set $U := \sqrt{A^tA}$ and note that U is symmetric positive definite and thus invertible. Set $R := AU^{-1}$. Then

$$R^tR = U^{-t}A^tAU^{-1} = U^{-t}U^2U^{-1} = \mathrm{Id},$$

and $\det R = \det A / \det U > 0$. Hence $R \in SO(d)$ and $A = RU$.
To prove uniqueness we assume that $A = \tilde{R}\tilde{U}$ with $\tilde{R} \in SO(d)$ and $\tilde{U}$ symmetric and positive definite. Then

$$A^tA = (\tilde{U}R^t)R\tilde{U} = \tilde{U}^2.$$

Hence, the definition of the square root yields $\tilde{U} = U$. Since $\tilde{U}$ is invertible, we get $R = \tilde{R}$.
Next, we present the argument for the decomposition in the case $\det A = 0$. Define $U = \sqrt{A^tA}$. Since U is symmetric, the kernel of U and the range of U form an orthogonal decomposition of $\mathbb{R}^d$, i.e. the linear spaces

$$\mathcal{R} := \mathrm{range}(U) = \{Ux \;:\; x \in \mathbb{R}^d\}, \qquad \mathcal{K} := \mathrm{kernel}(U) = \{x \in \mathbb{R}^d \;:\; Ux = 0\,\}$$

are orthogonal and the direct sum of both is equal to $\mathbb{R}^d$. Indeed, since U is symmetric and positive semidefinite, U is diagonalizable by an orthonormal matrix $Q = (q_1, \dots, q_d) \in O(d)$, i.e. $U = QDQ^t$ where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_i \geq 0$. Note that $\{q_i\}_{i=1,\dots,d}$ form an orthonormal basis, and $\mathcal{R}$ is spanned by those vectors q_i with $\lambda_i > 0$, while $\mathcal{K}$ is spanned by those vectors q_i with $\lambda_i = 0$. From the diagonalization of U we deduce that $U|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ is bijective. Next, let $R : \mathbb{R}^d \to \mathbb{R}^d$ denote the unique linear map satisfying

$$Rq_i := \begin{cases} A(U|_{\mathcal{R}})^{-1}q_i & \text{if } \lambda_i > 0 \\ \tilde{q}_i & \text{else,} \end{cases}$$

where $\{\tilde{q}_i\}$ denotes an orthornormal basis of the orthogonal complement of range(A). We claim that R is an isometry, i.e. $(Rv)\cdot(Rw) = v \cdot w$ for all $v, w \in \mathbb{R}^d$. To that end, let

$$v = v_{\mathcal{R}} + v_{\mathcal{K}} \qquad \text{and} \qquad w = w_{\mathcal{R}} + w_{\mathcal{K}}$$

denote the unique orthogonal decomposition with $v_{\mathcal{R}}, w_{\mathcal{R}} \in \mathcal{R}$ and $v_{\mathcal{K}}, w_{\mathcal{K}} \in \mathcal{K}$. By the definition of R we have

$$Rv_{\mathcal{R}} \cdot Rw_{\mathcal{K}} = Rw_{\mathcal{R}} \cdot Rv_{\mathcal{K}} = 0 \qquad \text{and} \qquad Rv_{\mathcal{K}} \cdot Rv_{\mathcal{K}} = v_{\mathcal{K}} \cdot v_{\mathcal{K}}.$$

Hence

$$\begin{aligned}(Rv)\cdot(Rw) &= A(U|_{\mathcal{R}})^{-1}v_{\mathcal{R}}\cdot A(U|_{\mathcal{R}})^{-1}w_{\mathcal{R}}+v_{\mathcal{K}}\cdot w_{\mathcal{K}}.\\ &= A^tA(U|_{\mathcal{R}})^{-1}v_{\mathcal{R}}\cdot (U|_{\mathcal{R}})^{-1}w_{\mathcal{R}}+v_{\mathcal{K}}\cdot w_{\mathcal{K}}\\ &= \sqrt{A^tA}(U|_{\mathcal{R}})^{-1}v_{\mathcal{R}}\cdot \sqrt{A^tA}(U|_{\mathcal{R}})^{-1}w_{\mathcal{R}}+v_{\mathcal{K}}\cdot w_{\mathcal{K}}\\ &= v_{\mathcal{R}}\cdot w_{\mathcal{R}}+v_{\mathcal{K}}\cdot w_{\mathcal{K}}=v\cdot w.\end{aligned}$$

Hence, R is a rigid affine deformation and $R\in O(d)$. But switching the sign of one of the $\tilde{q}_i$'s in the definition of R, we may achieve $\det R=1$, and thus $R\in SO(d)$. It remains to show the identity $A=RU$. For $v=v_{\mathcal{R}}+v_{\mathcal{K}}$ we have (since the kernel of A and U coincide)

$$Av=Av_{\mathcal{R}}=A(U|_{\mathcal{R}})^{-1}Uv_{\mathcal{R}}=RUv_{\mathcal{R}}=RUv.$$

□

Lemma 2.12 (Singular value decomposition)**.** *Let $A\in\mathbb{R}^{d\times d}$. Then there exist $R,Q\in SO(d)$ and $\lambda_1,\dots,\lambda_d\in\mathbb{R}$ with $|\lambda_1|\le\lambda_2\le\ldots\le\lambda_d$ such that*

$$A=R\,\mathrm{diag}(\lambda_1,\dots,\lambda_d)Q.$$

The numbers $(\lambda_i)_{i=1,\dots,d}$ are uniquely determined by A.

Beweis. If $\det A\ge 0$ this follows from the polar factorization $A=RU$ and the fact that the symmetric matrix U can be written as $U=Q^tDQ$ with $Q\in SO(d)$ and a diagonal matrix $D=\mathrm{diag}(\lambda_1,\dots,\lambda_d)$. In the case $\det A<0$ we apply the previous argument to the matrix $B=AP$ where P denotes the reflection $P=\mathbf{Id}-2(e_1\otimes e_1)$. Then $P\in O(d)$ and $\det P=-1$ and thus $\det B>0$. Since $P^2=\mathbf{Id}$ we get

$$A=R\,\mathrm{diag}(\mu_1,\dots,\mu_d)QP$$

with $0<\mu_1\le\ldots\le\mu_d$. Hence,

$$A=R\,\mathrm{diag}(\mu_1,\dots,\mu_d)P(PQP)=R\,\mathrm{diag}(-\mu_1,\dots,\mu_d)(PQP).$$

Since $PQP\in SO(d)$ this yields the desired decomposition. □

Remark 2.13. (**Exercise 2.13**). *Consider the situation of Lemma 2.12. One can easily check that*

1. *$\lambda_1^2,\dots,\lambda_d^2$ are the Eigenvalues of A^tA*
2. *$|\lambda_1|,\dots,|\lambda_d|$ are the singular values of A*
3. *$\det A=\lambda_1\cdots\lambda_d$*
4. *$|A|^2=\sum_{i=1}^d\lambda_i^2$ and $\|A\|=\lambda_d$*
5. *$A\in SO(d)$ if and only if $\{\lambda_1,\dots,\lambda_d\}=\{1\}$.*

Remark 2.14. *A matrix $A \in \mathbb{R}^{d\times d}$ of the form*

$$A = \mathbf{Id} + (\lambda - 1) e \otimes e$$

with $\lambda > 0$ and e unit vector is called a ***stretch*** *in direction e. With help of the polar factorization one can show that any affine deformation $x \mapsto Fx$ can be written as the composition of a rotation and d stretches in orthogonal directions, i.e. there exists an orthonormal basis $q_1, \ldots, q_d$, positive numbers $\lambda_1, \ldots, \lambda_d$ and a rotation $R \in SO(d)$ such that*

$$F = RU_d \cdots U_1$$

where $U_i = \mathbf{Id} + (\lambda_i - 1) q_i \otimes q_i$, (see problem sheet 2). The numbers $\lambda_1, \ldots, \lambda_d$ are the Eigenvalues of $\sqrt{F^t F}$.

Transformation of surface integrals and tensors under deformation. Next, we discuss the transformation of surface integrals (and vectorfields on surfaces) under deformation. While changes of volume integrals are captured by the right Cauchy-Green tensor $C = F^t F$, for surfaces the cofactor matrix is of interest:

Definition 2.15 (Cofactor matrix)**.** *Let $A \in \mathbb{R}^{d\times d}$ with entries A_{ij}. Then the cofactor matrix $\mathrm{cof}A \in \mathbb{R}^{d\times d}$ is defined by*

$$(\mathrm{cof}A)_{ij} := (-1)^{i+j} \det B^{ij}.$$

Here $B^{ij} \in \mathbb{R}^{(d-1)\times(d-1)}$ denotes the matrix obtained from A by deleting the ith row and jth column.
(Note: $\det B^{ij}$ is also called a subdeterminant or minor of A).

Remark 2.16 (Laplace's expansion formula)**.** *Let A denote a $d \times d$-matrix with entries A_{ij}. Laplace's expansion formula from linear algebra can expressed in the form*

$$\det A \;=\; \sum_{j=1}^{d} A_{ij}(\mathrm{cof}A)_{ij} = \sum_{i=1}^{d} A_{ij}(\mathrm{cof}A)_{ij}.$$

It is therefore also called cofactor expansion.

Lemma 2.17. (Exercise 2.17) *Let $A, B \in \mathbb{R}^{d\times d}$. Then*

(i) $(\det A)\mathbf{Id} = A^t \mathrm{cof}A$ *(Cramer's rule),*

(ii) $\det(\mathrm{cof}A) = (\det A)^{d-1}$

(iii) $\mathrm{cof}A = (\det A)A^{-t}$*, if A is invertible,*

(iv) $\mathrm{cof}A = A$*, if $A \in SO(d)$,*

(v) $\mathrm{cof}(AB) = (\mathrm{cof}A)(\mathrm{cof}B)$*,*

(vi) $\mathrm{cof}(A^t) = (\mathrm{cof}A)^t$*,*

(vii) $\mathrm{cof}(A^{-1}) = (\mathrm{cof}A)^{-1}$ *if A is invertible.*

Lemma 2.18. (**Exercise 2.18.** *Transformation of the normal). Let Ω be a domain, $\varphi : \Omega \to \mathbb{R}^d$ a C^1-deformation, and $U \subset\subset \Omega$ a C^1-domain. Then $U^\varphi := \varphi(U)$ is a C^1-domain. Moreover, if $\nu(x)$ denotes the outer normal to ∂U at $x \in \partial U$, then $x^\varphi := \varphi(x) \in \partial\varphi(U)$ and*

$$\nu^\varphi(x^\varphi) := \frac{(\mathrm{cof}\, D\varphi(x))\nu(x)}{|(\mathrm{cof}\, D\varphi(x))\nu(x)|},$$

is the outer normal to $\partial\varphi(U)$ at x^φ.

It is convenient to derive the transformation rule for surface integrals by combining the transformation rule for volume integrals and the divergence theorem. For this and future purposes we consider matix fields $T : \Omega \to \mathbb{R}^{d\times d}$ and its transformation under deformation.

Definition 2.19 (Divergence of a matrix field). *We define the divergence of a matrix field $T(x) \in \mathbb{R}^{d\times d}$ row wise as*

$$\mathrm{div}\, T(x) \in \mathbb{R}^d, \qquad (\mathrm{div}\, T(x))_i = \sum_{j=1}^d \partial_j T_{ij}(x).$$

Note that by integration by parts, we have for all $\varphi \in C_c^\infty(\Omega)$ and $T : \Omega \to \mathbb{R}^{d\times d}$ sufficiently smooth,

$$-\int_\Omega T\nabla\varphi\, dx = -\int_\Omega T_{ij}\partial_j\varphi\, dx = -\int_\Omega \partial_j(T_{ij}\varphi) + \int_\Omega (\partial_j T_{ij})\varphi\, dx = \int_\Omega (\mathrm{div}\, T)\varphi\, dx.$$

Lemma 2.20 (Piola's identity). *Let $\Omega \subset \mathbb{R}^d$ be open, $f : \Omega \to \mathbb{R}^d$ be twice continuously differentiable. Then*

$$\mathrm{div}\,\mathrm{cof} Df(x) = 0 \qquad \text{for all } x \in \Omega.$$

Beweis. We only give a proof for $d = 3$. In that case we have

$$(\mathrm{cof} Df(x))_{ij} = \partial_{j+1}f_{i+1}(x)\partial_{j+2}f_{i+2}(x) - \partial_{j+2}f_{i+1}(x)\partial_{j+1}f_{i+2}(x),$$

where the indices are counted modulo 3. Hence,

$$\begin{aligned}
(\mathrm{div}\,\mathrm{cof} Df(x))_i &= \sum_{j=1}^d \partial_j(\mathrm{cof} Df)_{ij} \\
&= \sum_{j=1}^d \partial_j\Big(\partial_{j+1}f_{i+1}(x)\partial_{j+2}f_{i+2}(x) - \partial_{j+2}f_{i+1}(x)\partial_{j+1}f_{i+2}(x)\Big) \\
&= \sum_{j=1}^d \partial_j\partial_{j+1}f_{i+1}(x)\partial_{j+2}f_{i+2}(x) + \partial_{j+1}f_{i+1}(x)\partial_j\partial_{j+2}f_{j+2}(x) \\
&\qquad -\partial_j\partial_{j+2}f_{i+1}(x)\partial_{j+1}f_{i+2}(x) - \partial_{j+2}f_{i+1}(x)\partial_j\partial_{j+1}f_{i+2}(x) \\
&\overset{\text{relabel}}{=} \sum_{j=1}^d \partial_j\partial_{j+1}f_{i+1}(x)\partial_{j+2}f_{i+2}(x) + \partial_{j+1}f_{i+1}(x)\partial_j\partial_{j+2}f_{j+2}(x) \\
&\qquad -\partial_{j+1}\partial_j f_{i+1}(x)\partial_{j+2}f_{i+2}(x) - \partial_{j+1}f_{i+2}(x)\partial_{j+2}\partial_{j+1}f_{i+2}(x) \\
&= 0.
\end{aligned}$$

□

Definition 2.21 (Piola transform)**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain, $\varphi : \overline{\Omega} \to \mathbb{R}^d$ a C^1-deformation, $T^\varphi : \varphi(\bar{\Omega}) \to \mathbb{R}^{d\times d}$. Then the map $T : \bar{\Omega} \to \mathbb{R}^{d\times d}$ defined by the identity*

$$T(x) = T^\varphi(\varphi(x))\mathrm{cof} D\varphi(x). \tag{2.4}$$

is called the Piola transform of T^φ (under φ).

Lemma 2.22. *Let $\Omega \subset \mathbb{R}^d$ be a domain, $\varphi : \overline{\Omega} \to \mathbb{R}^d$ a C^1-deformation, $T \in C^1(\Omega; \mathbb{R}^{d\times d})$. Then (for all $x \in \Omega$)*

$$\operatorname{div} T(x) = \det D\varphi(x)(\operatorname{div} T^\varphi)(x^\varphi), \qquad x^\varphi := \varphi(x)$$

provided $T^\varphi \in C^1(\varphi(\Omega); \mathbb{R}^{d\times d})$ is related with T by the Piola transform (2.4)*.*

Beweis.

$$\begin{aligned}
(\operatorname{div} T(x))_i &= \partial_j T_{ij}(x) = \partial_j \Big(T^\varphi_{ik}(\varphi(x))(\mathrm{cof} D\varphi(x))_{kj}\Big) \\
&\overset{\text{Piola identity}}{=} \partial_j \Big(T^\varphi_{ik}(\varphi(x))\Big)(\mathrm{cof}(D\varphi(x)))_{kj} \\
&= \det D\varphi(x)\partial_j \Big(T^\varphi_{ik}(\varphi(x))\Big)(D\varphi(x)^{-t})_{kj} \\
&\overset{\partial_j(f\circ\varphi)=\partial_\ell f(\varphi)\partial_j\varphi^\ell}{=} \det D\varphi(x)\partial_\ell T^\varphi_{ik}(\varphi(x))(D\varphi(x))_{\ell j}(D\varphi(x)^{-t})_{kj} \\
&= \det D\varphi(X)\partial_\ell T^\varphi_{ik}(\varphi(x))\delta_{\ell k} \\
&= \det D\varphi(x)[(\operatorname{div} T^\varphi)(\varphi(x))]_i.
\end{aligned}$$

□

Now we are ready to discuss the transformation rule for the surface integral.

Remark 2.23. *Let $\Omega \subset \mathbb{R}^d$ be a C^k- (resp. Lipschitz-) domain and $\varphi : \bar{\Omega} \to \mathbb{R}^d$ be a C^k-deformation (resp. C^1-deformation). To consider surface integrals and the outer normal on $\partial\varphi(\Omega)$, we require $\varphi(\Omega)$ to be a C^k- (resp. Lipschitz-) domain. A sufficient condition for this is that φ can be extended to a deformation on a slightly larger domain. We therefore will assume from time to time that $\varphi : \bar{U} \to \mathbb{R}^d$ is a C^k-deformation (resp. C^1-deformation) for a larger set $U \subset \mathbb{R}^d$ that is open and that covers $\bar{\Omega}$ (i.e. $\Omega \Subset U$).*

Theorem 2.24 (Transformation of surface integrals)**.** *Let $U \subset \mathbb{R}^d$ be a domain, $\Omega \Subset U$ a Lipschitz domain, and $\varphi : \bar{U} \to \mathbb{R}^d$ a C^1-deformation.*

(a) Let $T^\varphi \in C^1(\varphi(\bar{\Omega}); \mathbb{R}^{d\times d})$ and $T \in C^1(\bar{\Omega}; \mathbb{R}^{d\times d})$ be related by the Piola transform (2.4)*. Then*

$$\int_{\varphi(\Omega)} \operatorname{div} T^\varphi(x^\varphi)\, dx^\varphi = \int_\Omega \operatorname{div} T(x)\, dx.$$

(b) Let $T^\varphi \in L^1(\partial\varphi(\Omega); \mathbb{R}^{d\times d})$ and $T \in L^1(\partial\Omega; \mathbb{R}^{d\times d})$ be related by the Piola transform (2.4) *(for a.e. $x \in \partial\Omega$). Then*

$$\int_{\partial\varphi(\Omega)} T^\varphi \nu^\varphi\, \mathcal{H}^{d-1} = \int_{\partial\Omega} T\nu\, \mathcal{H}^{d-1},$$

where ν^φ (resp. nu) denotes the outer unit normal to $\partial\varphi(\Omega)$ (resp. $\partial\Omega$).

(c) Let $g \in L^1(\partial\varphi(\Omega);\mathbb{R}^d)$ and $f \in L^1(\partial\varphi(\Omega))$. Then

$$\begin{aligned}\int_{\partial\varphi(\Omega)} g(x^\varphi)\mathrm{d}\mathcal{H}^{d-1}(x^\varphi) &= \int_{\partial\Omega} g(\varphi(x))|\mathrm{cof}D\varphi(x)\nu(x)|\, d\mathcal{H}^{d-1}(x)\\ \int_{\partial\varphi(\Omega)} f(x^\varphi)\nu^\varphi(x^\varphi)\mathrm{d}\mathcal{H}^{d-1}(x^\varphi) &= \int_{\partial\Omega} f(\varphi(x))\mathrm{cof}D\varphi(x)\nu(x)\, d\mathcal{H}^{d-1}(x).\end{aligned}$$

Beweis. Proof of (a): By assumption $\varphi : \Omega \to \varphi(\Omega)$ is a C^1-diffeomorphism, thus

$$\int_{\varphi(\Omega)} \operatorname{div} T^\varphi(x^\varphi)\, dx^\varphi = \int_\Omega (\operatorname{div} T^\varphi)(\varphi(x)) \det D\varphi(x)\, dx = \int_\Omega \operatorname{div} T(x)\, dx.$$

Application of the divergence theorem yields the identity of (b). By an approximation argument, (b) remains valid for L^1-matrix fields defined on the boundary (first extend T^φ to a matrix field defined in a neighbourhood of $\varphi(\partial\Omega)$ and then regularize the extension via convolution).
Argument for (c). Set $T^\varphi := g \otimes \nu^\varphi$. Then $g = T^\varphi \cdot \nu^\varphi$ and Piola's indentity yields

$$(T \cdot \nu)(x) = g(x^\varphi) \otimes \nu^\varphi(x^\varphi)\mathrm{cof}D\varphi(x)\nu(x).$$

combined with the formula for ν^φ we get

$$(T \cdot \nu)(x) = g(\varphi(x)\frac{\mathrm{cof}D\varphi(x)\nu(x) \cdot \mathrm{cof}D\varphi(x)\nu(x)}{|\mathrm{cof}D\varphi(x)\nu(x)|} = g(\varphi(x))|\mathrm{cof}D\varphi(x)\nu(x)|,$$

and the first identity follows from (b). The second identity is a special case of the first: $g := f\nu^\varphi$. □

2.2 Spannungsprinzip von Euler und Cauchy

Auf einen Körper mit deformierter Konfiguration $\varphi(\Omega)$ wirken

- Kontaktkräfte (*contact forces*) zwischen einem (beliebigen) Teil des Körpers $A \subset \varphi(\Omega)$ und dessen Komplement $\varphi(\Omega) \setminus A$,
- (externe) Oberflächenkräfte (*surface forces*) die auf Teile des Randes $\Gamma^\varphi \subset \partial\varphi(\Omega)$ des Körpers wirken,
- (externe) Volumenkräfte (*volume forces*).

Die statische Theorie der Kontinuumsmechanik basiert auf der Annahme, dass sich der Körper nicht veformt und bewegt, falls sich diese Kräfte im Gleichgewicht befinden. In der klassischen Kontinuumsmechanik trifft man Annahmen über die Gestalt zulässiger Kräfte.

Definition 2.25 (Kräftesystem)**.** *Sei $\Omega \subset \mathbb{R}^d$ Gebiet und $\varphi : \overline{\Omega} \to \mathbb{R}^d$ eine Deformation. Ein* Kräftesystem *ist ein Paar (t^φ, f^φ) so dass*

- *$t^\varphi : \varphi(\Omega) \times S^{d-1} \to \mathbb{R}^d$ is Borel-messbar and lokal beschränkt,*

- $x^\varphi \mapsto t^\varphi(x^\varphi, \nu)$ *ist stetig für alle* $\nu \in S^{d-1}$,
- $f^\varphi : \varphi(\Omega) \to \mathbb{R}^d$ *ist stetig.*

Wir behandeln externe Oberflächenkräfte später.

Axiom 2.26 (Spannungsprinzip von Cauchy und Euler)**.** *Sei* $\Omega \subset \mathbb{R}^d$ *Gebiet und* $\varphi : \overline{\Omega} \to \mathbb{R}^d$ *eine Deformation. Ein Körper nehme im deformierten Zustand die Konfiguration* $\varphi(\Omega)$ *ein.*

(a) Dann existiert eine Kräftesystem (t^φ, f^φ) *so dass für alle (Lipschitz) Gebiete* $A \Subset \varphi(\Omega)$*, die auf* A *wirkende Gesamtkraft durch*

$$F^\varphi(A) := \int_A f^\varphi(x^\varphi) dx^\varphi + \int_{\partial A} t^\varphi(x^\varphi, \nu^\varphi(x^\varphi))\, d\mathcal{H}^{d-1}(x^\varphi),$$

gegeben ist, und das auf A *wirkende Gesamtdrehmoment durch*

$$M^\varphi(A) := \int_A x^\varphi \wedge f^\varphi(x^\varphi)\, dx^\varphi + \int_{\partial A} x^\varphi \wedge t^\varphi(x^\varphi, \nu^\varphi(x^\varphi))\, d\mathcal{H}^{d-1}(x^\varphi),$$

gegeben ist. (Notation $a \wedge b := a \otimes b - b \otimes a$*).*

(b) Wir sagen der Körper mit deformierter Konfiguration $\varphi(\Omega)$ *befindet sich im* **Kräfte und Drehmomentgleichgewicht***, falls für alle (Lipschitz) Gebiete* $A \Subset \varphi(\Omega)$*,*

$$F^\varphi(A) = 0, \qquad M^\varphi(A) = 0.$$

Remark 2.27. *Das Prinzip von Cauchy und Euler formuliert die Annahme,*

- *dass ein Teilkörper* $\varphi(\Omega) \setminus A$ *auf ein Testvolumen* $A \Subset \varphi(\Omega)$ *Kontaktkräfte ausübt, die über den gemeinsamen Rand* ∂A *übertragen werden;*
- *dass die Abhängigkeit der Kontaktkraft vom Testvolumen* A *an einem Punkt* $x^\varphi \in \partial A$ *nur als Funktion der äußeren Normalen* ν^φ *gegeben ist (und nicht auch von der Krümmung von* ∂A *oder anderen geometrischen Größen abhängt).*
- *dass sich ein deformierter Körper genau dann in einem (statischen) Gleichgewichtszustand befindet, wenn Kräfte- und Momentengleichgewicht für alle Testvolumen* $A \Subset \varphi(\Omega)$ *gegeben ist.*

Example 2.28 (Volumenkraft)**.** *Sei* $\rho^\varphi : \varphi(\Omega) \to \mathbb{R}_+$ *die Massendichte in der deformierten Konfiguration. Dann beschreibt* $f^\varphi(x^\varphi) := -g\rho^\varphi(x^\varphi)e_3$ *(mit* g *Gravitationskonstante) die Gravitationskraft(dichte) (falls* e_3 *"nach unten zeigt").*

Theorem 2.29 (Cauchy's Theorem (existence of stress))**.** *Sei* $\Omega \subset \mathbb{R}^d$ *ein Gebiet,* $\varphi : \overline{\Omega} \to \mathbb{R}^d$ *eine Deformation. Es gelte Axiom A1 und sei* $t^\varphi(x^\varphi, \nu)$ *stetig differenzierbar in* $x^\varphi \in \varphi(\Omega)$ *(für alle* $\nu \in S^{d-1}$*) und stetig in* $\nu \in S^{d-1}$ *(für alle* $x^\varphi \in \varphi(\Omega)$*). Dann existiert eine Abbildung*

$$T^\varphi : \varphi(\Omega) \to \mathcal{L}(\mathbb{R}^d) \equiv \mathbb{R}^{d\times d},$$

so dass $t^\varphi(x^\varphi, \nu) = T^\varphi(x^\varphi)\nu$ *und*

$$-\operatorname{div} T^\varphi = f^\varphi \quad \text{in } \varphi(\Omega) \tag{2.5a}$$

$$T^\varphi = (T^\varphi)^t \quad \text{in } \varphi(\Omega). \tag{2.5b}$$

Definition 2.30 (Cauchy'scher Spannungstensor)**.** *Die Abbildung T^φ aus Theorem 2.29 heißt* Cauchy'sche Spannungstensor*. Die Abbildung t^φ heißt Cuachy'scher Spannungsvektor.*

Remark 2.31. *The proof of Theorem 2.29 relies on Cauchy's Tetrahedron argument. We recall some basic facts on the geometry of simplices. Fix an orthonormal basis (ONB) $b_1, \dots, b_d \in \mathbb{R}^d$ and a unit vector $k \in S^{d-1}$ with $k \cdot b_i > 0$ for all $i = 1, \dots, d$.*

- $\triangle := \{\, x \in \mathbb{R}^d \,:\, x \cdot b_i > 0,\, x \cdot k < 1 \,\}$ *defines a simplex with faces*

$$\begin{aligned} S_0 &:= \partial\triangle \cap \{x \,:\, x \cdot k = 1\} \\ S_i &:= \partial\triangle \cap \{x \,:\, x \cdot b_i = 0\,\} \qquad (i = 1, \dots, d) \end{aligned}$$

- $\mathcal{H}^d(\triangle) = \frac{1}{d}\mathcal{H}^{d-1}(S_i)\frac{1}{k\cdot b_i} = \frac{1}{d}\mathcal{H}^{d-1}(S_0)$ *for $i = 1, \dots, d$, and thus*

$$\mathcal{H}^{d-1}(S_i) = (k \cdot b_i)\mathcal{H}^{d-1}(S_0). \tag{2.6}$$

Proof of Theorem 2.29. **Step 1.** Claim: For any ONB $b_1, \dots, b_d$ and any $k \in S^{d-1}$ with $k \cdot b_i \geq 0$ (for all i) we have

$$t^\varphi(x^\varphi, k) = -\sum_{i=1}^{d}(k \cdot b_i)t^\varphi(x^\varphi, -b_i). \tag{2.7}$$

Argument: We first consider the case $k \cdot b_i > 0$ for all $i = 1, \dots, d$. In that case $\triangle := \{x : x \cdot b_i > 0,\, x \cdot k < 1\}$ defines a non-empty simplex. Conservation of linear momentum applied to the test volume $A := x^\varphi + \delta\triangle \Subset \varphi(\Omega)$ (with $x^\varphi \in \varphi(\Omega)$ and $0 < \delta \ll 1$) reads

$$\begin{aligned} -\int_{x^\varphi+\delta\triangle} f^\varphi \, d\mathcal{H}^d &= \int_{x^\varphi+\delta\partial\triangle} t^\varphi(y, \nu^\varphi(y))\, d\mathcal{H}^{d-1}(y) \\ &= \sum_{i=0}^{d-1}\int_{x^\varphi+\delta S_i} t^\varphi(y, \nu_i)\, d\mathcal{H}^{d-1}(y), \end{aligned}$$

where (as shown in Remark 2.31)

$$\nu_0 = k, \qquad \nu_i = -b_i.$$

A change of coordinates ($y \rightsquigarrow x^\varphi + \delta y$) and dividing by δ^{d-1} yields

$$-\delta\int_{\triangle} f^\varphi(x^\varphi + \delta y)\,\mathrm{d}y \;=\; \sum_{i=0}^{d-1}\int_{S_i} t^\varphi(x^\varphi + \delta y, \nu_i)\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

Since f^φ and $t^\varphi(\cdot, \nu_i)$ are continuous, the limit $\delta \downarrow 0$ yields

$$0 = \sum_{i=0}^{d}\mathcal{H}^{d-1}(S_i)t^\varphi(x^\varphi, \nu_i).$$

Hence, combined with (2.6), we get the identity

$$t^\varphi(x^\varphi, \nu_0) = -\sum_{i=1}^d (k \cdot b_i) t^\varphi(x^\varphi, \nu_i).$$

Now the claim follows from the definition of ν_i.
For the case $k \cdot b_i \geq 0$ (for $i = 1, \dots, d$) consider a sequence $(k_j) \subset S^{d-1}$ satisfying $k_j \cdot b_i > 0$ and $k_j \to k$. Then the statement follows from the statement for k_j and the continuity of $t^\varphi(x^\varphi, \cdot)$.

Step 2. Claim: For all $k \in S^{d-1}$ we have $t^\varphi(x^\varphi, k) = -t^\varphi(x^\varphi, -k)$.

Argument: Extend k to an ONB $b_1 = k, b_2, \dots, b_d$. Then Step 1 applied with k and $(b_1 = k, \dots, b_d)$ yields

$$t^\varphi(x^\varphi, k) = -t^\varphi(x^\varphi, -k) - \underbrace{\sum_{i=2}^d (k \cdot b_i) t^\varphi(x^\varphi, -b_i)}_{=0}.$$

Step 3 – Definition of T^φ. Define

$$T^\varphi(x^\varphi) := \sum_{i=1}^d t^\varphi(x^\varphi, e_i) \otimes e_i.$$

We claim that for all $k \in S^{d-1}$ we have

$$t^\varphi(x^\varphi, k) = T^\varphi(x^\varphi) k$$

Argument: Let $k \in S^{d-1}$ and set

$$\sigma_i := \begin{cases} 1 & k \cdot e_i \geq 0, \\ -1 & k \cdot e_i < 0. \end{cases}$$

Then $b_i = \sigma_i e_i$ defines an ONB with $k \cdot b_i \geq 0$. Hence, Step 1 and Step 2 yield

$$t^\varphi(x^\varphi, k) \overset{\text{Step 1}}{=} -\sum_{i=1}^d (k \cdot b_i) t^\varphi(x^\varphi, -b_i) \overset{\text{Step 2}}{=} \sum_{i=1}^d (k \cdot e_i) t^\varphi(x^\varphi, e_i) = T^\varphi(x^\varphi) k.$$

Step 4 – Proof of (2.5a)**.** Since $t^\varphi(\cdot, \nu)$ is continuously differentiable, we conclude that T^φ is continuously differentiable. Hence, conservation of linear momentum, $\int_A f^\varphi + \int_{\partial A} t^\varphi(x^\varphi, \nu^\varphi) = 0$, combined with Step 3 and the divergence theorem implies

$$\begin{aligned} 0 &= \int_A f^\varphi \, d\mathcal{H}^d + \int_{\partial A} T^\varphi \nu^\varphi \, d\mathcal{H}^{d-1} \\ &= \int_A f^\varphi + \operatorname{div} T^\varphi \, \mathcal{H}^d. \end{aligned}$$

Since this is true for arbitrary test volumes $A \Subset \varphi(\Omega)$, (2.5a) follows.

(Note that up to now we did not use the conservation of angular momentum!)

Step 5 – Proof of (2.5b)**.** Set $w(x^\varphi) := (x^\varphi_i e_j - x^\varphi_j e_i)$ and note that we have $(x^\varphi \wedge a) \cdot (e_i \otimes e_j) = a \cdot w(x^\varphi)$ for any $a \in \mathbb{R}^d$. Hence, by the balance of total torque we obtain

$$\begin{aligned} 0 &= \int_A f^\varphi \cdot w + \int_{\partial A} T^\varphi \nu \cdot w \, d\mathcal{H}^{d-1} \\ &= \int_A f^\varphi \cdot w + \int_{\partial A} (T^\varphi)^t w \cdot \nu \, d\mathcal{H}^{d-1} \\ &= \int_A f^\varphi \cdot w + \operatorname{div}((T^\varphi)^t w) \, d\mathcal{H}^d. \end{aligned}$$

With $\operatorname{div}((T^\varphi)^t w) = \operatorname{div} T^\varphi \cdot w + T \cdot Dw$ and $(Dw)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta j} - \delta_{\alpha j}\delta_{\beta i}$, we obtain

$$0 = \int_A (f^\varphi + \operatorname{div} T^\varphi) \cdot w + \int_A T^\varphi_{ij} - T^\varphi_{ji} \, d\mathcal{H}^d.$$

By Step 3, the first integral vanishes, and thus $\int_A T^\varphi_{ij} - T^\varphi_{ji} \, d\mathcal{H}^d = 0$. Since T^φ is continuous, and since this is true for all $A \Subset \varphi(\Omega)$ (Lipschitz) domains, we conclude that $T^\varphi_{ij} = T^\varphi_{ji}$ in $\varphi(\Omega)$. □

Remark 2.32 ((**Exercise 2.32**))**.** *Die Gleichgewichtsgleichung für Kräfte* (2.5a) *gilt auch unter schwächeren Regularitätsvoraussetzungen an t^φ: Sei $\Omega \subset \mathbb{R}^d$ ein Gebiet, $\varphi : \overline{\Omega} \to \mathbb{R}^d$ eine Deformation. Sei $f^\varphi : \varphi(\Omega) \to \mathbb{R}^d$ lokal integrierbar, und $t^\varphi : \varphi(\Omega) \times S^{d-1} \to \mathbb{R}^d$ ein Abbildung, die getrennt stetig ist (d.h. $t^\varphi(\cdot, \nu)$ (resp. $t^\varphi(x^\varphi, \cdot)$) ist stetig für alle $\nu \in S^{d-1}$ (resp. für alle $x^\varphi \in \varphi(\Omega)$). Weiter gelte das Axiom des Kräftegleichgewichts:*

$$\int_A f^\varphi + \int_{\partial A} t^\varphi(x^\varphi, \nu^\varphi(x^\varphi)) \, d\mathcal{H}^{d-1}(x^\varphi) = 0$$

für alle (Lipschitz) Gebiete $A \Subset \varphi(\Omega)$. Dann erfüllt der Cauchy'sche Spannunstensor

$$T^\varphi : \varphi(\Omega) \to \mathcal{L}(\mathbb{R}^d), \qquad T^\varphi(x^\varphi) := \sum_{i=1}^d t^\varphi(x^\varphi, e_i) \otimes e_i,$$

die distributionelle Erhaltungsgleichung

$$-\operatorname{div} T^\varphi \;=\; f^\varphi \qquad \textit{in } \mathcal{D}'(\varphi(\Omega)). \tag{2.8}$$

Die obige Schreibweise bedeuted "im Sinne von Distributionen" und meint

$$\forall \eta \in C^\infty_c(\varphi(\Omega), \mathbb{R}^d) \,:\, \int_{\varphi(\Omega)} T^\varphi \cdot D\eta = \int_{\varphi(\Omega)} f^\varphi \cdot \eta.$$

Example 2.33 (Spannungszustände und deren Visualisierung)**.** *Der Cauchy'sche Spannungstensor T^φ und Spannungsvektor t^φ stehen in der Relation*

$$t^\varphi_i(x^\varphi, e_j) = T^\varphi_{ij}(x^\varphi).$$

Diese wird zur Visualisierung von Spannungszuständen verwendet. Im Folgenden sei $d = 3$, $x^\varphi \in \varphi(\Omega)$ und $T = T^\varphi(x^\varphi)$.

(a) $T = \alpha e_1 \otimes e_1$. *Für* $\alpha > 0$ *handelt es sich im eine reine* **Zugspannung** *in Richtung* e_1. *Für* $\alpha < 0$ *um eine Druckspannung.*

(b) $T = -pI$, $p > 0$. *Es handelt sich um einen* ***hydrostatischen Druck***.

(c) $T = \alpha(e_2 \otimes e_1 + e_1 \otimes e_2)$. *Es handelt sich um eine reine* **Schubspannung**.

Der Cauchy'sche Spannungstensor T^φ ist symmetrisch und besitzt damit eine Orthonormalbasis aus Eigenvektoren.

Definition 2.34 (Normalspannung, Schubspannung,Hauptspannungen). *Sei* $T = T^\varphi(x^\varphi)$ *der Cauchy'sche Spannungstensor in* x^φ.

- *Die Eigenvektoren von* T *heißen Hauptspannungsrichtungen und die zugehörigen Eigenwerte* τ_i *Hauptspannungen.*
- *Verschwindet im Falle* $d = 3$ *eine der Hauptspannungen, so spricht man von einem ebenen Spannungszustand.*
- *Sei* $\nu \in S^{d-1}$. *Dann heißt* $T_N := T \cdot (\nu \otimes \nu) = T\nu \cdot \nu \in \mathbb{R}$ **Normalspannung** *in Richtung* ν *und der Vektor* $T_S := T\nu - T_N\nu \in \mathbb{R}^d$ **Schubspannung** *oder* **Tangentialspannung** *zu* ν.

Remark 2.35 ((**Exercise 2.35**)). *Sei* $d = 3$. *Seien* $\tau_1 \geq \tau_2 \geq \tau_3$ *die Hauptspannungen eines Cauch'schen Spannungstensors* T. *Dann gilt:*

- τ_1 *(resp.* τ_3*) ist durch die Maximale (resp. Minimale) Normalspannung* $\max_{\nu \in S^2} T \cdot (\nu \otimes \nu)$ *(resp.* $\min_{\nu \in S^2} T \cdot (\nu \otimes \nu)$*) gegeben.*
- *Falls* $\tau_1 > \tau_2 > \tau_3$, *so ist die maximale Schubspannung durch* $\frac{1}{2}(\tau_1 - \tau_3)$ *gegeben und das Maximum wird in Richtung*

$$\nu = \pm \frac{e_1 + e_3}{|e_1 + e_3|}$$

  eingenommen.

Remark 2.36 (externe Oberflächenkräfte). *Wir betrachten die Situation von Theorem 2.29 und nehmen an, dass auf einen Teil des Randes* $\Gamma^\varphi \subset \partial\varphi(\Omega)$ *eine Oberflächenkraft mit Dichte* $g^\varphi : \Gamma^\varphi \subset \partial\varphi(\Omega) \to \mathbb{R}^d$ *wirkt. Zur Modellierung des Gleichgewichtszustandes fordern wir zusätzlich, dass* T^φ *stetig auf* Γ^φ *fortgesetzt werden kann und die Randbedingung* $T^\varphi(x^\varphi)\nu^\varphi(x^\varphi) = g^\varphi(x^\varphi)$ *für* $x^\varphi \in \Gamma^\varphi$ *gilt.*

Example 2.37 (Druckkraft). *Auf den deformierten Körper wirke eine (räumlich konstante) Druckkraft. Zur Modellierung setzen wir* $\Gamma^\varphi := \partial\varphi(\Omega)$ *und* $g^\varphi(x^\varphi) := -p\nu^\varphi(x^\varphi)$ *mit* $p > 0$.

Definition 2.38 (Gleichgewichtsgleichung, deformierten Konfiguration). *Sei* $\Omega \subset \mathbb{R}^d$ *ein Gebiet,* $\varphi : \bar{\Omega} \to \mathbb{R}^d$ *Deformation. Seien Kraftdichten* $g^\varphi : \Gamma^\varphi \to \mathbb{R}^d$ *(mit* $\Gamma^\varphi \subset \partial\varphi(\Omega)$*)*

und $f^\varphi : \varphi(\Omega) \to \mathbb{R}^d$ gegeben. Es bezeichne $T^\varphi : \varphi(\Omega) \cup \Gamma^\varphi \to \mathbb{R}^{d\times d}$ den Cauchy'schen Spannungstensor. Das System

$$\begin{aligned} -\operatorname{div} T^\varphi &= f^\varphi \qquad \text{in } \varphi(\Omega),\\ T^\varphi &= (T^\varphi)^t \qquad \text{in } \overline{\varphi(\Omega)},\\ T^\varphi \nu^\varphi &= g^\varphi \qquad \text{auf } \Gamma^\varphi, \end{aligned} \tag{2.9}$$

heißt Gleichgewichtsgleichung (in der deformierten Konfiguration).

Das System (2.9) ist bzgl. der Koordinaten x^φ formuliert. Diese hängen von der a priori unbekannten Deformation φ ab. Es ist daher von Nutzen das System in der Referenzkonfiguration zu formulieren.

Definition 2.39. *Sei $\Omega \subset \mathbb{R}^d$ Gebiet, $\varphi : \bar{\Omega} \to \mathbb{R}^d$ Deformation. T^φ Cauchy'scher Spannungstensor.*

- *Die Abbildung $T : \bar{\Omega} \to \mathbb{R}^{d\times d}$,*

  $$T(x) := T^\varphi(x^\varphi)\operatorname{cof} D\varphi(x), \qquad x^\varphi := \varphi(x)$$

  *heißt **Erster Piola-Kirchhoff-Spannungstensor**.*

- *Die Abbildung $\Sigma : \bar{\Omega} \to \mathbb{R}^{d\times d}_{\text{sym}}$,*

  $$\Sigma(x) := D\varphi(x)^{-1} T(x) = D\varphi^{-1} T^\varphi(x^\varphi)\operatorname{cof} D\varphi(x), \qquad x^\varphi := \varphi(x)$$

  *heißt **Zweiter Piola-Kirchhoff-Spannungstensor**.*

Theorem 2.40 (Transformation in die Referenzkonfiguraiton)**.** *Sei $\Omega \subset \mathbb{R}^d$ ein Gebiet, $\Gamma \subset \partial\Omega$, sei $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^d)$ Deformation mit $\det D\varphi > 0$ auf $\bar{\Omega}$. Seien Kraftdichten $g^\varphi : \Gamma^\varphi \to \mathbb{R}^d$ (mit $\Gamma^\varphi := \varphi(\Gamma)$) und $f^\varphi : \varphi(\Omega) \to \mathbb{R}^d$ gegeben. Es bezeichne T^φ den Cauchy'schen Spannungstensor. Setze*

$$f(x) := \det D\varphi(x) f^\varphi(\varphi(x)) \qquad (x \in \Omega), \qquad g(x) := |\operatorname{cof} D\varphi(x)\nu(x)| g^\varphi(\varphi(x)), \qquad (x \in \Gamma).$$

Es bezeichne T (resp. Σ) den ersten (resp. zweiten) Piola-Kirchoff-Spannungstensor. Dann ist (2.9) formal äquivalent zum System

$$\begin{aligned} -\operatorname{div} T &= f \qquad \text{in } \Omega,\\ T D\varphi^t &= D\varphi T^t \qquad \text{in } \bar{\Omega},\\ T\nu &= g \qquad \text{auf } \Gamma, \end{aligned} \tag{2.10}$$

und zum System

$$\begin{aligned} -\operatorname{div}(D\varphi\Sigma) &= f \qquad \text{in } \Omega,\\ \Sigma &= \Sigma^t \qquad \text{in } \bar{\Omega},\\ D\varphi\Sigma\nu &= g \qquad \text{auf } \Gamma, \end{aligned} \tag{2.11}$$

Beweis. Beweis siehe Übung. Formal äquivalent bedeuted, dass alle involvierten Abbildungen und Gebiete hinreichend regulär sind. □

Definition 2.41. *Die System* (2.10) *und* (2.11) *heißen Gleichgewichtsgleichungen in der Referenzkonfiguraiton.*

Theorem 2.42 (Variationsformulierungen der Gleichgewichtsgleichungen)**.** *Wir betrachten die Situation von Satz 2.40. Dann ist*

- (2.9) *formal äquivalent zur Variationsformulierung:* T^φ *ist symmetrisch und*

$$\int_{\Omega^\varphi} T^\varphi \cdot D\eta = \int_{\Omega^\varphi} f^\varphi \cdot \eta + \int_{\Gamma^\varphi} g^\varphi \cdot \eta \, d\mathcal{H}^{d-1}, \tag{2.12}$$

  for all $\eta \in C^1(\bar{\Omega}^\varphi, \mathbb{R}^d)$ *with* $\eta = 0$ *on* $\partial\Omega^\varphi \setminus \Gamma^\varphi$.

- (2.11) *formal äquivalent zur Variationsformulierung:* Σ *ist symmetrisch und*

$$\int_{\Omega} D\varphi\Sigma \cdot D\eta = \int_{\Omega} f \cdot \eta + \int_{\Gamma} g \cdot \eta \, d\mathcal{H}^{d-1}, \tag{2.13}$$

  for all $\eta \in C^1(\bar{\Omega}, \mathbb{R}^d)$ *with* $\eta = 0$ *on* $\partial\Omega \setminus \Gamma$.

Beweis. We only discuss (2.9). First note that the identity $\operatorname{div}((T^\varphi)^t\eta) = \operatorname{div} T^\varphi \cdot \eta + T^\varphi \cdot D\eta$ and the divergence Theorem 1.11 yields for all $\eta \in C^1(\bar{\Omega}^\varphi, \mathbb{R}^d)$ with $\eta = 0$ on $\partial\Omega^\varphi \setminus \Gamma^\varphi$,

$$\begin{aligned} \int_{\Omega^\varphi} T^\varphi \cdot D\eta &= \int_{\Omega^\varphi} \operatorname{div}((T^\varphi)^t\eta) - (\operatorname{div} T^\varphi) \cdot \eta \, d\mathcal{H}^d \\ &= \int_{\partial\Omega^\varphi} \eta \cdot T^\varphi \nu^\varphi \, d\mathcal{H}^{d-1} - \int_{\Omega^\varphi} (\operatorname{div} T^\varphi) \cdot \eta. \end{aligned} \tag{2.14}$$

Argument (2.9) $\Rightarrow$ (2.12): (2.14) turns into

$$\int_{\Omega^\varphi} T^\varphi \cdot D\eta \;=\; \int_{\Gamma^\varphi} \eta \cdot g^\varphi \, d\mathcal{H}^{d-1} + \int_{\Omega^\varphi} f^\varphi \cdot \eta.$$

Argument (2.12) $\Rightarrow$ (2.9): (2.14) applied with $\eta \in C_c^\infty(\varphi(\Omega), \mathbb{R}^d)$ yields $\int_{\Omega^\varphi} (-\operatorname{div} T^\varphi) \cdot \eta = \int_{\Omega^\varphi} T^\varphi \cdot D\eta$, and thus with (2.12),

$$\int_{\Omega^\varphi} (f^\varphi + \operatorname{div} T^\varphi) \cdot \eta \;=\; 0.$$

If f^φ and $\operatorname{div} T^\varphi$ are continuous, this implies the first identity in (2.9). Combined with (2.14), for all $\eta \in C^1(\Omega^\varphi, \mathbb{R}^d)$ with $\eta = 0$ on $\partial\Omega^\varphi \setminus \Gamma^\varphi$, we get

$$\int_{\Omega^\varphi} T^\varphi \cdot D\eta \;=\; \int_{\Gamma^\varphi} \eta \cdot T^\varphi \nu^\varphi \, d\mathcal{H}^{d-1} + \int_{\Omega^\varphi} f^\varphi \cdot \eta.$$

Hence, with (2.12) we obtain

$$\int_{\Gamma^\varphi} \eta \cdot (g^\varphi - T^\varphi \nu^\varphi) \, d\mathcal{H}^{d-1} = 0.$$

If these testfunctions are dense in $L^1(\Gamma^\varphi)$, and $g^\varphi - T^\varphi\nu^\varphi$ is continuous, the second identity in (2.9) follows. □

An der Transformationsregel für dei Kräfte sieht man, dass Kraftdichten in der Referenzkonfiguration im Allgemeinem von der gesuchten Deformation φ abhängen, also folgende Form haben:

$$f(x) = \hat{f}(x, \varphi(x), D\varphi(x)) \qquad g(x) = \hat{g}(x, D\varphi(x))$$

mit Abbildungen $\hat{f} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ und $\hat{g} : \Gamma \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$.

Definition 2.43 (Tote Last). *Eine Kraft $f(x) = \hat{f}(x, \varphi(x), D\varphi(x))$ die unabhängig von der Deformation φ ist, heißt tote Last.*

Example 2.44. *Die Schwerkraft $f^\varphi(x^\varphi) := -g\rho^\varphi(x^\varphi)e_3$ (mit ρ^φ Massendichte in der deformierten Konfiguration) ist eine Tote Last, denn es gilt gemäß des Axioms der Massenerhaltung unter Deformation, dass*

$$\rho(x) = \det D\varphi(x)\rho^\varphi(\varphi(x)),$$

wobei ρ die Massendichte in der Referenzkonfiguration bezeichnet. Also

$$f(x) = \det D\varphi(x)f^\varphi(\varphi(x)) = -g\rho(x)e_3.$$

2.3 Constitutive laws

Die Gleichgewichtsgleichung (2.11) reicht noch nicht aus um den Spannungstensor Σ und die Deformation (bzw. den Cauchy Verzerrungstensor C) berechnen zu können: In $d = 3$, haben wir 12 unbekannte Funktionen, aber nur 6 Gleichungen. Es fehlt noch das Materialgesetz, welches einen Zusammenhang von Spannung und Verzerrung herstellt.

Definition 2.45 (Elastic material). *A material with reference domain $\Omega \subset \mathbb{R}^d$ is called*

- elastic *if there exists a map $\hat{T}^D : \bar{\Omega} \times GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ such that $T^\varphi(\varphi(x)) = \hat{T}^D(x, D\varphi(x))$. The map $\hat{T}^D$ is called constitutive function (or response function). (The superscript D indicates that it yields the stress tensor is in the deformed configuraiton).*
- linearly elastic*, if it is elastic and $\hat{T}^D(x, F)$ is affine in F.*
- elastic and homogeneous*, if it is elastic and $\hat{T}^D(x, F)$ is independent of x. Otherwise the material is called heterogeneous.*

Remark 2.46.
- *The form of the response function is based on*
  - *experiments*
  - *formal derivations from atomistic or microstructural models*
  - *numerical simulations*
- *Yet, general properties for the response function can be deduced from symmetry considerations (see the discussion below)*
- *Note that the form of the response function $\hat{T}^D$ changes, if we describe the body w.r.t. to a different reference configuration.*

We restrict for simplicity of notation to the homogeneous case. Yet, everything extends in the obvious way to heterogeneous materials.

2.3.1 Frame indifference

Zur Motivation der nachfolgenden Definition sei $d = 3$. Bei der Beschreibung eines deformierten Körpers wird zunächst eine Referenzkonfiguration Ω eingeführt, wobei $x \in \Omega$ als "Beschriftung eines Materialpunktes" des Körpers gesehen werden kann. Der deformierte Körper lebt im Euklidischen Punktraum $\mathbb{E}^3$. Die Positionen eines Materialpunktes x in der deformierten Konfiguration wird durch eine Deformation φ beschrieben. Hierbei haben wir $\varphi(x)$ mit einem Vektor im $\mathbb{R}^3$ identifiziert. D.h. wir haben uns stillschweigend auf ein Bezugssytem $(O, P_1, \ldots, P_2) \in \mathbb{E}^3$ geeinigt (mit $vec(O, P_i) \in \mathbb{V}^3$ normiert und orthogonal) und die Position $O + (\varphi(x) \cdot e_i)\, vec(O, P_i)$ mit dem Vektor $\varphi(x) \in \mathbb{R}^3$ identifiziert. Das gleiche gilt für die Vektoren $f^\varphi, g^\varphi, t^\varphi, \nu^\varphi \in \mathbb{V}^3$, z.B. haben wir den Vektor $(t^\varphi \cdot e_i)\, vec(O, P_i)$ mit $t^\varphi \in \mathbb{R}^3$ identifiziert. Die Wahl des Bezugssystems ist hierbei beliebig. Ändern wir das Bezugssystem, so ändert Position von Materialpunkten und Vektoren in der deformierten Konfiguration nicht, jedoch die Beschreibung durch Vektoren im $\mathbb{R}^3$. Insbesondere, seien $(O, P_1, \ldots, P_d)$ und $(O^*, P_1^*, \ldots, P_2^*)$ zwei ortonormale Bezugssystem mit gleicher Orientierung.

- Eine Punkt P im Euklidischen Raum sei bzgl. $(O, P_1, \ldots, P_d)$ (resp. $(O^*, P_1^*, \ldots, P_d^*)$) durch $p \in \mathbb{R}^3$ (bzw. $p^* \in \mathbb{R}^3$) beschrieben, also
$$P = O + p_i\, vec(O, P_i) = O^* + p_i^*\, vec(O, P_i^*)$$
- Eine Vektor $F \in \mathbb{V}^d$ sei durch $f \in \mathbb{R}^3$ (bzw. $f^* \in \mathbb{R}^3$) beschrieben, also
$$F = f_i\, vec(O, P_i) = f_i^*\, vec(O, P_i^*).$$

Dann existiert $c \in \mathbb{R}^d$ und $R \in SO(d)$, so dass
$$p^* = Rp + c \qquad \text{und} \qquad f^* = Rf.$$

Axiom 2.47 (Objektivität)**.** *Ein Material mit Referenzkonfiguration Ω heißt objektiv, falls für alle $R \in SO(d)$, $c \in \mathbb{R}^d$, und Deformationen $\varphi, \varphi^* : \bar{\Omega} \to \mathbb{R}^d$ mit $\varphi^* = R\varphi + c$ für die Cauchy'schen Spannungsvektoren gilt:*
$$t^{\varphi^*}(\varphi^*(x), R\nu) = Rt^\varphi(\varphi(x), \nu), \qquad x \in \bar{\Omega}, \nu \in S^{d-1}.$$

Theorem 2.48. *Für ein elastisches Material mit Antwortfunktion $\hat{T}^D$ gilt das Axiom der Objektivität genau dann, wenn*
$$\hat{T}^D(x, RF) = R\hat{T}^D(x, F)R^t \tag{2.15}$$
für alle $R \in SO(d)$ und $F \in \mathbb{R}_+^{d\times d}$.

Beweis. Sei $\varphi(x) = Fx$ Deformation und $\varphi^* := R\varphi$ mit $R \in SO(d)$, $F \in \mathbb{R}^{d\times d}$. Dann gilt
$$T^{\varphi^*}(\varphi^*(x))R\nu = t^{\varphi^*}(\varphi^*(x), R\nu) = Rt^\varphi(\varphi(x), \nu) = RT^\varphi(\varphi(x))\nu,$$
also
$$\hat{T}^D(x, RF)R\nu = \hat{T}^D(x, D\varphi^*(x))R\nu = R\hat{T}^D(x, D\varphi(x))\nu = R\hat{T}^D(x, F)\nu.$$
□

Definition 2.49. *Ein elastische Konstitutivgesetz $\hat{T}^D$ heißt objektiv (oder frame indifferent), falls* (2.15) *erfüllt ist.*

The constitutive law can equivalently be stated on the level of the Piola-Kirchhoff stress tensor. To be definite, let $\hat{T}^D$ (resp. $\hat{T}$ and $\hat{S}$) denote the response function for the Cauchy stress tensor T^φ (resp. the first and second Piola-Kirchhoff-Stresstensor T and Σ). Then

$$\hat{T}(F) := \hat{T}^D(F)\mathrm{cof}F \qquad \text{and} \qquad \hat{\Sigma}(F) := F^{-1}\hat{T}^D(F)\mathrm{cof}F. \tag{2.16}$$

Corollary 2.50 ((Exercise 2.50))**.** *Für ein elastisches Material sind äquivalent:*

(i) das Material ist objektiv,

(ii) $\hat{T}^D(RF) = R\hat{T}^D(F)R^t$ *for all* $R \in SO(d)$ *and* $F \in GL_+(d)$

(iii) $\hat{T}(RF) = R\hat{T}(F)$ *for all* $R \in SO(d)$ *and* $F \in GL_+(d)$

(iv) $\hat{\Sigma}(RF) = \Sigma(F)$ *for all* $R \in SO(d)$ *and* $F \in GL_+(d)$

(v) there exists a map $\tilde{\Sigma} : \mathbb{R}^{d\times d}_{sym} \cap GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ *such that* $\tilde{\Sigma}(F^tF) = \hat{\Sigma}(F)$ *for all* $F \in GL_+(d)$.

2.3.2 Material symmetries

Consider a test volume, say $U = B_r(0)$, of an elastic material. We compare the following 2 experiments:

- We affinely deform the body, i.e. $x \mapsto Fx$.
- We first rotate the body and then affinely deform the body, i.e. $x \mapsto FRx$.

In general, both experiments yield different stresses.

Definition 2.51. *An elastic materials has the point group* $G \subset GL_+(d)$*, if* $\hat{T}^D(F) = \hat{T}^D(Fg)$ *for all* $F \in GL_+(d)$ *and* $g \in G$*. In particular, we call the material isotropic, if it has the point group* $SO(d)$*.*

Examples.

G	material
$SO(3)$	isotropic solid
discrete subgroup of $SO(3)$	crystalline solid

Remark. Strictly speaking, a point group is a subgroup of $(O(d), \cdot)$ - the group of all isometries with matrix multiplication as the group action. More generally, we consider groups $G \subset GL_+(d)$ that leave the origin invariant. For example the set

$$\{\, F = \text{simple stretch} \,\} = \{\, \mathbf{Id} + (\lambda - 1)e \otimes e \;:\; \lambda > 0,\, e \in S^{d-1} \,\}$$

is not a group, since it is not closed under multiplication. On the other hand for $e \in S^{d-1}$ fixed, the set

$$\{\, F = \text{simple stretch} \,\} = \{\, \mathbf{Id} + (\lambda - 1)e \otimes e \;:\; \lambda > 0 \,\}$$

is an admissible group in the sense above.
In the following we study isotropic materials, i.e. $G = SO(d)$. We recall some definitions from linear algebra. Let $A \in \mathbb{R}^{d\times d}$. The polynomial

$$\chi(\omega) := \det(A - \omega\mathbf{Id}) = (-\omega)^d + \sum_{k=1}^{d} (-\omega)^{d-k} I_k(A)$$

is called the *characteristic polynomial* of A, the coefficients $I_k(A)$, $k = 1, \ldots, d$, are called the *principal invariants* of A. For convenience we introduce the notation $\mathcal{I}(A) := (I_1(A), \ldots, I_d(A))$.

Lemma 2.52. *The mapping $\mathbb{R}^{d\times d} \ni A \mapsto \mathcal{I}(A)$ is invariant in the sense that for all $A \in \mathbb{R}^{d\times d}$ we have*

$$\mathcal{I}(F^{-1}AF) = \mathcal{I}(A) \qquad \text{for all invertible } F \in \mathbb{R}^{d\times d}. \tag{2.17}$$

Corollary 2.53. *Let $\beta_0, \ldots, \beta_{d-1}$ denote real-valued functions defined on $\mathcal{I}(\mathbb{R}^{d\times d}_{sym}) \subset \mathbb{R}^d$. Set*

$$\bar{T} : \mathbb{R}^{d\times d}_{sym,+} \to \mathbb{R}^{d\times d}_{sym}, \qquad \bar{T}(B) := \sum_{k=0}^{d-1} \beta_k(\mathcal{I}(B))B^k. \tag{2.18}$$

Then

$$\hat{T}^D(F) := \bar{T}(FF^t)$$

defines a frame-indifferent and isotropic response function for the Cauchy stress.

Beweis. For frame indifference we need to check

$$\hat{T}^D(RF) = R\hat{T}^D(F)R^t$$

for all $R \in SO(d)$ and $F \in GL_+(d)$. Indeed,

$$\begin{aligned}
\hat{T}^D(RF) &= \bar{T}(RFF^tR^t) = \sum_{k=0}^{d-1} \beta_k(\underbrace{\mathcal{I}(RFF^tR^t)}_{=\mathcal{I}(FF^t)})\ \underbrace{(RFF^tR^t)^k}_{=R(FF^t)^kR^t} \\
&= R\left(\sum_{k=0}^{d-1} \beta_k(\mathcal{I}(FF^t))(FF^t)^k\right)R^t \\
&= R\hat{T}^D(F)R^t.
\end{aligned}$$

Regarding isotropy we note that

$$\hat{T}^D(FR) = \bar{T}(FR(FR)^t) = \bar{T}(FF^t) = \hat{T}^D(F)$$

for all $R \in SO(d)$ and $F \in GL_+(d)$. □

The representation theorem of Rivlin-Ericksen (1955) shows that the inverse is also true, i.e. any isotropic, frame-indifferent response function admits a representation of the form (2.18).

Theorem 2.54 (Rivlin-Ericksen Representation Theorem)**.** *Let $d \geq 3$ and $\hat{T}^D : GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ a map. Then the following properties are equivalent*

(i) $\hat{T}^D$ is isotropic (i.e. $\hat{T}^D$ has the point group $SO(d)$) and satisfies frame indifference.

(ii) There exist functions $\beta_0, \dots, \beta_{d-1} : \mathbb{R}^d \to \mathbb{R}$ such that $\hat{T}^D(F) = \bar{T}(FF^t)$ with

$$\bar{T} : \mathbb{R}^{d\times d}_{sym,+} \to \mathbb{R}^{d\times d}_{sym}, \qquad \bar{T}(B) := \sum_{k=0}^{d-1} \beta_k(\mathcal{I}(B))B^k.$$

In order to prove this theorem we need some auxiliary results from linear algebra. For $A \in \mathbb{R}^{d\times d}$ symmetric we denote by $\mathcal{J}(A) = (\lambda_1, \dots, \lambda_d)$ the list of ordered eigen values of A, i.e. there exists $Q = (q_1, \dots, q_d) \in SO(d)$ with $Aq_i = \lambda_i q_i$ for $i = 1, \dots, d$ and $\lambda_1 \leq \dots \leq \lambda_d$.

Lemma 2.55. *Set $\mathcal{I}(\mathbb{R}^{d\times d}_{sym}) := \{\, \mathcal{I}(A) \,:\, A \in \mathbb{R}^{d\times d}_{sym} \,\}$ and $\mathcal{J}(\mathbb{R}^{d\times d}_{sym}) := \{\, \mathcal{J}(A) \,:\, A \in \mathbb{R}^{d\times d}_{sym} \,\}$. There exists a bijection $\iota : \mathcal{J}(\mathbb{R}^{d\times d}_{sym}) \to \mathcal{I}(\mathbb{R}^{d\times d}_{sym})$ such that*

$$\mathcal{I}(A) = \iota(\mathcal{J}(A)) \qquad \textit{for all } A \in \mathbb{R}^{d\times d}_{sym}.$$

Beweis. Thanks to Corollary 2.53 we only need to prove (i) $\Rightarrow$ (ii). For conveniene, we will assume in the last step that $\hat{T}^D$ is additionally continuous. This assumption is not necessary (see Step 5).

Step 1. For $B \in \mathbb{R}^{d\times d}_{sym,+}$ set $\bar{T}(B) := \hat{T}^D(\sqrt{B})$. We claim that for all $B \in \mathbb{R}^{d\times d}_{sym,+}$ and $R \in SO(d)$ we have

$$\hat{T}^D(F) = \bar{T}(FF^t) \tag{2.19}$$

$$\bar{T}(RBR^t) = R\bar{T}(B)R^t. \tag{2.20}$$

Argument for (2.19): By the polar factorization and isotropy we have

$$\hat{T}^D(F) = \hat{T}^D(\sqrt{FF^t}R) = \hat{T}^D(\sqrt{FF^t}) = \bar{T}(FF^t).$$

Argument for (2.19): Thanks to frame indifference we have

$$\bar{T}(RBR^t) = \bar{T}(RB^{\frac{1}{2}}(RB^{\frac{1}{2}})^t) = \hat{T}^D(RB^{\frac{1}{2}}) = R\hat{T}^D(B^{\frac{1}{2}})R^t = R\bar{T}(B)R^t.$$

Step 2. Fix $B \in \mathbb{R}^{d\times d}_{sym,+}$. Claim: If $Q \in SO(d)$ diagonalizes B, then Q diagonalizes $\bar{T}(B)$. Argument: Let $Q = (q_1, \dots, q_d) \in SO(d)$ and $\lambda_1 \leq \dots \leq \lambda_d$ satisfy $Bq_i = \lambda_i q_i$ for $i = 1, \dots, d$. We only need to show that

$$q_i \cdot \bar{T}(B)q_j = 0 \qquad \text{for all } i \neq j. \tag{2.21}$$

To that end fix three distinct indices i, j, k and consider the projection

$$P := \sum_{\ell=1}^{d} \sigma_\ell(q_\ell \otimes q_\ell), \qquad \sigma_\ell := \begin{cases} -1 & \ell = j \text{ or } \ell = k \\ 1 & \text{else.} \end{cases}$$

By construction we have $P \in SO(d)$ and $P^t = P$. We claim that

$$P^t B P = B. \tag{2.22}$$

Indeed:

$$P^t B P q_\ell = \sigma_\ell P^t B q_\ell \sigma_\ell \lambda_\ell P q_\ell = \sigma_\ell^2 \lambda_\ell q_\ell = B q_\ell.$$

Hence,

$$P^t \bar{T}(B) P \overset{(2.20)}{=} \bar{T}(P^t B P) = \bar{T}(B),$$

and thus

$$q_i \cdot \bar{T}(B) q_j = q_i \cdot (P^t \bar{T}(B) P q_j) = (P q_i) \cdot (\bar{T}(B) P q_j) = -q_i \cdot \bar{T}(B) q_j.$$

Hence, (2.21) follows.

Step 3. Claim: Fix $\lambda_1, \ldots, \lambda_d > 0$. By Step 2 there exist $\mu_1, \ldots, \mu_d \in \mathbb{R}$ such that

$$\bar{T}(\operatorname{diag}(\lambda_1, \ldots, \lambda_d)) = \operatorname{diag}(\mu_1, \ldots, \mu_d).$$

We claim that for any permutation $\pi : \{1, \ldots, d\} \to \{1, \ldots, d\}$ we have

$$\bar{T}(\operatorname{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(d)})) = \operatorname{diag}(\mu_{\pi(1)}, \ldots, \mu_{\pi(d)}).$$

The claim follows from (2.20) and the following

Auxiliary lemma: For $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ diagonal matrix and $\pi : \{1, \ldots, d\} \to \{1, \ldots, d\}$ permutation, we set $\pi(D) := \operatorname{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(d)})$. Then for any permutation π there exists $Q \in SO(d)$ such that

$$\pi(D) = Q^t D Q$$

for all diagonal matrices D. *Argument:* For $\sigma_i \in \{\pm 1\}$ $(i = 1, \ldots, d)$ consider

$$Q = \sum_{i=1}^{d} \sigma_i (e_{\pi(i)} \otimes e_i).$$

Then $Q \in O(d)$, since (by orthogonality) we have

$$Q^t Q = \sum_{i,j} \sigma_i \sigma_j (e_{\pi(i)} \otimes e_i)^t (e_{\pi(j)} \otimes e_j) = \sum_{i,j} \sigma_i \sigma_j (e_{\pi(i)} \cdot e_{\pi(j)})(e_i \otimes e_j) = \sum_i \sigma_i^2 (e_i \otimes e_i) = \mathbf{Id}.$$

Moreover,

$$DQ = \sum_{i,j} \lambda_i \sigma_j (e_i \otimes e_i)(e_{\pi(j)} \otimes e_j) = \sum_j \lambda_{\pi(j)} \sigma_j e_{\pi(j)} \otimes e_j,$$

and thus

$$Q^t D Q = \sum_{i,j} \lambda_{\pi(j)} \sigma_j \sigma_i (e_i \otimes e_{\pi(i)})(e_{\pi(j)} \otimes e_j) = \sum_i \lambda_{\pi(i)} (e_i \otimes e_i) = \pi(D).$$

Since switching the sign of σ_1, means to swicht the sign of $\det Q$, we additionally have $Q \in SO(d)$ for a suitabel choice of the σ_i's.

Step 4. Set

$$\begin{aligned} \mathcal{D} &:= \{\, D \in \mathbb{R}^{d \times d} \,:\, D \text{ is diagonal and has } d \text{ distinct, positive eigenvalues} \,\}, \\ \mathcal{B} &:= \{\, B \in \mathbb{R}^{d \times d} \,:\, B \text{ has } d \text{ distinct positive eigenvalues} \,\} \end{aligned}$$

- Let $D \in \mathcal{D}$. By definition, the diagonal entries of D are distinct, and thus

$$\operatorname{span}\{D^0, D^1, \dots, D^{d-1}\} = \{\, D \in \mathbb{R}^{d\times d} \,:\, D \text{ is diagonal.}\,\}.$$

  Indeed, by appealing to the Vandermonde determinant one can show that the d-vectors formed by the diagonals of D^k, $k = 0, \dots, d-1$ are linearly independent.

- Since $\bar{T}(D)$ is diagonal by Step 2, we conclude that there exist coefficients $c_k(D)$, $k = 0, \dots, d$ such that

$$\bar{T}(D) = \sum_{k=0}^{d-1} c_k(D) D^k.$$

- From Step 3 we deduce that $c_k(D) = c_k(\pi(D))$ for any permutation π. Indeed: If $Q \in SO(d)$ denotes the rotation associated with π according to the auxiliary lemma, then by definition we have

$$\bar{T}(\pi(D)) = \sum_k c_k(\pi(D))\pi(D)^k = Q^t(\sum_k c_k(\pi(D))D^k)Q,$$

  and by (2.20) we have

$$\bar{T}(\pi(D)) = \bar{T}(Q^t D Q) = Q^t \bar{T}(D) Q.$$

  Hence,

$$\sum_k c_k(\pi(D))D^k = \bar{T}(D) = \sum_k c_k(D) D^k,$$

  and because $D^0, \dots, D^{d-1}$ are linearly independent, we deduce that $c_k(\pi(D)) = c_k(D)$.

- Since the diagonal entries of D are the eigenvalues of D, we can find a permutation such that $\pi(D) = \operatorname{diag}(\mathcal{J}_1(D), \dots, \mathcal{J}_d(D))$. Hence, $c_k(D)$ can be written as a function of $\mathcal{J}(D)$; and thanks to Lemma 2.55 there exists a function β_k with $c_k(D) = \beta_k(\mathcal{I}(D))$. Hence,

We claim that for any $B \in \mathcal{B}$ we have

$$\bar{T}(B) = \sum_{k=0}^{d-1} \beta_k(\mathcal{I}(B)) B^k. \tag{2.23}$$

We already know that (2.23) is valid in the diagonal case. Thanks to the spectral theorem there exists $R \in SO(d)$ such that $B = RDR^t$ with $D \in \mathcal{D}$. Hence,

$$\begin{aligned} \bar{T}(B) &= \bar{T}(RDR^t) = R\bar{T}(D)R^t \\ &= \sum_{k=0}^{d-1} \beta_k(\underbrace{\mathcal{I}(D)}_{=\mathcal{I}(RDR^t)}) \underbrace{RD^kR^t}_{=(RDR^t)^k} . \end{aligned}$$

Step 5. To finish the proof we need to show (2.23) for arbitrary positive definite symmetric matrices. Note that $\mathcal{B}$ is dense in $\mathbb{R}^{d\times d}_{sym,+}$. Hence, if $\hat{T}^D$ is continuous, (2.23) easily follows from a continuity argument. If $\hat{T}^D$ is not continuous, one conduct a case study where the different cases of eigenvalues with multiplicity larger 1 are treated seperately (e.g. see Proof of Theorem 3.6-1 in the "Ciarlet. Mathematical Elasticity Volume I"). □

Corollary 2.56. *Let $d \geq 3$. An elastic material is isotropic and frame indifferent, if and only if there exist functions $\gamma_0, \ldots, \gamma_{d-1} : \mathbb{R}^d \to \mathbb{R}$ such that*

$$\hat{\Sigma}(F) = \sum_{k=0}^{d-1} \gamma_k(\mathcal{I}(C))C^k$$

where $\hat{\Sigma}$ denotes the response function for the second Piola-Kirchhoff stress tensor and $C = F^tF$ the right Cauchy-Green strain tensor.

For the proof we recall the Cayley-Hamilton theorem, which states that every matrix A is the root of it's characteristic polynomial, i.e. $\chi_A(A) = 0$ or equivalently

$$(-A)^d + \sigma_{k=1}^d I_k(A)(-A)^{d-k}.$$

In particular, if A is invertible, then

$$A^{-1} = \frac{1}{I_d(A)}\left((-A)^{d-1} + I_1(A)(-A)^{d-2} + \ldots + I_{d-1}(A)\right),$$

i.e. A^{-1} can be expressed as a polynomial of $A^0, \ldots, A^{d-1}$ with coefficients that are functions of $\mathcal{I}(A)$.

Proof of Corollary 2.56. By the Piola-transform and the definition of Σ we have

$$\begin{aligned}\hat{\Sigma}(F) &= (\det F)F^{-1}\hat{T}^D(F)F^{-t} \\ &= (\det F)F^{-1}\left(\sum_{k=0}^{d-1} \beta_k(\mathcal{I}(FF^t))(FF^t)^k\right)F^{-t}.\end{aligned}$$

We have $\det F = (\det C)^{\frac{1}{2}}$, $I_d(C) = \chi_C(0) = \det C$, $\mathcal{I}(FF^t) = \mathcal{I}(F^tF) = \mathcal{I}(C)$, and $F-1(FF^t)^kF^{-t} = C^{-1}C^k$. Hence

$$\begin{aligned}\hat{\Sigma}(F) &= I_d(C)^{\frac{1}{2}}C^{-1}\sum_{k=0}^{d-1} \beta_k(\mathcal{I}(C))C^k \\ &= I_d(C)^{\frac{1}{2}}C^{-1}\beta_0(\mathcal{I}(C)) + \sum_{k=0}^{d-2} I_d(C)^{\frac{1}{2}}\beta_{k+1}(\mathcal{I}(C))C^k.\end{aligned}$$

This proves the statement, since $I_d(C)^{\frac{1}{2}}C^{-1}$ can be written as a polyomial of $C^0, \ldots, C^{d-1}$ with coefficients being functions of $\mathcal{I}(C)$. $\square$

2.3.3 The stress-strain relation close to the reference configuration.

Consider the *Green-St. Venant strain tensor*

$$E := \frac{1}{2}(C - \mathbf{Id}) = \frac{1}{2}(F^tF - \mathbf{Id}) \qquad (F = D\varphi)$$

associated with the deformation φ. We are interested in the stress-strain relation of an isotropic, frame-indifferent material, when E is small. For convenience we restrict to the case $d = 3$. As shown in an exercise the principal invariants of $C \in \mathbb{R}^{d\times d}_{sym}$ can be expressed with help of C's eigenvalues $(\lambda_1, \ldots, \lambda_3) = \mathcal{J}(C)$:

$$I_1(C) = \lambda_1 + \lambda_2 + \lambda_3, \qquad I_2(C) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \qquad I_3(C) = \lambda_1\lambda_2\lambda_3. \tag{2.24}$$

Lemma 2.57. *Let $d = 3$. The principal invariants satisfy the expansion*

$$\begin{aligned} I_1(\mathbf{Id} + 2E) &= 3 + 2\,\mathrm{trace}(E) \\ I_2(\mathbf{Id} + 2E) &= 3 + 4\,\mathrm{trace}(E) + o(E) \\ I_3(\mathbf{Id} + 2E) &= 1 + 2\,\mathrm{trace}(E) + o(E) \end{aligned} \tag{2.25}$$

for all $E \in \mathbb{R}^{d\times d}_{sym}$.

Beweis. By the spectral theorem there exists $Q \in O(d)$ such that $Q^t E Q = \Lambda$ where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a diagonal matrix. Since

$$\mathcal{I}(\mathbf{Id} + 2E) = \mathcal{I}(Q^t(\mathbf{Id} + 2\Lambda)Q) = \mathcal{I}(\mathbf{Id} + 2\Lambda),$$

and

$$\mathrm{trace}(E) = \mathrm{trace}(Q\Lambda Q^t) = \mathrm{trace}(\Lambda),$$

it suffices to check (2.25) for E replaced by Λ. Since the diagonal entries of $\mathbf{Id} + 2\Lambda$ are precisely the entries of $\mathcal{J}(\mathbf{Id} + 2\Lambda)$, (2.25) follows from (2.24) by a straight forward calculation. □

Theorem 2.58. *Let $d = 3$. Let $\hat{\Sigma}$ denote the response function of an isotropic and frame indifferent elastic material, i.e. there exist $\gamma_0, \ldots, \gamma_2$ such that*

$$\hat{\Sigma}(F) = \sum_{k=0}^{2} \gamma_k(\mathcal{I}(C))C^k \qquad (\textit{with } C = F^t F).$$

Suppose that the coefficients $\gamma_0, \ldots, \gamma_2$ are differentiable at $(3, 3, 1)$. Then there exists scalars p, λ and μ such that

$$\hat{\Sigma}(F) = -p\mathbf{Id} + \lambda(\mathrm{trace}\, E)\mathbf{Id} + 2\mu E + o(E) \tag{2.26}$$

for all $F \in GL_+(d)$ and $E = \frac{1}{2}(F^t F - \mathbf{Id})$.

Beweis. From (2.25) and the differentiability of λ_k we learn that

$$\gamma_k(\mathcal{I}(\mathbf{Id} + 2E))) = \underbrace{\gamma_k(3, 3, 1)}_{=:\gamma_k^0} + \underbrace{(\nabla\gamma_k(3, 3, 1) \cdot (2, 4, 2))}_{=:\gamma_k^1}\mathrm{trace}(E) + o(E).$$

Hence, (since $E^2 = o(E)$ and $\mathrm{trace}(E)E = o(E)$)

$$\begin{aligned} \hat{\Sigma}(F) &= \sum_{k=0}^{2} \gamma_k(\mathcal{I}(\mathbf{Id} + 2E))(\mathbf{Id} + 2E)^k \\ &= (\gamma_0^0 + \gamma_0^1\,\mathrm{trace}(E))\mathbf{Id} \\ &\quad + \gamma_1^0(\mathbf{Id} + 2E) + \gamma_1^1\,\mathrm{trace}(E)\mathbf{Id} \\ &\quad + \gamma_2^0(\mathbf{Id} + 4E) + \gamma_2^1\,\mathrm{trace}(E)\mathbf{Id} \ + \ o(E). \end{aligned}$$

Obviously, the RHS can be brought into the form of the statement. □

Definition 2.59. *Consider an elastic material with response function $\hat{T}^D$. A reference configuration is called a* natural state, *if it is stress free, i.e.* $\hat{T}^D(\mathbf{Id}) = 0$.

Remark 2.60. *In Theorem 2.58 the material's reference configuration is a natural state, if and only if* $p = 0$.

Definition 2.61 (Lamé constants)**.** *Let* $d = 3$. *Consider an isotropic, frame-indifferent elastic material whose reference configuration is a natural state, i.e.* (2.26) *holds with* $p = 0$. *Then* λ *and* μ *are called the material's* Lamé constants.

Remark 2.62. *The elastic behavior of elastic materials (in particular isotropic materials) is often described in terms of elastic moduli, which are quantities that capture ratios of the form* $\frac{\textit{stress}}{\textit{strain}}$ *up to first order. E.g.:*

- *Young's modulus*[4] $E = \frac{\textit{tensile stress}}{\textit{extensional stretch}}$
- *Shear modulus*[5] $= G = \frac{\textit{shear stress}}{\textit{shear strain}}$
- *Bulk modulus*[6] $= K = \frac{\textit{pressure}}{\textit{decrease of volume}}$

These moduli can be expressed in terms of the material's Lamé constants (and vice versa); see exercise.

2.3.4 Hyperelastic materials

Definition 2.63. *An elastic material is called* hyperelastic, *if* $\hat{T}(F)$*—the response function for the 1st Piolo-Kirchhoff-stress tensor has a potential, i.e. if there exists a function* $W \in C^1(GL_+(d); \mathbb{R})$ *such that*

$$\hat{T}(F) = \nabla W(F).$$

The function W *is called stored energy density.*

Examples 2.64. *(i) St. Venant-Kirchhoff material. The map*

$$W(F) := \frac{\lambda}{2}(\operatorname{trace} E)^2 + \mu \operatorname{trace}(E^2), \qquad E := \frac{F^t F - \mathbf{Id}}{2}.$$

defines a hyperelastic energy density. Exercise: Check that the material obeys frame-indifference, has a stress-free reference state, is isotropic and has Lamé constants λ *and* μ.

(ii) The map

$$W(F) := \frac{1}{2} \operatorname{trace}(F^t F) + \frac{1}{\det F}$$

defines a hyperelastic, isotropic and frame-indifferent material Exercise: Show that W *is minimized at* $SO(d)$.

[4] deutsch: E-Modul, Elastizitätsmodul, Youngscher Modul
[5] deutsch: Schubmodul
[6] deutsch: Kompressionsmodul

(iii) *Example (ii) is a simple example of an* Ogden material*:*

$$W(F) = \sum_{i=1}^{M} a_i \operatorname{trace}(F^t F)^{\gamma_i/2} + \sum_{j=1}^{N} \operatorname{trace}(\operatorname{cof}(F^t F))^{\delta_j/2} + g(\det F)$$

with $a_i > 0$, $\gamma_i \geq 1$, $\delta_i \geq 1$ *and* g *is a smooth convex function with the property* $\lim_{\epsilon \downarrow 0} g(\epsilon) = \infty$.

3 Calculus of Variations and nonlinear elasticity

To motivate the content of this chapter consider a hyperelastic material with

- reference configuration $\Omega \subset \mathbb{R}^d$ (smooth bounded domain),
- stored energy function W,
- a volume force (assumed to be a dead load) $f : \Omega \to \mathbb{R}^d$,
- a surface force (assumed to be a dead load) $g : \partial\Omega \setminus \Gamma \to \mathbb{R}^d$ where $\Gamma \subset \partial\Omega$ is relatively open.

Given a map $\psi : \Gamma \to \mathbb{R}^d$, we seek a deformation φ that solves the boundary value problem

$$\begin{cases} -\operatorname{div}(\hat{T}(D\varphi) = f & \text{in } \Omega, \\ \varphi = \psi & \text{on } \Gamma, \\ \hat{T}(D\varphi)\nu = g & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{3.1}$$

It describes the equilibrium shape of an elastic body that is exposed to a dead load f, a surface force g and has a part of the boundary Γ that is deformed according to the map ψ.

Problem (3.1) is nonlinear, since

- $F \mapsto \hat{T}(F)$ is (typically) nonlinear and
- the space of deformations is non-convex.

In contrast to linear problems, we therefore cannot expect properties such as uniqueness, continuous dependence on the given data, or the existence of smooth solutions. In this chapter we aim to prove existence for (3.1) by means of the *direct methods of the Calculus of Variations*. It is based on the idea to replace (3.1) by a suitable minimization problem and to recover (3.1) as the first order optimality condition (the so called Euler-Lagrange equation). We associate with (3.1) the *total energy functional*,

$$I(\varphi) := \int_\Omega W(D\varphi)\,\mathrm{d}x - \int_\Omega f \cdot \varphi\,\mathrm{d}x - \int_{\partial\Omega\setminus\Gamma} g \cdot \varphi\,\mathrm{d}x \tag{3.2}$$

The Dirichlet boundary condition $\varphi = \psi$ on Γ is encoded in the space of admissible deformations

$$\mathcal{A} = \{\, \varphi \in C^2(\overline{\Omega}, \mathbb{R}^d) \,:\, \det \nabla\varphi > 0 \text{ in } \overline{\Omega} \text{ and } \varphi = \psi \text{ on } \Gamma \,\}.$$

(Note that we relaxed the notion of "deformation": Every $\varphi \in \mathcal{A}$ is locally invertible, but possibly not globally.) We assume $\mathcal{A}$ to be non-empty.

Observation. Suppose $\varphi \in \mathcal{A}$ minimizes I, i.e.

$$I(\varphi) \leq I(\tilde{\varphi}) \qquad \text{for all } \tilde{\varphi} \in \mathcal{A}.$$

Then *formally* φ solves (3.1).

Argument. We say θ is an admissible test-function, if $\theta \in C^2(\overline{\Omega}, \mathbb{R}^d)$ and $\theta = 0$ on $\partial\Omega \setminus \Gamma$. Note that for any test-function and $s > 0$ we have $\varphi + s\theta \in \mathcal{A}$, provided s is sufficiently small.
Consider the function $h(s) := I(\varphi + s\theta)$. Then we formally have

$$h'(0) = \int_\Omega \nabla W(D\varphi) \cdot D\theta \, dx - \int_\Omega f \cdot \theta \, \mathrm{d}x - \int_{\partial\Omega} g \cdot \theta \, \mathrm{d}\mathcal{H}^{d-1}.$$

(For a rigorous argument we would to check that we are allowed to interchange differentiation and integration, which requires additional assumptions on W). On the other hand, by minimality of φ we have

$$h'(0) = \lim_{s \downarrow 0} \frac{1}{s}(I(\varphi + s\theta) - I(\varphi)) \geq 0$$

Since we may replace θ by $-\theta$, we deduce that

$$h'(0) = 0.$$

Since (formally)

$$\begin{aligned} \int \nabla W(D\varphi) \cdot D\theta &= \int \hat{T}(D\varphi) \cdot D\theta = \int \operatorname{div}(\hat{T}(D\varphi)^t \theta) - \operatorname{div} \hat{T}(D\varphi) \cdot \theta \\ &= \int_{\partial\Omega} \hat{T}(D\varphi)\nu \cdot \theta - \int_\Omega \operatorname{div} \hat{T}(D\varphi) \cdot \theta, \end{aligned}$$

we infer

$$0 = h'(0) = \int_{\partial\Omega} (\hat{T}(D\varphi)\nu - g) \cdot \theta - \int_\Omega (\operatorname{div} \hat{T}(D\varphi) + f) \cdot \theta.$$

Since this is true for all admissible test functions, we formally get (3.1). □

The goal of this chapter is to prove existence of minimizers for I following the direct method and J.Ball's concept of polyconvexity. As a corollary we shall get existence of a weak solution to (3.1). The chapter's programme is as follows:

- Direct method and lower semicontinuity
- Existence of minimizers for integral functionals

  $$\varphi \mapsto \int_\Omega f(D\varphi) \, \mathrm{d}x$$

  with convex integrands f. Application to linearized elasticity.
- Existence of minimizers for polyconvex materials

3.1 The direct method of the Calculus of Variations

Throughout this section X denotes a metric space with metric d_X.

Definition 3.1. *A function $I : X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semicontinuous in (X, d_X) at a point $x \in X$ if for every sequence (x_k) converging to x in X we have*

$$I(x) \le \liminf_{k \to \infty} I(x_k).$$

We say I is lower semicontinuous on (X, d_X) if it is lower semicontinuous at every point $x \in X$.

Example 3.2.
- *Consider $X = \mathbb{R}$ with $d_X(x, y) := |x - y|$. Let $f_1 \in C(\mathbb{R})$, $a \in \mathbb{R}$. Then*
$$f(x) := \begin{cases} f_1(x) & x \neq 0 \\ a & x = 0 \end{cases}$$
is lsc, if and only if $a \le f_1(0)$.
- *Consider a (possibly uncountable) family $\mathcal{F}$ of l.s.c. functions on (X, d_X). Then the function*
$$f_+(x) := \sup_{f \in \mathcal{F}} f(x), \qquad x \in X$$
is l.s.c.

Lemma 3.3 (Exercise 3.3). *Let $\Omega \subset \mathbb{R}^d$ open and bounded, and $1 \le p < \infty$. Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be Borel-measurable and suppose that there exists $C > 0$ s.t.*

$$g(F) \ge -C(1 + |F|^p) \qquad \text{for all } F \in \mathbb{R}^m.$$

Consider the functional $I : L^p(\Omega, \mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$,

$$I(u) := \int_\Omega g(u(x))\,\mathrm{d}x.$$

Then I is lsc on $L^p(\Omega, \mathbb{R}^m)$ (w.r.t. strong convergence in $L^p(\Omega, \mathbb{R}^m)$), if and only if g is l.s.c. on $\mathbb{R}^m$.

Remark 3.4.
- *It is easy to check that a function I defined on a metric space (X, d_X) is l.s.c. if and only if*
$$\text{for every } t \in \mathbb{R} \text{ the sublevels } \{x \in X \,:\, I(x) \le t\} \text{ are closed.} \tag{3.3}$$
- *The notion of lower semicontinuity can extended to general topological spaces: A function is called lsc if it satisfies* (3.3)*, while a function that satisfies Definition 3.1 is called sequentially lower semicontinuous.*

Theorem 3.5 (Weierstrass)**.** *Let $K \subset X$ be compact, and $I : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Then there exists $x_0 \in K$ such that*

$$I(x_0) = \min_{x \in K} I(x).$$

Beweis. W.l.o.g. we may assume that I is not identically $+\infty$ on K, so that $\inf_K I < +\infty$. Let (x_k) denote a sequence in K such that

$$\lim_{k\to\infty} I(u_k) = \inf_K I.$$

(Convention, for a sequence (a_k) in $\mathbb{R}$ we write $\lim\limits_{k\to\infty} a_k = -\infty$ if for all $a \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that $a_k < a$ for all $k > K$.) Since K is compact there exists a subsequence (x_{k_ℓ}) that converges to some $x_0 \in K$, and so

$$\inf_K I \leq I(x_0) \leq \liminf_{\ell\to\infty} I(x_{k_\ell}) = \lim_{k\to\infty} I(x_k) = \inf_K I.$$

Hence, x_0 is the desired minimizer. □

Tonelli's direct method. Let V be a set and let $I : V \to (-\infty, \infty]$ be not identically equal to $+\infty$. The direct method provides conditions on V and I that ensure the existence of a minimizer for I. The procedure consists of three steps: **Step 1.** Consider a minimizing sequence (x_k) in V, that is, a sequence s.t.

$$\lim_{k\to\infty} I(x_k) = \inf_{x\in V} I(x).$$

Step 2. Find a metric (topology) on V such that

(a) (x_k) admits a convergent subsequence (x_{k_ℓ}) with limit x_0,

and

(b) I is (sequentially) lower semicontinuous at x_0.

Step 3. Conclude that x_0 is a minimizer of I, because

$$\inf I \leq I(x_0) \leq \liminf_{\ell\to\infty} I(x_{k_\ell}) = \lim_{k\to\infty} I(x_k) = \inf_V I.$$

Remark 3.6.

- *Note that properties (a) and (b) are competing: The weaker the topology, the easier to find convergent subsequences(i.e. compactness), the harder to prove lsc.*
- *Typical situation: V is a reflexive and separable Banach Space, and I admits a minimizing sequence that is contained in a closed and bounded set, say K. Then, for minimization we may restrict I to K. Since K is bounded and closed, it is compact w.r.t. weak convergence in V. Hence, the natural topology to choose is the weak-topology, and we are left to check that I is lsc w.r.t. weak convergence in K. Note that on K the topology of weak-convergence is metrizable!*
  In the next section we will discuss this general setting in the specific situations, when
  $$V = L^p(\Omega, \mathbb{R}^m), \qquad I(u) := \int_\Omega g(u(x))\,\mathrm{d}x, \qquad 1 < p < \infty$$

and

$$V = W^{1,p}(\Omega, \mathbb{R}^m), \qquad I(u) := \int_\Omega g(Du(x))\,\mathrm{d}x.$$

We will discuss properties (in particular convexity conditions) of g that ensure lower semicontinuity of I (w.r.t. weak convergence).

In the remainder of this paragraph we recall some basic notions from functional analysis.

3.2 Analytical Toolbox II: Weak convergence in Sobolev spaces

In this section X denotes a (real) Banach space. We recall some important facts.

- We denote by X^* its dual space, i.e. X^* is the linear space of $\mathbb{R}$-linear, bounded functionals $L : X \to \mathbb{R}$.
- X^* equipped with the operatornorm $\|L\| := \sup_{\substack{x\in X^* \\ \|x\|=1}} \frac{|\langle L,x\rangle|}{\|x\|}$ is a Banach space.
- If X is a (real) Hilbert space, say $X = H$, then the map

$$j : H \to H^*, \qquad \langle j(x), y\rangle := (x, y)$$

  is an isometric isomorphism. (Riesz' Representation Theorem)

Definition 3.7. *Let X be a Banach space with dual X^*. We say a sequence $(x_j) \subset X$ weakly converges in X to a limit $x \in X$ (notation: $x_j \rightharpoonup x$ weakly in X), if*

$$\forall L \in X^* : \qquad < L, x_j > \to < L, x > .$$

We say a sequence $(L_j) \subset X^*$ weakly-$*$ converges in X^* to a limit $L \in X^*$ (notation: $L_j \overset{*}{\rightharpoonup} L$ weakly-$*$ in X^*), if

$$\forall x \in X : \qquad < L_j, x > \to < L, x > .$$

We recall some important facts.

- **(Weak convergence $\Rightarrow$ boundedness)**. Consider $x_j \rightharpoonup x$ weakly in X (resp. $L_j \overset{*}{\rightharpoonup} L$ weak-* in X^*), then (x_k) (resp. (L)) is a bounded sequence.
- **(Lower semicontinuity of the norm w.r.t. Weak convergence)**. Consider $x_j \rightharpoonup x$ weakly in X, then

$$\|x\| \le \liminf_{j\to\infty} \|x_j\|.$$

  and the same holds for weak-* convergence.
- **(Weak convergence of bounded sequences.)** Suppose (x_k) is a bounded sequence in X. Let $\mathcal{D} \subset X^*$ denote a dense subset. Then $x_k \rightharpoonup x$, if and only if $< L, x_k > \to < L, x >$ for all $L \in \mathcal{D}$.
- **(Weak-* compactness — Theorem of Banach-Alaoglu)**. The unit disc $\{L \in X^* : \|L\|_* \le 1\}$ is compact in the weak-$*$ topology.
- **(Equivalence of weak and weak-* convergence in reflexive spaces).** Let X be a reflexive Banach space. Then

$$x_j \rightharpoonup x \qquad \Leftrightarrow \qquad x_j \overset{*}{\rightharpoonup} x.$$

- **(Metrizability of weak convergence).** Let be X^* be separable, and $B \subset X$ a bounded set. Then B endowed with the weak topology is metrizable.

- **(Equivalence of closures of convex sets).** Let $K \subset X$ be convex. Then K is closed in the strong topology if and only if K is closed in the weak topology.
- **(Strong convergence $\Rightarrow$ weak (weak-$*$) convergence).**
- **(Products of strongly $\times$ weakly convergent sequences converge).** Let $x_j \rightharpoonup x$ weakly in X and $L_j \to L$ strongly in X^*, then $\langle L_j, x_j\rangle \to \langle L, x\rangle$. Let $x_j \to x$ strongly in X and $L_j \overset{*}{\rightharpoonup} L$ weakly-* in X^*, then $\langle L_j, x_j\rangle \to \langle L, x\rangle$.

Exercise 3.8. *Prove that products of strongly $\times$ weakly convergent sequences converge.*

Examples 3.9. *(a) Let X be a finite dimensional vector space, say $X \equiv \mathbb{R}^d$. Then $X^* \equiv \mathbb{R}^d$, and weak-* convergence = weak convergence = strong convergence.*

(b) $X = L^p(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open, $1 \le p < \infty$. Then $X^* \equiv L^q(\Omega)$ with $q = \frac{p}{p-1}$. More precisely, for any $L \in (L^p(\Omega))^*$ there exists a unique $\varphi \in L^q(\Omega)$, s.t.

$$< L, u >= \int_\Omega u(x)\varphi(x)\,dx, \qquad \text{for all } u \in L^p(\Omega).$$

(Recall that elements in $L^p(\Omega)$ are strictly speaking equivalence classes of functions that coincide almost everywhere).

Exercise 3.10. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^d$ open, and $(u_k) \subset L^p(\Omega)$ be bounded. Show that TFAE:*

- $u_k \rightharpoonup u$ *weakly* $L^p(\Omega)$,
- $\int_Q u_k \to \int_Q u$ *for all cubes* $Q \subset \Omega$,
- $\int_\Omega u_k\varphi \to \int_\Omega u\varphi$ *for all* $\varphi \in C_c^\infty(\Omega)$.

Exercise 3.11. *Let $1 \le p \le \infty$ and $u \in L^p_{\text{loc}}(\mathbb{R}^d)$. Assume that u is periodic, i.e., there exists $k \in \mathbb{Z}^d$ with $k_i^d > 0$ such that $u(x + nk) = u(x)$ for all $n \in \mathbb{N}$ and a.e. x. Consider $u_\varepsilon(x) := u(\frac{x}{\varepsilon})$. Then for all $\Omega \subset \mathbb{R}^d$ open and bounded we have $u_\varepsilon \rightharpoonup \bar{u} := \fint_{[0,k)} u\,dy$ weakly in $L^p(\Omega)$ for $1 \le p < \infty$ and weakly star in $L^\infty(\Omega)$ if $p = \infty$.*
Proof the statement in the case $1 < p < \infty$.

Exercise 3.12. *(a) Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $e \in \mathbb{R}^d$. Consider*

$$u_k(x) := \sin(k(x \cdot e))$$

Decide (in dependence of e) if (u_k) strongly or weakly converges in $L^2(\Omega)$ as $k \to \infty$ and compute its limit.

(b) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $0 \le \varphi \not\equiv 0$. Consider

$$u_k(x) := k^\alpha \varphi(\frac{x}{k}).$$

Decide (in dependence of $\alpha \in \mathbb{R}$) if (u_k) strongly or weakly converges in $L^1(\mathbb{R}^d)$ as $k \to \infty$ and compute its limit.
Consider

$$v_k(x) := \varphi(x + ke_1).$$

Decide if (u_k) weakly converges in $L^p(\mathbb{R}^d)$ as $k \to \infty$ (in dependence on $1 \le p < \infty$) and compute its limit.

Exercise 3.13 (Weakly closed $\neq$ strongly closed). *Let* $d = 3$. $X = L^2((0,1), \mathbb{R}^{d\times d})$, $K = \{F \in X \,:\, F(x) \in SO(d)$ *a.e.* $\}$. *Show that*

(a) K *is closed w.r.t. strong convergence.*

(b) K *is not closed w.r.t. weak convergence.*

Lemma 3.14 (Mazur's Lemma). *Let* $(X, \|\cdot\|)$ *be a Banach space and let* (x_k) *be a sequence in* X *that converges weakly to some* x_0 *in* X. *Then* x_0 *is the strong limit of convex combinations of the elements of* (x_k). *More precisely, there exists a sequence* (y_k) *such that*

$$y_k \in conv(x_k, x_{k+1}, x_{k+2}, \ldots) \text{ for all } k, \text{ and } \lim_{k\to\infty} \|y_k - x_0\| = 0.$$

Remark

- Let C be a subset of a vector space X. A convex combination y of elements of C is a (finite) sum of the form
$$y = \sum_{\ell=1}^{N} \alpha_\ell x_\ell \quad \text{with} \quad x_\ell \in C, \qquad \alpha_\ell \geq 0, \qquad \sum_{\ell}^{N} \alpha_\ell = 1, \qquad N \text{ finite.}$$
We write $\text{conv}(C)$ for the set of all convex combinations of C. A set is called convex, if $C = \text{conv}(C)$.
- A function $f : C \to \mathbb{R} \cup \{+\infty\}$ is convex, if C and
$$f(\sum_{\ell=1}^{N} \alpha_\ell x_\ell) \leq \sum_{\ell=1}^{N} \alpha_\ell f(x_\ell)$$
for every convex combination $\sum_{\ell=1}^{N} \alpha_\ell x_\ell$ of C.

Beweis. We only sketch the proof. Note that $C_k := \overline{\text{conv}\{x_k, x_{k+1}, \ldots\}}^{\|\cdot\|_X}$ is a convex subset of X. Since (x_k) is weakly convergent (and thus bounded), C_k is convex, bounded and stronly closed. Hence, by a standard result of functional analysis, C_k is closed w.r.t. the weak topology. Hence, $x_0 \in C_k$. Since C_k is the strong closure of the convex hull of $\{x_k, x_{k+1}, \ldots\}$, we can find a sequence $(y_\ell^{(k)})_{\ell\in\mathbb{N}}$ in $\text{conv}(x_k, x_{k+1}, \ldots)$ such that $\|y_\ell^{(k)} - x_0\|_X \to 0$. Let $n(k) \in \mathbb{N}$ denote an index s.t. $\|y_{n(k)}^{(k)} - x_0\|_X \leq \frac{1}{k}$. Hence, the sought for sequence can be obtained as the diagonal sequence $y_k := y_{n(k)}^{(k)}$. $\square$

Exercise 3.15. *Let* $1 \leq p < \infty$, $q := \frac{p}{p-1}$, $\Omega \subset \mathbb{R}^d$ *open,* $X := W^{1,p}(\Omega)$. *Consider*

$$\Lambda : L^q(\Omega) \times L^q(\Omega; \mathbb{R}^d) \to X^*, \qquad < \Lambda(f, F), \varphi >:= \int_\Omega \varphi f + \nabla\varphi \cdot F. \tag{3.4}$$

Show that Λ *defines a bounded, linear operator. Show that* Λ *is surjective in the case* $p = 2$.

Theorem 3.16 (Compact embeddings)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

- *(Rellich-Kondrachov). Let $1 \leq p < d$. Then $W^{1,p}(\Omega) \subset L^q(\Omega)$ is* ***compactly embedded*** *for all $1 \leq q < \frac{pd}{d-p}$, i.e., $u_k \rightharpoonup u$ weakly in $W^{1,p}$ implies $u_k \to u$ strongly in L^q.*

- *(Morrey's + Arzelá-Ascoli). Let $d < p \leq \infty$. Then $W^{1,p}(\Omega) \subset L^p(\Omega)$ is* ***compactly embedded****.*

For the argument see Evans' texbook "Partial Differential Equations".

Exercise 3.17. *Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $1 < p < \infty$. Let $g : \mathbb{R} \to \mathbb{R}$ be lower-semicontinuous, and $g(s) \geq -C(1 + |s|^p)$ for all $s \in \mathbb{R}$. Consider*

$$I : L^p(\Omega) \to \mathbb{R} \cup \{+\infty\}, \qquad I(u) := \begin{cases} \int_\Omega g(u(x))\, dx & \text{if } \|\nabla u\|_{L^p(\Omega)} \leq 1 \text{ and } \int_\Omega u = 0 \\ +\infty & \text{else.} \end{cases}$$

Show that I is lower-semicontinuous in $L^p(\Omega)$. Prove the existence of a minimizer in $W^{1,p}(\Omega)$.

3.3 Convex integral functionals

Theorem 3.18 (Seq. lower semicontinuity for convex integral functionals)**.** *Let $g : \mathbb{R}^m \to \mathbb{R} \cup +\infty$ be convex, lower semicontinuous and bounded from below, let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then for any sequence (u_k) in $L^1(\Omega, \mathbb{R}^m)$ with*

$$u_k \rightharpoonup u \qquad \textit{weakly in } L^1(\Omega, \mathbb{R}^m)$$

we have

$$\liminf_{k\to\infty} \int_\Omega g(u_k(x))\, dx \geq \int_\Omega g(u(x))\, dx. \tag{3.5}$$

Beweis. Wlog we may assume that $g \geq 0$, otherwise consider $g(\cdot) - \inf g$. We only need to consider the case

$$m := \liminf_{k\to\infty} I(u_k) < \infty,$$

since otherwise the statement is trivial. By passing to a subequence we may assume that $m = \lim_{k\to\infty} I(u_k)$. Thanks to Mazur's Lemma there exists a sequence

$$v_k \in \mathrm{conv}(u_k, u_{k+1}, u_{k+2}, \ldots) \text{ for all } k$$

such that $\|v_k - u\|_{L^1} \to 0$ as $k \to \infty$. By passing to a subsequence (not relabeled) we may assume that $v_k(x) \to u(x)$ for almost every $x \in \Omega$. Now lsc of g and Fatou's lemma yield

$$\int_\Omega g(v(x))\, dx \overset{\text{lsc}}{\leq} \int_\Omega \liminf_{k\to\infty} g(v_k(x)) \overset{\text{Fatou}}{\leq} \liminf_{k\to\infty} \int_\Omega g(v_k(x))\, dx. \tag{3.6}$$

Since v_k is a convex combination of $\{u_\ell\}_{\ell\geq k}$, we have

$$v_k = \sum_{\ell \geq k} \alpha_\ell^{(k)} u_\ell,$$

where $\{\alpha_\ell^{(k)}\}$ are non-negative and $\sum_{\ell\geq k} \alpha_\ell^{(k)} = 1$. By convexity of g, we thus get

$$\int_\Omega g(v_k(x))\, dx \leq \sum_{\ell\geq k} \alpha_\ell^{(k)} \int_\Omega g(u_\ell(x))\, dx. \tag{3.7}$$

Now, let $\delta > 0$. Since $I(u_k) \to m$, there exists $K \in \mathbb{N}$ s.t. $I(u_k) \leq m + \delta$ for all $k \geq K$. Hence,

$$\forall k \geq K \ : \qquad \sum_{\ell\geq k} \alpha_\ell^{(k)} \int_\Omega g(u_\ell(x))\, dx \leq \sum_{\ell\geq k} \alpha_\ell^{(k)} (m+\delta) = m + \delta.$$

Since $\delta > 0$ is arbitrary, we get

$$\limsup_{k\to\infty} \sum_{\ell\geq k} \alpha_\ell^{(k)} \int_\Omega g(u_\ell(x))\, dx \leq m.$$

Combined with (3.6), (3.7) and the definition of m, statement (3.5) follows. $\square$

Corollary 3.19. *Let $g : \mathbb{R}^d \to \mathbb{R}$ be convex, continuous and suppose that*

$$\frac{1}{c}|F|^p - c \le g(F) \le c(1 + |F|^p)$$

for some $1 < p < \infty$ and $c > 0$. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. For $u_0 \in W^{1,p}(\Omega)$ consider

$$\mathcal{A} := \{\, u \in W^{1,p}(\Omega) \,:\, u - u_0 \in W_0^{1,p}(\Omega) \,\}.$$

Then for all $f \in L^q(\Omega)$ ($q = \frac{p}{p-1}$) the functional

$$I(u) := \int_\Omega g(\nabla u(x))\,dx - \int_\Omega f(x)u(x)\,dx$$

admits a minimizer in $\mathcal{A}$.

Beweis. **Step 1.** A priori estimate.

Claim: There exists $C > 0$ s.t.

$$\|u\|^p_{W^{1,p}(\Omega)} \le C(I(u) + 1) \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Argument: In the following C denotes a generic constant, that might change from line to line, but is independend of u. By the triangle inequality and the Poincaré inequality we have

$$\begin{aligned}
\left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}} &\overset{\triangle\text{-ineq.}}{\le} \left(\int_\Omega |u - u_0|^p\right)^{\frac{1}{p}} + \left(\int_\Omega |u_0|^p\right)^{\frac{1}{p}} \\
&\overset{\text{Poincaré}}{\le} C\left(\int_\Omega |\nabla u - \nabla u_0|^p\right)^{\frac{1}{p}} + \left(\int_\Omega |u_0|^p\right)^{\frac{1}{p}} \\
&\le C\left(\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} + \|u_0\|^p_{W^{1,p}(\Omega)}\right) \\
&\overset{\triangle\text{-ineq.}}{\le} C\left(\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} + 1\right).
\end{aligned} \tag{3.8}$$

By coercivity of g we have

$$\begin{aligned}
\int_\Omega |\nabla u|^p\,dx &\le c\int_\Omega g(\nabla u)\,dx + c^2|\Omega| \\
&= c\left(I(u) + \int_\Omega f(x)u(x)\,dx\right) + c^2|\Omega|.
\end{aligned}$$

The combination of Hölder's inequality with exponents $(p, q = \frac{p}{p-1})$, Young's inequality in form of $ab \le \frac{\epsilon}{p}a^p + \frac{1}{\epsilon q}b^q$ (with $\epsilon > 0$ arbitrary) and (3.8) yields

$$\begin{aligned}
\int_\Omega f(x)u(x)\,dx &\le \frac{\epsilon}{p}\|u\|^p_{L^p(\Omega)} + \frac{1}{\epsilon q}\|f\|^q_{L^q(\Omega)} \\
&\le C\frac{\epsilon}{p}\left(\int_\Omega |\nabla u|^p\,dx + 1\right) + \frac{1}{\epsilon q}\|f\|^q_{L^q(\Omega)}.
\end{aligned}$$

and thus

$$\int_\Omega |\nabla u|^p \, dx \quad \leq \quad C\left(I(u) + \frac{\epsilon}{p}\left(\int_\Omega |\nabla u|^p \, dx + 1\right) + \frac{1}{\epsilon q}\|f\|^q_{L^q(\Omega)} + 1\right).$$

With the choice $\epsilon = \frac{p}{2C\epsilon}$ we can absorb the middle-term into the left-hand side and get

$$\frac{1}{2}\int_\Omega |\nabla u|^p \, dx \leq C(I(u) + 1)$$

for a suitable constant $C > 0$ that might depend on Ω, p, u_0 and f, but not on u. Now the claimed estimate follows from (3.8).
Step 2. Extraction of a weakly convergent minimizing sequence.

Since $\{u \in \mathcal{A} \,:\, I(u) < \infty\} \neq \emptyset$, there exists (u_k) in $\mathcal{A}$ s.t.

$$\lim_{k\to\infty} I(u_k) = \inf_{\mathcal{A}} I.$$

By Step 1 (u_k) is bounded in $W^{1,p}(\Omega)$. Hence, there exists a subsequence (not relabeled) s.t.

$$u_k \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega).$$

Since $\mathcal{A}$ is an affine (and thus convex), closed subspace of $W^{1,p}(\Omega)$, it is weakly closed. Hence, the $u \in \mathcal{A}$. Moreover, from Theorem 3.18 we deduce that

$$\liminf_{k\to\infty} \int_\Omega g(u_k(x)) \, dx \geq \int_\Omega g(u(x)) \, dx,$$

and since $f \in L^q(\Omega) = (L^p(\Omega))^*$, we have

$$\int_\Omega f(x) u_k(x) \, dx \to \int_\Omega f(x) u(x) \, dx.$$

Hence,

$$\liminf_{k\to\infty} I(u_k) \geq I(u).$$

Step 3. (Conlcusion).
u is a minimizer, since

$$\inf_{\mathcal{A}} I \leq I(u) \leq \liminf_{k\to\infty} I(u_k) = \inf_{\mathcal{A}} I.$$

□

Example 3.20. $g(F) := \frac{1}{p}|F|^p$. *Then there exists a unique* $u \in W^{1,p}_0(\Omega)$ *that minimizes*

$$\int_\Omega \frac{1}{p}|\nabla u(x)|^p - u(x) f(x) \, dx.$$

If u *is* C^2 *in* Ω, *then*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega.$$

The operator $\operatorname{div}(|\nabla u|^{p-2}u)$ *is called the* p*-Laplacian.*

In the case of dimension one or codimension one, the inverse of Theorem 3.18 is also true. We discuss the case of codimension= 1:

Lemma 3.21. *Let $1 \le p < \infty$, $\Omega \subset \mathbb{R}^d$ open and bounded. Consider*

$$I : W^{1,p}(\Omega) \to \mathbb{R}, \qquad I(u) := \int_\Omega f(\nabla u(x))\, dx.$$

Suppose that I is lower-semicontinuous w.r.t. weak convergence in $W^{1,p}(\Omega)$. Then f is convex.

Beweis. Let $a, b \in \mathbb{R}^d$, $\lambda \in (0,1)$. We need to show that

$$f(\lambda a + (1-\lambda b)) \le \lambda f(a) + (1-\lambda) f(b).$$

Set

$$c = \lambda a + (1-\lambda) b, \qquad v = b - a.$$

Then $a = c + (\lambda - 1)v$ and $b = c + \lambda v$. Let $h : \mathbb{R} \to \mathbb{R}^d$ denote a 1-periodic function with

$$h(t) = \begin{cases} \lambda - 1 & \in [0,\lambda] \\ \lambda & \in (\lambda, 1) \end{cases}, \qquad H(t) := \int_0^t h(s)\, ds.$$

Then $H \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and is 1-periodic. Consider

$$u_k(x) := c \cdot x + \frac{1}{k} H(k(x \cdot v)).$$

Then $u_k \in W^{1,\infty}(\Omega)$, and $\nabla u_k(x) = c + h(k(x \cdot v)v$. Note that h only takes values in $\{\lambda, \lambda - 1\}$, and thus $\nabla_k u$ only takes values in $\{a, b\}$. Moreover,

$$\begin{aligned} |\{\nabla u_k = a\}| &= |\{x \in \Omega \,:\, k(x \cdot v) \in [0,\lambda)\}| \to \lambda |\Omega| \\ |\{\nabla u_k = b\}| &= |\{x \in \Omega \,:\, k(x \cdot v) \in [\lambda, 1)\}| \to (1-\lambda)|\Omega|, \end{aligned}$$

and one can show that

$$\nabla u_k \overset{\rightharpoonup}{\lambda}\, a + (1-\lambda) b = c \qquad \text{weakly-* in } L^\infty(\Omega).$$

Similarly, we get

$$f(\nabla u_k) \overset{*}{\rightharpoonup} \lambda f(a) + (1-\lambda) f(b).$$

Thus,

$$\big(\lambda f(a) + (1-\lambda) f(b)\big) = \lim_{k \to \infty} \fint_\Omega f(\nabla u_k) \ge \fint_\Omega f(c) = f(c) = f(\lambda a + (1-\lambda) b).$$

□

Remark 3.22. *The statement is wrong for vector-valued integral functionals, e.g. we shall see that $\int_\Omega f(Du)\, dx$ ($u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$) for $f(F) := \det(F)$ is lower-semicontinuous, but $f = \det$ is not convex. However, the above argument can be easily adapted to show that integrands f of lower-semicontinuous integral functionals are rank-one-convex, i.e. $\lambda f(a) + (1-\lambda) f(b) \le f(\lambda a + (1-\lambda) b)$ for all $\lambda \in [0,1]$ and $a, b \in \mathbb{R}^{d \times n}$ with rank$(a-b) = 1$.*

Remark 3.23. *The above results do not apply to nonlinear elasticity, because convexity of the integrand f is not compatible with the fundamental assumptions of nonlinear elasticity. To solve the balance equations for a hyperelastic material with stored energy density W, we would like to minimize the functional*

$$\int_\Omega W(Du) - b \cdot u \, dx$$

Frame indifference implies that $W(RF) = W(R)$ for all $R \in SO(d)$. Moreover W should attain its minimum at an invertible matrix F_0 (usually $F_0 = Id$). Such a function W can not be convex. In 1977 John Ball introduced the concept of polyconvex functions to extend the direct method of the calculus of variations to integral functionals that are compatible with nonlinear elasticity.

3.4 Minors and weak continuity

We seek for maps $m : \mathbb{R}^{d\times d} \to \mathbb{R}$ that are weakly continuous in the sense that

$$Du_k \rightharpoonup Du \qquad \Rightarrow \qquad m(Du_k) \rightharpoonup m(Du). \tag{3.9}$$

Exercise 3.24. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $m : \mathbb{R}^k \to \mathbb{R}$ be affine. Then $m : L^p(\Omega; \mathbb{R}^k) \to L^p(\Omega)$ is weakly contininuous, i.e.,*

$$u_k \rightharpoonup u \text{ weakly in } L^p \qquad \Rightarrow \qquad m(u_k) \rightharpoonup m(u) \text{ weakly in } L^p.$$

One can show that the inverse is also true, i.e. any weakly continuous map $m : L^p(\Omega; \mathbb{R}^k) \to L^p(\Omega)$ (satisfying some growth condition) must be affine. (This follows from the fact that weakly lower semicontinuous functionals must be convex). It turns out that there exist some specific nonlinear maps that are continuous w.r.t. weak convergence, when we restrict to weakly convergent *gradients*, as in (3.9).

Definition 3.25 (Minors)**.** *Let $F \in \mathbb{R}^{d\times d}$ and $k \in \{1, \dots, d\}$.*

- *A $k \times k$-submatrix of F is a matrix obtained from F by deleting $(d-k)$-rows and columns.*

- *A $k \times k$-minor (or subdeterminant) of a matrix F is a determinant of a $k \times k$-submatrix of F.*
- *We write $M(F)$ for the vector of all minors of F.*

Remark 3.26. *Let F be a $d \times d$-matrix.*

- *It has only a single $d \times d$-minor, that is, the* $\det F$.
- *It has d^2 1×1-minors, namely, the entries of F.*
- *It has d^2 $(d-1) \times (d-1)$-minors. Up to a sign, they coincide with the entries of* $\mathrm{cof} F$.

- *It has $\binom{d}{k}^2$ submatrices of size $k \times k$, and thus*

$$M(F) \in \mathbb{R}^{\tau(d)}, \qquad \tau(d) := \sum_{k=1}^{d} \binom{d}{k}^2.$$

- *We did not fix a specific ordering for $M(F)$. This may vary from context to context, e.g. for $d = 2$ we set*

$$M(F) = (F, \det F) \in \mathbb{R}^{2\times 2} \times \mathbb{R} \cong \mathbb{R}^5$$

- *In dimension $d = 3$ we may set(modulo a sign) $M(F) = (F, \mathrm{cof} F, \det F) \in \mathbb{R}^{19}$*

The goal of this section is to show that minors are weakly continuous. (In fact, one can show that any map that is weakly continuous w.r.t. gradient fields, is a linear combination of minors).

Definition 3.27 (Null-Lagrangian)**.** *A function $f \in C^2(\mathbb{R}^{d\times d})$ is called a Null-Lagrangian, if*

$$-\operatorname{div}(\nabla f(Du)) = 0 \qquad \text{for all } u \in C^2(\mathbb{R}^d, \mathbb{R}^d).$$

Lemma 3.28. *(a) If $f \in C^2(\mathbb{R}^{d\times d})$ is a Null-Lagrangian, then*

$$\int_\Omega f(Du(x))\,dx = \int_\Omega f(Dv(x))\,dx$$

for all $u, v \in C^2(\overline{\Omega}, \mathbb{R}^d)$ with $u - v \in C^2_c(\Omega, \mathbb{R}^d)$. (Here $\Omega \subset \mathbb{R}^d$ denotes an arbitrary open and bounded set).

(b) Suppose $f \in C^2(\mathbb{R}^{d\times d})$ satisfies

$$\int_B f(Du(x))\,dx = \int_B f(Dv(x))\,dx$$

for all $u, v \in C^2(\overline{B}, \mathbb{R}^d)$ with $u - v \in C^2_c(B, \mathbb{R}^d)$ (and a ball $B \subset \mathbb{R}^d$). Then f is a Null-Lagrangian.

Beweis. • **Argument for (a).** Consider

$$h(t) := \int_\Omega f(tDu(x) + (1-t)Dv(x))\,dx.$$

Note that

$$h'(t) = \int_\Omega \nabla f(tDu(x) + (1-t)Dv(x)) \cdot (Du(x) - Dv(x))\,dx.$$

Since $u - v \in C^2_c(\Omega, \mathbb{R}^d)$, an integration by parts yields

$$h'(t) = -\int_\Omega \operatorname{div}(\nabla f(tDu(x) + (1-t)Dv(x))) \cdot (u(x) - v(x))\,dx \overset{f \text{ is N.-L.}}{=} 0.$$

Hence,

$$\int_\Omega f(Du(x))\,dx = h(1) = h(0) = \int_\Omega f(Dv(x))\,dx.$$

- **Argument for (b).** Let $\theta \in C_c^\infty(B, \mathbb{R}^d)$. Set $h(t) := \int_B f(Du(x) + tD\theta(x))\,dx$. Then by assumption h is constant, and thus

$$0 = h'(0) = \int_B \nabla f(Du(x)) \cdot D\theta(x)\,dx = -\int_B \operatorname{div}(\nabla f(Du(x))) \cdot \theta(x)\,dx.$$

Since $\theta \in C_c^\infty(B, \mathbb{R}^d)$ and $x \mapsto \operatorname{div}(\nabla f(Du(x)))$ is continuous, we deduce that $-\operatorname{div}(\nabla f(Du(x))) = 0$ for all $x \in B$. Hence, f is a Null-Lagrangian.

□

Lemma 3.29 (Determinant is Null-Lagrangain)**.**
$\det(\cdot)$ *is a Null-Lagrangian.*

Beweis. We recall that $\nabla \det(F) = \operatorname{cof} F$.
Thus, we we only need to show

$$-\operatorname{div}\operatorname{cof} Du = 0$$

for all $u \in C^2$. Since $\operatorname{cof} F = \nabla \det(F)$, we get from the Leibniz identity for the determinant the identity

$$(\operatorname{cof} F)_{ab} = \sum_{\substack{\sigma \in S_d \\ \sigma(a)=b}} \operatorname{sgn}\sigma \prod_{i \neq a} F_{i,\sigma(i)}.$$

Hence,

$$\partial_j (\operatorname{cof} Du)_{aj} \quad = \quad \sum_{\substack{\sigma \in S_d \\ \sigma(a)=j}} (\operatorname{sgn}\sigma) \quad \underbrace{\partial_j \left(\prod_{i \neq a} \partial_{\sigma(i)} u_i \right)}_{\sum_{i \neq a} \partial_{\sigma(a)} \partial_{\sigma(i)} u_i \prod_{k \neq a,i} \partial_{\sigma(k)} u_k} \qquad .$$

Hence,

$$\sum_{j=1}^d \partial_j (\operatorname{cof} Du)_{aj} \quad = \quad \underbrace{\sum_{i \neq a} \sum_{j=1}^d \sum_{\substack{\sigma \in S_d \\ \sigma(a)=j}}}_{=\sum_{\sigma \in S_d}} \underbrace{(\operatorname{sgn}\sigma) \partial_{\sigma(a)} \partial_{\sigma(i)} u_i \prod_{k \neq a,i} \partial_{\sigma(k)} u_k}_{=: f(\sigma,a,i)}.$$

By symmetry of the summand f the right-hand side is zero; indeed: We have $f(\sigma, a, i) = -f(\tilde{\sigma}, a, i)$ with

$$\tilde{\sigma}(\ell) := \begin{cases} \sigma(i) & \ell = a \\ \sigma(a) & \ell = i \\ \sigma(\ell) & \text{else,} \end{cases}$$

and thus

$$\sum_{\sigma \in S_d} f(\sigma, i, a) = \sum_{\substack{\sigma \in S_d \\ \sigma(a) < \sigma(i)}} f(\sigma, i, a) + \sum_{\substack{\sigma \in S_d \\ \sigma(a) > \sigma(i)}} f(\sigma, i, a) = \sum_{\substack{\sigma \in S_d \\ \sigma(a) < \sigma(i)}} f(\sigma, i, a) + f(\tilde{\sigma}, i, a) = 0.$$

□

Lemma 3.30 (Exercise 3.30). *Let F be a $d \times d$-matrix. Then every minor F corresponds up to a sign to a partial derivative of* $\det(F)$. *More precisely, if $\ell \in \{0, \dots, d-1\}$ and $m(F)$ denotes the $(d-\ell)\times(d-\ell)$-minor associated with the submatrix obtained by deleting the rows with index*

$$1 \le i_1 < \cdots < i_\ell \le d$$

and columns with index

$$1 \le j_1 < \cdots < j_\ell \le d,$$

then

$$m(F) = \pm \frac{\partial^\ell \det(F)}{\partial F_{i_1,j_1} \dots \partial F_{i_\ell,j_\ell}}.$$

Corollary 3.31. *Let $m(F)$ denote a minor of $F \in \mathbb{R}^{d\times d}$. Then $F \mapsto m(F)$ is a Null-Lagrangian.*

Beweis. In view of Lemma 3.29 there exists a partial derivative

$$\tilde{\partial}_F := \pm \frac{\partial^\ell}{\partial F_{i_1,j_1} \cdots \partial F_{i_\ell,j_\ell}}$$

s.t.

$$m(F) = \tilde{\partial}_F \det(F). \tag{*}$$

For $u \in C^2(\bar{B}, \mathbb{R}^d)$ and $F \in \mathbb{R}^{d\times d}$ set

$$I(F,u) := \int_B \det(F + Du(x))\, dx.$$

Since $\det(\cdot)$ is a Null-Lagrangian we have (see Lemma 3.28)

$$I(F, u+\varphi) = I(F,u) \qquad \text{for all } \varphi \in C_c^\infty(B, \mathbb{R}^d).$$

Hence, $\tilde{\partial}_F I(F, u+\varphi) = \tilde{\partial}_F I(F,u)$. Interchanging differentiation and integration yields

$$\int_B \tilde{\partial}_F \det(F + Du(x) + D\varphi(x))\, dx = \int_B \tilde{\partial}_F \det(F + Du(x))\, dx.$$

Hence, for $F = 0$ identity $(*)$ yields

$$\int_B m(Du(x) + D\varphi(x))\, dx = \int_B m(Du(x))\, dx,$$

and thus $m(\cdot)$ is a Null-Lagrangian thanks to Lemma 3.28. □

Theorem 3.32 (Weak continuity of Minors)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $q > d$, and assume that*

$$u_k \rightharpoonup u \qquad \textit{weakly in } W^{1,q}(\Omega, R^d).$$

Then

$$\det Du_k \rightharpoonup \det Du \qquad \textit{weakly in } L^{q/d}(\Omega).$$

Moreover, if m denotes a $\ell \times \ell$-minor, then

$$m(Du_k) \rightharpoonup m(\det Du) \qquad \textit{weakly in } L^{q/\ell}(\Omega).$$

Proof of Theorem 3.32. We first prove the statement for the determinant.
Step 1. Claim: For all cubes $Q \subset \Omega$ and all $\phi, \psi \in W^{1,q}(Q, R^d)$, $\phi - \psi \in W_0^{1,q}(Q, \mathbb{R}^d)$ we have

$$\int_Q \det(D\phi)\, dx = \int_Q \det(D\psi)\, dx. \tag{*}$$

Indeed, this follows from a density argument: Since $C^\infty(\overline{Q}, \mathbb{R}^d)$ (resp. $C_c^\infty(Q, \mathbb{R}^d)$) is dense in $W^{1,q}(Q, R^d)$ (resp. in $W_0^{1,q}(Q, R^d)$), we can find $\phi_k, \psi_k \in C^\infty(\overline{Q}, \mathbb{R}^d)$ s.t. $\phi_k - \psi_k \in C_c^\infty(Q, \mathbb{R}^d)$ and

$$\phi_k \to \phi \qquad \text{strongly in } W^{1,q}(Q, \mathbb{R}^d)$$

and

$$\psi_k \to \psi \qquad \text{strongly in } W^{1,q}(Q, \mathbb{R}^d).$$

Since

$$|\det(F + G) - \det(F)| \leq C(1 + |F + G|^{d-1} + |F|^{d-1})|G|$$

(as follows from the fact that $\det(F)$ is a polynomial of degree d), we conclude that

$$\begin{aligned} & |\int_Q \det(D\phi_k) - \det(D\phi)\, dx| \qquad (3.10)\\ \leq\ & \int_Q |\det(D\phi + (D\phi_k - D\phi)) - \det(D\phi)|\, dx \\ \leq\ & C \int_Q (1 + |D\phi_k|^{d-1} + |D\phi|^{d-1})|D\phi_k - D\phi|\, dx \\ \overset{\text{Hölder}}{\leq}\ & C \left(\int_Q (1 + |D\phi_k|^d + |D\phi|^d) dx \right)^{\frac{d-1}{d}} \left(\int_Q |D\phi_k - D\phi|^d\, dx \right)^{\frac{1}{d}}. \end{aligned}$$

Here and below $C > 0$ denotes a generic constant that might change from line to line, but only depends on d, q and Ω. Since $d \leq q$, the first factor on the RHS is bounded uniformly in k, while the second factor vanishes as $k \to \infty$. Hence, we conclude that $\int_Q \det(D\phi_k)\, dx \to \int_Q \det(D\phi)\, dx$ and similarly $\int_Q \det(D\psi_k)\, dx \to \int_Q \det(D\psi)\, dx$. Since $\det(\cdot)$ is a Null-Lagrangian, we have $\int_Q \det(D\phi_k)\, dx = \int_Q \det(D\psi_k)\, dx$ thanks to Lemma 3.28, and the claimed identity follows.

Step 2. Let Q be a cube in Ω. We claim that

$$\int_Q \det Du_k\, dx \to \int_Q \det Du\, dx.$$

Argument: Note that if we had $u_k = u$ on ∂Q, then the statement would follows from Step 1. In order to able to apply Step 1 we introduce a cut-off function: For $\eta \in C_c^\infty(Q)$ with $0 \leq \eta \leq 1$ consider

$$v_k = u + \eta(u_k - u).$$

Note that $v \in W_0^{1,q}(\Omega, \mathbb{R}^d)$, and thus

$$\int_Q \det Dv_k\, dx = \int_Q \det Du\, dx$$

by Step 1. Hence,

$$\limsup_{k\to\infty} |\int_Q \det(Du_k) - \det(Du)|\,dx \leq \limsup_{k\to\infty} |\int_Q \det(Du_k) - \det(Du)|\,dx.$$

We claim that

$$\limsup_{k\to\infty} \int_Q |\det(Dv_k) - \det(Du_k)|\,dx \leq C\|1-\eta\|_{L^{\frac{dq}{q-d}}}. \tag{*}$$

Note that we can make the RHS arbitrarily small by chosing a suitable cut-off function. Therefore, the combination of the previous two estimates yields the claim. Argument for $(*)$: From $v_k - u_k = (1-\eta)(u-u_k)$, we deduce that

$$|Dv_k - Du_k| \leq (1-\eta)|Du - Du_k| + |\nabla\eta||u-u_k|,$$

and thus

$$\begin{aligned}\|Dv_k - Du_k\|_{L^d(Q)} &\leq \|(1-\eta)|Du-Du_k|\|_{L^d(\Omega)} + \||\nabla\eta||u-u_k|\|_{L^d(\Omega)}\\ &\overset{\text{Hölder } (\frac{q}{q-d},\frac{q}{d})}{\leq} \|1-\eta\|_{L^{\frac{dq}{d-q}}}\|Du-Du_k\|_{L^q(\Omega)} + \|\nabla\eta\|_{L^{\frac{dq}{d-q}}}\|u-u_k\|_{L^q(\Omega)}.\end{aligned}$$

From $u_k \rightharpoonup u$ in $W^{1,q}$ we deduce that

(**) $\|Du_k\|_{L^q}$ is bounded,

(***) $u_k \to u$ stronly in L^q (by compact embedding!).

Thus,

$$\limsup_{k\to\infty} \|Dv_k - Du_k\|_{L^d(Q)} \leq C\|1-\eta\|_{L^{\frac{dq}{d-q}}}.$$

A calculation similar to (3.10) yields

$$\int_Q |\det(Dv_k) - \det(Du_k)|\,dx| \leq C\left(\int_Q |Dv_k - Du_k|^d\,dx\right)^{\frac{1}{d}}.$$

The combination of the last two estimates yields $(*)$.
Step 3. We conclude by a density argument.
Let X be the space of finite linear combinations of charateristic functions of cubes in Ω, i.e.

$$X = \left\{\varphi := \sum_{\ell=1}^N \alpha_\ell \chi_{Q_\ell} \,:\, \alpha_\ell \in \mathbb{R},\ Q_\ell \text{ cube in } \Omega,\ N\in\mathbb{N}\right\}.$$

Thanks to Step 2 we have

$$\int_\Omega \varphi \det Du_k\,dx \to \int_\Omega \varphi \det Du\,dx \qquad \text{for all } \varphi\in X.$$

Since $(\det Du_k)$ is bounded in $L^{q/d}$, and X is dense in $(L^{q/d})^*$, the previous identity holds for all $\varphi\in(L^{q/d})^*$, i.e. $\det Du_k \rightharpoonup \det Du$ weakly in $L^{q/d}$.
Step 4. The proof for general $\ell\times\ell$-minors $m(F)$ is similar. Indeed,

- $m(F)$ is a Null-Lagrangian, and
- $m(F)$ is a polynomial of degree ℓ, and thus

$$|m(F+G)-m(F)| \leq C(1+|F+G|^{\ell-1}+|G|^{\ell-1})|G|.$$

$\square$

An alternative proof of Theorem 3.32 relies on the identity

$$\det(Du) = \frac{1}{d}\operatorname{div}((\operatorname{cof} Du)^t u)$$

which holds for all smooth u, and is valid in the distributional sense for arbitrary $u \in W^{1,d}(\Omega, R^d)$, i.e.

$$\int_\Omega \det(Du)\varphi\, dx = -\frac{1}{d}\int_\Omega (\operatorname{cof} Du)^t u \cdot \nabla\varphi\, dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{3.11}$$

In the following restrict to $d=2$ and first prove (3.11) and secondly Theorem 3.32 with help of (3.11). In two dimensions we have for smooth u

$$\operatorname{cof} Du = \begin{pmatrix} \partial_2 u_2 & -\partial_1 u_2 \\ -\partial_2 u_1 & \partial_1 u_1 \end{pmatrix}, \qquad (\operatorname{cof} Du)^t u = \begin{pmatrix} u_1\partial_2 u_2 - u_2\partial_2 u_1 \\ -u_1\partial_1 u_2 + u_2\partial_1 u_1 \end{pmatrix}$$

and thus

$$\operatorname{div}((\operatorname{cof} Du)^t u) = 2\det Du.$$

Hence, for $u \in C^\infty(\Omega,\mathbb{R}^2) \cap W^{1,2}(\Omega,\mathbb{R}^2)$ and $\varphi \in C_c^\infty(\Omega)$ we have by integration by parts

$$\int_\Omega (\det Du)\varphi\, dx = -\frac{1}{2}\int_\Omega (\operatorname{cof} Du)^t u \cdot \nabla\varphi\, dx$$

By density the identity remains valid for $u \in W^{1,2}(\Omega,\mathbb{R}^2)$.
Suppose now that $u^{(k)} \rightharpoonup u$ weakly in $W^{1,q}(\Omega,\mathbb{R}^2)$ for some $q > d = 2$. Then

(a) $\operatorname{cof} Du^{(k)} \rightharpoonup \operatorname{cof} Du$ weakly in L^q

(b) $u^{(k)} \to u$ strongly in L^q

(c) $\det Du^{(k)} \rightharpoonup \delta$ weakly in $L^{q/2}$ for a subsequence (not relabeled) and some $\delta \in L^{q/2}$.

We claim that $\delta = \det Du$. Thanks to (c) it suffices to show

$$\int_\Omega (\det Du^{(k)})\varphi\, dx \to \int_\Omega (\det Du)\varphi\, dx$$

for all $\varphi \in C_c^\infty(\Omega,\mathbb{R}^2)$. With help of (3.11) we have

$$\int_\Omega (\det Du^{(k)})\varphi\, dx = -\frac{1}{2}\int_\Omega (\operatorname{cof} Du^{(k)})^t u^{(k)} \cdot \nabla\varphi\, dx$$

Thanks to (b) we have $u^{(k)} \cdot \nabla\varphi \to u \cdot \nabla\varphi$ strongly in $L^q(\Omega)$, and combined with (a) we get

$$-\frac{1}{2}\int_\Omega (\operatorname{cof} Du^{(k)})^t u^{(k)} \cdot \nabla\varphi \, dx = -\frac{1}{2}\int_\Omega (\operatorname{cof} Du)^t u \cdot \nabla\varphi \, dx.$$

Note that the right-hand side is equal to $\int_\Omega (\det Du)\varphi \, dx$, which completes the argument.

Remark. The minors are the only functions that are continuous w.r.t. weak convergence of gradients. More precisely, if $\Omega \subset \mathbb{R}^d$ is open and bounded and if $f : \mathbb{R}^{d\times d} \to \mathbb{R}^d$ is continuous and satisfies the property

$$u_k \overset{*}{\rightharpoonup} u \qquad \text{weak star in } W^{1,\infty}(\Omega, \mathbb{R}^d) \qquad \Rightarrow \qquad f(Du_k) \overset{*}{\rightharpoonup} f(Du) \text{ weak star in } L^\infty(\Omega).$$

Then f is a constant plus a linear combination of minors.

3.5 Polyconvexity and Sir John Ball's Existence Theorem

Definition 3.33 (Polyconvexity). *A function $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$ is called polyconvex, if there exists a convex function*

$$g : \mathbb{R}^{\tau(d)} \to \mathbb{R} \cup \{+\infty\}$$

such that

$$f(F) = g(M(F)) \qquad \text{for all } F \in GL_+(d).$$

An elastic material is called polyconvex, if its stored energy function W is polyconvex (more precisely, there exists a polyconvex function $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$ such that $W(F) = f(F)$ for all $F \in GL_+(d)$.)

Examples

- $W(F) = \frac{1}{2}|F|^2 + \frac{1}{\det F}$ is polyconvex. Indeed, we have $W(F) = g(F, \det F)$ for all $F \in GL_+(d)$ where

$$g(F, d) := \begin{cases} \frac{1}{2}|F|^2 + \frac{1}{d} & \text{for } d > 0 \\ +\infty & \text{else.} \end{cases}$$

- g might not be unique, e.g. $d = 2$

$$g(F, d) := |F|^2 + 2d \text{ and } \tilde{g}(F, d) = |F|^2 + 2\det F$$

  are both convex and $g(F, \det F) = \tilde{g}(F, \det F)$ for all $F \in \mathbb{R}^{d\times d}$.
  The definition says "there exists a convex g" and not "all possible gs are convex"; e.g. $\bar{g}(F, d) := |F|^2 - 4\det F + 6d$ is not convex, but $F \mapsto \bar{g}(F, \det F) = g(F, \det F)$ is polyconvex.

Theorem 3.34 (J. Ball's existence result). *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $W \in C(GL_+(d), \mathbb{R})$ be polyconvex such that*

$$W(F) \to \infty \qquad \textit{when } \det F \to 0$$

and

$$W(F) \geq c|F|^q - \frac{1}{c}$$

for some $c > 0$ and $q > d$.
For a given $\varphi_0 \in W^{1,q}(\Omega, \mathbb{R}^d)$ we define

$$\mathcal{A} := \{ \varphi \in W^{1,q}(\Omega, \mathbb{R}^d) \,:\, \varphi - \varphi_0 \in W_0^{1,q}(\Omega, \mathbb{R}^d) \text{ and } \det D\varphi > 0 \text{ almost everywhere },\}$$

and assume that there is $\varphi \in \mathcal{A}$ with $\int_\Omega W(D\varphi)\,dx < \infty$. Then for all $f \in L^p(\Omega, \mathbb{R}^d)$ ($p = \frac{q}{q-1}$) the functional

$$I(\varphi) = \int_\Omega W(D\varphi) - f \cdot \varphi \, dx$$

has a minimizer in $\mathcal{A}$.

Jun 28, 2018

Proof of Theorem 3.34. **Step 1.** Extraction of a Minimizing Sequence.

Since $\{\varphi \in \mathcal{A} \,:\, I(\varphi) < \infty \} \neq \emptyset$, there exists a minimizing sequence, i.e. (φ_k) in $\mathcal{A}$ s.t.

$$\lim_{k\to\infty} I(\varphi_k) = \inf_{\mathcal{A}} I < \infty.$$

Step 2. A priori estimate and compactness.

By an argument similar to Step 1 in the proof of Corollary 3.19 we deduce that there exists $C > 0$ s.t.

$$\|\varphi\|^q_{W^{1,q}(\Omega,\mathbb{R}^d)} \leq C(I(\varphi) + 1) \qquad \text{for all } \varphi \in \mathcal{A}.$$

Hence, the minimizing sequence (φ_k) is bounded in $W^{1,q}$, and thus

$$\varphi_k \rightharpoonup \varphi \qquad \text{weakly in } W^{1,q} \tag{3.12}$$

for a subsequence (not relabeled) and $\varphi \in W^{1,q}(\Omega, \mathbb{R}^d)$.
Step 3. (Weak continuity of minors).
From (3.12) and Theorem (3.32) we learn that the list of minors weakly converges, i.e.

$$M(D\varphi_k) \rightharpoonup M(D\varphi) \qquad \text{weakly in } L^{q/d}(\Omega, \mathbb{R}^{\tau(d)}).$$

Step 4. (Weak lsc. of convex integral functionals).
Since W is polyconvex, there exists a convex g s.t. $W(F) = g(M(F))$. Hence, Step 3 and Theorem 3.18 yield

$$\liminf_{k\to\infty} \int_\Omega g(M(D\varphi_k))\,dx \geq \int_\Omega g(M(D\varphi))\,dx.$$

Moreover,

$$\int_\Omega f \cdot \varphi_k \, dx \to \int_\Omega f \cdot \varphi \, dx.$$

Hence,

$$\liminf_{\ell\to\infty} I(\varphi_k) \geq \int_\Omega g(M(D\varphi)) - f\cdot\varphi\,\mathrm{d}x.$$

Step 5. Claim: $\varphi \in \mathcal{A}$. Since $\varphi - \varphi_0 \in W_0^{1,q}(\Omega,\mathbb{R}^d)$ (cf. Step 2 in the proof of Corollary 3.19), we only need to argue that $\det D\varphi(x) > 0$ for almost every $x \in \Omega$. Proof by contradiction: Let $E := \{\, x \in \Omega \,:\, \det D\varphi(x) \leq 0 \,\}$ and assume that $\int_E dx > 0$. From $\det D\varphi_k > 0$ a.e. and $\det D\varphi_k \rightharpoonup \det D\varphi$ weakly in $L^1(\Omega)$, we get

$$\lim_{k\to\infty}\int_E |\det D\varphi_k|\,dx = \lim_{k\to\infty}\int_\Omega \chi_E \det D\varphi_k\,dx = \int_\Omega \chi_E \det D\varphi\,dx \leq 0.$$

Hence, $\det D\varphi_k \to 0$ in $L^1(E)$, and thus (for a subsequence, not relabeled) $\det D\varphi(x) \to 0$ a.e. in E. Hence, by Egorov's theorem there exists a measurable subset $E' \subset E$ with $\int_{E'} dx \geq \frac{1}{2}\int_E dx > 0$ s.t. $\det D\varphi_k \to 0$ uniformly in E'. Hence, since $W(F) \to \infty$ as $\det F \to 0$, we deduce that

$$\int_{E'} W(D\varphi_k)\,dx \to \infty,$$

in contradiction to the boundedness of $I(\varphi_k)$. We conclude that E is a null-set, and thus $\varphi \in \mathcal{A}$.

Step 6. Conclusion. From Step 4 and Step 5 we deduce that

$$\liminf_{k\to\infty} I(\varphi_k) \geq I(\varphi)$$

and $\varphi \in \mathcal{A}$. Hence,

$$\inf_{\mathcal{A}} I \leq I(\varphi) \leq \liminf_{k\to\infty} I(\varphi_k) = \inf_{\mathcal{A}} I,$$

and thus $\varphi \in \mathcal{A}$ is a minimizer.

□

4 Asymptotic analysis and Γ-convergence

Motivation

4.1 Γ-convergence

Literaturempfehlung: A. Braides - Handbook of Γ-Convergence; G. DalMaso Γ-convergence.

Definition 4.1. *Sei (X,d) ein metrischer Raum, $\mathcal{F},\mathcal{F}_j : X \to \mathbb{R}\cup\{\pm\infty\}$, $j \in \mathbb{N}$. Die Folge $(\mathcal{F}_j)$ $\Gamma(d)$-konvergiert zu $\mathcal{F}$ (Abkürzung: $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$) genau dann, wenn für alle $x \in X$ gilt*

1. *Für alle $(x_j) \subset X$ mit $x_j \to x$, gilt*

$$\liminf_{j\to\infty} \mathcal{F}_j(x_j) \geq \mathcal{F}(x).$$

Bezeichnung: $\liminf$-*Ungleichung*

2. *Es existiert* $(x_j) \subset X$ *mit* $x_j \to x$ *und*

$$\limsup_{j\to\infty} \mathcal{F}_j(x_j) \leq \mathcal{F}(x).$$

Bezeichnung: lim sup-*Ungleichung bzw. Existenz einer Wiederherstellungsfolge (recovery sequence)*

Bemerkung: In der Definition kann die lim sup-*Ungleichung durch die Bedingung ersetzt werden, dass für alle* $x \in X$ *eine Folge* (x_j) *mit* $x_j \to x$ *existiert, sodass*

$$\lim_{j\to\infty} \mathcal{F}_j(x_j) = \mathcal{F}(x).$$

Example 4.2. *Sei* $X = \mathbb{R}$, $d(x, y) = |x - y|$.

1. *Wir betrachten*

$$\mathcal{F}_j(x) = \begin{cases} -1, & x = \frac{1}{j}, \\ 0, & x \neq \frac{1}{j}. \end{cases}$$

Die Folge $(\mathcal{F}_j)$ Γ-*konvergiert zu* $\mathcal{F}$ *gegeben durch*

$$\mathcal{F}(x) = \begin{cases} -1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Argument:

(a) *(*lim inf *Ungleichung) Sei* $x \neq 0$ *und sei* (x_j) *mit* $x_j \to x$ *gegeben. Dann existiert ein* $J \in \mathbb{N}$ *sodass*

$$\begin{aligned} & |x_j - x| < \frac{|x|}{2} \text{ und } \frac{|x|}{2} > \frac{1}{j} \text{ für alle } j \geq J \\ \implies \quad & x_j \neq \frac{1}{j} \text{ für alle } j \geq J \\ \implies \quad & \liminf_{j\to\infty} \mathcal{F}_j(x_j) \geq 0 = \mathcal{F}(x). \end{aligned}$$

Sei $x = 0$ *und sei* (x_j) *mit* $x_j \to x$ *gegeben. Da* $\mathcal{F}_j(x_j) \geq -1$ *für alle* $j \in \mathbb{N}$ *gilt*

$$\liminf_{j\to\infty} \mathcal{F}_j(x_j) \geq -1 = \mathcal{F}(x).$$

(b) *(*lim sup-*Ungleichung) Sei* $x \neq 0$. *Setze* $x_j = x \neq 0$, *dann gilt* $\limsup_{j\to\infty} \mathcal{F}_j(x) = 0 = \mathcal{F}(x)$.
Sei nun $x = 0$. *Wähle* $x_j = \frac{1}{j}$, *dann konvergiert* x_j *gegen* 0 *und*

$$\limsup_{j\to\infty} \mathcal{F}_j(x_j) = \limsup_{j\to\infty} -1 = -1 = \mathcal{F}(x).$$

2. *Eine Variation des Beispieles zeigt, dass*

$$\mathcal{F}_j(x) = \begin{cases} -1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

gegen

$$\mathcal{F}(x) = \begin{cases} -1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

Γ-*konvergiert.*

3. *Betrachte nun*

$$\mathcal{F}_j(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Es gilt $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$ *mit* $\mathcal{F} \equiv 0$, *denn*

(a) *(*$\liminf$*-Ungleichung) Sei* $x \in \mathbb{R}$, *und* $x_j \to x$, *dann ist* $\liminf_{j\to\infty} \mathcal{F}_j(x_j) \geq 0 = \mathcal{F}(x)$

(b) *(*$\limsup$*-Ungleichung) Sei* $x \neq 0$. *Für* $x_j = x$ *gilt* $x_j \to x$ *und* $\lim_{j\to\infty} \mathcal{F}(x_j) = 0 = \mathcal{F}(x)$. *Sei* $x = 0$. *Für* $x_j = \frac{1}{j}$ *gilt* $x_j \to x$ *und*

$$\limsup_{j\to\infty} \mathcal{F}_j(x_j) = 0 = \mathcal{F}(x).$$

Remark 4.3. *Es gilt im Allgemeinen nicht*

- $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F} \quad \Longrightarrow \quad (-\mathcal{F}_j) \xrightarrow{\Gamma} -\mathcal{F}$,
- $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$, $\mathcal{G}_j \xrightarrow{\Gamma} \mathcal{G} \quad \Longrightarrow \quad \mathcal{F}_j + \mathcal{G}_j \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}$.

Theorem 4.4. *Seien* $\mathcal{F}, \mathcal{G}, \mathcal{F}_j, \mathcal{G}_j : X \to \mathbb{R} \cup \{\pm\infty\}$. *Es gelte* $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$ *und* $\mathcal{G}$ *konvergiere stetig zu* $\mathcal{G}$, *das heißt*

$$\forall x, (x_j) \to x : \quad \lim_{j\to\infty} \mathcal{G}_j(x_j) = \mathcal{G}(x).$$

Dann gilt

$$\mathcal{F}_j + \mathcal{G}_j \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}.$$

Beweis. ($\liminf$-Ungleichung) Fixiere $x \in X$. Sei $(x_j) \subset X$ mit $x_j \to x$. Dann gilt

$$\begin{aligned} \liminf_{j\to\infty}(\mathcal{F}(x_j) + \mathcal{G}_j(x_j)) &\geq \liminf_{j\to\infty} \mathcal{F}_j(x_j) + \lim_{j\to\infty} \mathcal{G}_j(x_j) \\ &\geq \mathcal{F}(x) + \mathcal{G}(x). \end{aligned}$$

(lim sup-Ungleichung) Fixiere $x \in X$. Wegen $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$ existiert eine Folge $(\tilde{x}_j)$ mit $\tilde{x}_j \to x$ und $\limsup_{j\to\infty} \mathcal{F}(x_j) \leq \mathcal{F}(x)$. Dann

$$\begin{aligned}\limsup_{j\to\infty}(\mathcal{F}_j(\tilde{x}_j) + \mathcal{G}_j(\tilde{x}_j)) &\leq \limsup_{j\to\infty} \mathcal{F}_j(\tilde{x}_j) + \lim_{j\to\infty} \mathcal{G}_j(\tilde{x}_j)) \\ &\leq \mathcal{F}(x) + \mathcal{G}(x).\end{aligned}$$

□

Definition 4.5. *Sei* $\mathcal{F}_j : X \to \mathbb{R} \cup \{\pm\infty\}$. *Die Folge* $(\mathcal{F}_j)_{j\in\mathbb{N}}$ *heißt gleichgradig mildly koerziv (englisch: equi-mildly coercive), wenn es eine nichtleere, kompakte Menge* $K \subseteq X$ *gibt, sodass für alle* $j \in \mathbb{N}$

$$\inf_{x\in K} \mathcal{F}_j(x) = \inf_{x\in X} \mathcal{F}_j(x).$$

Theorem 4.6. *Sei* $\mathcal{F}_j, \mathcal{F} : X \to \mathbb{R}\cup\{\pm\infty\}$, $(\mathcal{F}_j)$ *gleichgradig mildly koerziv und* $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$. *Dann existiert ein* $x_0 \in X$, *sodass*

$$\mathcal{F}(x_0) = \min_{x\in X} \mathcal{F}(x) = \lim_{j\to\infty} \inf_{x\in X} \mathcal{F}_j(x).$$

Sei (x_j) *eine Folge von fast Minimierern (englisch: almost minimizers), das heißt*

$$\mathcal{F}_j(x_j) \leq \inf_{x\in X} \mathcal{F}_j(x) + o(1).$$

Dann besitzt jede Teilfolge von (x_j) *eine konvergente Teilfolge und jeder Häufungspunkt minimiert* $\mathcal{F}$.

Beweis. Wähle $(x_j) \subseteq K$, so dass

$$\liminf_{j\to\infty} \mathcal{F}_j(x_j) = \liminf_{j\to\infty} \inf_{x\in X} \mathcal{F}_j(x).$$

Wähle eine Teilfolge (x_{j_k}), sodass $x_{j_k} \to \bar{x} \in X$ und

$$\lim_{k\to\infty} \mathcal{F}_{j_k}(x_{j_k}) = \liminf_{j\to\infty} \mathcal{F}_j(x_j).$$

Sei

$$\tilde{x}_j = \begin{cases} x_{j_k}, & j = j_k \\ \bar{x} & j \notin \{j_k\}_k. \end{cases}$$

Dann gilt $\tilde{x}_j \to \bar{x}$,

$$\begin{aligned}\inf_{x\in X} \mathcal{F}(x) \leq \mathcal{F}(\bar{x}) &\leq \liminf_{j\to\infty} \mathcal{F}_j(\tilde{x}_j) \\ &\leq \lim_{j\to\infty} \mathcal{F}_{j_k}(x_{j_k}) \\ &= \liminf_{j\to\infty} \inf_{x\in X} \mathcal{F}_j(x) \\ &\leq \limsup_{j\to\infty} \inf_{x\in X} \mathcal{F}_j(x) \\ &\overset{(*)}{\leq} \inf_{x\in X} \mathcal{F}(x).\end{aligned}$$

Begründung für die letzte Ungleichung (*): Sei $\inf_{x\in X} \mathcal{F}(x) \in \mathbb{R}$. Für jedes $\delta > 0$ existiert $\hat{x}^\delta \in X$, sodass

$$\mathcal{F}(\hat{x}^\delta) \le \inf_{x\in X} \mathcal{F}(x) + \delta.$$

Aus der lim sup-Ungleichung folgt die Existenz von $(\hat{x}_j^\delta)$ mit $\hat{x}_j^\delta \to \hat{x}^\delta$ und

$$\limsup_{j\to\infty} \inf_{x\in X} \mathcal{F}_j(x) \le \limsup_{j\to\infty} \mathcal{F}_j(\hat{x}_j^\delta) \le \mathcal{F}(\hat{x}^\delta) \le \inf_{x\in X} \mathcal{F}(x) + \delta.$$

Ungleichung (*) folgt aus der Beliebigkeit von $\delta > 0$.
Sei $\inf_{x\in X} \mathcal{F}(x) = -\infty$. Für jedes $k \in \mathbb{N}$ existiert $\hat{x}^k \in X$, so dass

$$\mathcal{F}(\hat{x}^k) \le -k.$$

Aus der lim sup-Ungleichung folgt die Existenz von $(\hat{x}_j^k)$ mit $\hat{x}_j^k \to \hat{x}^k$ und

$$\limsup_{j\to\infty} \inf_{x\in X} \mathcal{F}_j(x) \le \limsup_{j\to\infty} \mathcal{F}_j(\hat{x}_j^k) \le \mathcal{F}(\hat{x}^k) \le -k.$$

Ungleichung (*) folgt aus der Beliebigkeit von $k \in \mathbb{N}$. □

Example 4.7. *1. Sei $\mathcal{F}_j(x) = \sin(jx)$. Es gilt $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$ wobei $\mathcal{F} \equiv -1$.*

2. *Sei $\mathcal{G}_j(x) = \sin(jx) + x^2 + 1$. Es gilt $\mathcal{G}_j \xrightarrow{\Gamma} \mathcal{G}$ mit $\mathcal{G}(x) = x^2$. Beachte, dass $\mathcal{G}_j$ abzählbar viele lokale Minima hat und $\mathcal{G}$ nur ein lokales (und gleichzeitig globales) Minimum hat.*

3. *Sei $\mathcal{F}_j(x) = (-1)^j \mathcal{F}(x)$ und $\mathcal{F}(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$. Die Folge $(\mathcal{F}_j)$ hat keinen Γ-Grenzwert.*

Für die Berechnung von Γ-limiten sind folgende Eigenschaften nützlich:

Theorem 4.8. *Sei $\mathcal{F}, \mathcal{F}_j : X \to \mathbb{R} \cup \{\pm\infty\}$. Aus $\mathcal{F}_j \xrightarrow{\Gamma} \mathcal{F}$ folgt die Unterhalbstetigkeit von $\mathcal{F}$.*

Example 4.9. *(Homogenisierung) In diesem Beispiel wollen wir verdeutlichen, dass der Γ-Grenzwert von der Wahl der Metrik abhängig ist.*
Sei $a : \mathbb{R} \to [\alpha, \beta]$ mit $0 < \alpha < \beta < +\infty$ 1-periodisch. Für $\ell > 0$ definieren $\mathcal{F}_j^\ell : H^1(0,1) \to [0, +\infty]$ durch

$$\mathcal{F}_j^\ell(u) = \begin{cases} \int_0^1 a(jx)(u'(x))^2\,dx & \text{falls } u(0) = 0 \text{ und } u(1) = \ell, \\ +\infty & \text{sonst.} \end{cases}$$

Es bezeichne

$$\overline{a} = \int_0^1 a(y)\,dy$$

das arithmetische Mittel und

$$\underline{a} := \left(\int_0^1 \frac{1}{a(y)}\, dy \right)^{-1}$$

das harmonische Mittel. Beachte: Es gilt $\underline{a} \leq \overline{a}$ und $\underline{a} = \overline{a}$ genau dann, wenn a konstant ist. Setze

$$\mathcal{F}^\ell(u) = \begin{cases} \overline{a} \int_0^1 (u'(x))^2\, dx & \textit{falls } u(0) = 0 \textit{ und } u(1) = \ell, \\ +\infty & \textit{sonst,} \end{cases}$$

und

$$\hat{\mathcal{F}}^\ell(u) = \begin{cases} \underline{a} \int_0^1 (u'(x))^2\, dx & \textit{falls } u(0) = 0 \textit{ und } u(1) = \ell, \\ +\infty & \textit{sonst.} \end{cases}$$

Wir untersuchen Γ-konvergenz bzgl. starker und schwacher Konvergenz in $H^1(0,1)$. Beachte: die schwache Topologie in $H^1(0,1)$ ist auf beschränkten Mengen metrisierbar. Es gilt:

1. *$\mathcal{F}_j^\ell \xrightarrow{\Gamma} \mathcal{F}^\ell$ bezüglich starker Konvergenz in $H^1(0,1)$. Dies folgt direkt aus $a_j \overset{*}{\rightharpoonup} \overline{a}$ in $L^\infty(0,1)$.*

2. *$\mathcal{F}^\ell$ ist nicht gleichgradig mild koerziv bzgl. starker Konvergenz in $H^1(0,1)$. (Argument siehe 8. unten).*

3. *$\mathcal{F}_j^\ell \xrightarrow{\Gamma} \hat{\mathcal{F}}^\ell$ bezüglich schwacher Konvergenz in $H^1(0,1)$. Der Beweis dieser Aussage erfordert mehr Arbeit und folgt aus späteren Vorträgen zur Homogenisierung.*

4. *Die Folge $\mathcal{F}_j^\ell$ ist gleichgradig mild koerziv bzgl. schwacher Konvergenz in $H^1(0,1)$. Argument: Setze $\hat{u}(x) := \ell x$. Die direkte Method zeigt, dass $u_j \in H^1(0,1)$ existiert mit $\mathcal{F}_j^\ell(u_j) = \min \mathcal{F}_j^\ell$. Es gilt nun*

$$\frac{\alpha}{2} \|u_j\|^2_{H^1((0,1))} \leq \alpha \int_0^1 |u_j'|^2\, dx \leq \mathcal{F}_j^\ell(u_j) \leq \mathcal{F}_j^\ell(\hat{u}) \leq \beta \ell^2.$$

   Also liegt u_j in der Menge

$$K := \{u \in H^1(0,1) \,:\, u(0) = u(1) - \ell = 0,\ \|u\|^2_{H^1(0,1)} \leq \frac{2}{\alpha} \beta \ell^2\}.$$

   Diese Menge ist abgeschlossen, beschränkt und konvez, also kompakt bzgl. schwacher Konvergenz in $H^1(0,1)$.

5. *Man kann zeigen: $\mathcal{F}_j^\ell$ und $\hat{\mathcal{F}}^\ell$ besitzen eindeutige Minimierer. Aus 3. & 4. und Satz 4.6 folgt nun: Minimiert u_j^ℓ (resp. $\hat{u}^\ell$) $\mathcal{F}_j^\ell$ (resp. $\hat{\mathcal{F}}^\ell$), so gilt $u_j^\ell \rightharpoonup \hat{u}^\ell$ schwach in $H^1(0,1)$.*

6. *Der (eindeutige) Minimierer von $\mathcal{F}_j^\ell$ ist explizit gegeben durch:*

$$u_j^\ell(x) = \ell\frac{\underline{a}}{j}\int_0^{jx}\frac{1}{a(y)}\,dy.$$

Begründung: $u_j^\ell(0) = 0$, $u_j^\ell(1) = \ell\frac{\underline{a}}{j}\int_0^j \frac{1}{a(y)}\,dy = \ell\underline{a}\underline{a}^{-1} = \ell$,

$$\begin{aligned}
&\mathcal{F}_j^\ell(u_j^\ell) = \ell^2\underline{a}^2\int_0^1 \frac{1}{a(jx)}\,dx = \underline{a}\ell^2 \quad \textit{und}\\
&\ell^2 = \left(\int_0^1 u'(x)\,dx\right)^2 \leq \left(\int_0^1 \frac{dx}{a(jx)}\right)\left(\int_0^1 a(jx)u'(x)^2\,dx\right)\\
&\quad\Rightarrow \quad \ell^2\left(\int_0^1 \frac{dx}{a(jx)}\right)^{-1} = \ell^2\underline{a} = \mathcal{F}_j(u_j^\ell) \leq \inf_{H^1(0,1)} \mathcal{F}_j^\ell.
\end{aligned}$$

7. *Aus dem Satz von Riemann Lebesgue folgt, dass $u_j^\ell \rightharpoonup \hat{u}^\ell$ schwach in $H^1(0,1)$ mit $\hat{u}^\ell(x) := \ell x$.*

8. *Falls a nicht konstant ist, so konvergiert u_j^ℓ nicht stark gegen $\hat{u}^\ell$ in $H^1(0,1)$. Begründung:*

$$\liminf_{j\to\infty} \|(u_j^\ell)' - (\hat{u}^\ell)'\|_{L^2(0,1)}^2 = \liminf_{j\to\infty} \ell^2\int_0^1 |\underline{a}\frac{1}{a(jx)} - 1|^2\,dx > 0.$$

9. *Falls a nicht konstant ist, so gilt $\overline{a} > \underline{a}$ und*

$$\min_u \mathcal{F}^\ell(u) = \overline{a}\ell^2 > \underline{a}\ell^2 = \min_u \hat{\mathcal{F}}^\ell(u) = \min_u \mathcal{F}_j^\ell(u) \textit{ für alle } j \in \mathbb{N}.$$

4.2 From nonlinear to linearized Elasticity

4.2.1 Formal derivation

We consider the variational problem of nonlinear elasticity

$$I(\varphi) = \int_\Omega W(D\varphi)\,\mathrm{d}x - \int_\Omega f\cdot\varphi\,\mathrm{d}x.$$

In situations when

- the reference configuration is a natural state, i.e. $W(\mathbf{Id}) \leq W(F)$ for all $F \in GL_+(d)$,
- the deformation φ is close to identity at the boundary, i.e. $\varphi(x) = x + \varepsilon u_0(x)$ for some given boundary displacement $u_0 : \partial\Omega \to \mathbb{R}^d$ and a small scaling parameter $\varepsilon > 0$,
- the volume force is small, i.e. $f = \varepsilon\tilde{f}$,

it is natural to expect that a minimizer φ_* of I takes the form

$$\varphi_*(x) = x + \varepsilon u(x) \tag{4.1}$$

with displacement $u : \Omega \to \mathbb{R}^d$ satisfying the boundary condition $u = u_0$ on $\partial\Omega$. If plug the expansion (4.1) into I, we formally get by a Taylor expanstion

$$\begin{aligned}
I(\varphi_*) &= \int_\Omega W(\mathbf{Id} + \varepsilon Du(x)) - \varepsilon\tilde{f} \cdot (x + \varepsilon u(x))\,\mathrm{d}x \\
&= \int_\Omega \Big(W(\mathbf{Id}) + \varepsilon \nabla W(\mathbf{Id}) \cdot Du(x) + \varepsilon^2 \frac{1}{2} D^2 W(\mathbf{Id})[Du, Du]\Big) + o(\varepsilon^2)\,\mathrm{d}x \\
&\quad - \int_\Omega \varepsilon \tilde{f} \cdot (x + \varepsilon u(x))\,\mathrm{d}x.
\end{aligned}$$

Thanks to the natural state assumption, we have $\nabla W(\mathbf{Id}) = 0$, and thus (up to terms of higher order),

$$\begin{aligned}
I(\varphi_*) &= \int_\Omega W(\mathbf{Id}) - \varepsilon \tilde{f} \cdot x\,\mathrm{d}x \\
&\quad + \varepsilon^2 \left(\frac{1}{2}\int_\Omega Q(Du) \; - \; \tilde{f} \cdot u\,\mathrm{d}x\right) + o(\varepsilon^2).
\end{aligned}$$

Here and below we use the short hand notation

$$Q(F) := D^2 W(\mathbf{Id})[F, F]$$

to denote the quadratic form associated with the Hessian of W at identity. Since the first integral on the right-hand is independent of u, we expect that for ε small, the displacement u in the expansion (4.1) is close to the minimizer to

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega Q(Du) \; - \; \tilde{f} \cdot u\,\mathrm{d}x \qquad \text{subject to } u = u_0 \text{ on } \partial\Omega. \tag{4.2}$$

This is the variational problem of linearized elasticity. Since Q is the quadratic form in the Taylor expansion of W around a minimum, Q is positive semidefinit and thus convex – which is a genuine simplification compared to the nonconvex stored energy function W used in nonlinear elasticity.
Note that we might represent Q with help of 4th order tensor $\mathbb{L}$ (cf. Problem Sheet 5, Problem 20):

$$Q(F) = F \cdot \mathbb{L}F, \qquad \mathbb{L}_{ij\alpha\beta} := \frac{\partial^2}{\partial F_{ij} \partial F_{\alpha\beta}} W(\mathbf{Id}).$$

We gather some properties of Q:

Lemma 4.10. *Suppose $W \in C^2(GL_+(d))$ is*

- *frame indifferent*
- *minimized at identity and $W(\mathbf{Id}) = 0$.*

Then for all $A \in \mathbb{R}^{d\times d}$ *we have*

$$W(\mathbf{Id}+\varepsilon A) = \varepsilon^2 \frac{1}{2} Q(A) + o(\varepsilon^2) \tag{4.3}$$

$$Q(A) \geq 0 \tag{4.4}$$

$$Q(\operatorname{skew} A) = 0 \tag{4.5}$$

Moreover, if W *is non-degenerate in the sense of* $W(F) \geq c \operatorname{dist}^2(F, SO(d))$ *(for some* $c > 0$*), then there exists* $c' > 0$ *s.t.*

$$Q(A) \geq c' |\operatorname{sym} A|^2. \tag{4.6}$$

Beweis. Argument for (4.3): Taylor expanstion.
Argument for (4.4): Since $W \geq 0$, we have

$$Q(A) = 2 \lim_{\varepsilon\downarrow 0} \frac{W(\mathbf{Id}+\varepsilon A)}{\varepsilon^2} \geq 0.$$

Argument for (4.5): By Polar factorization we have $\mathbf{Id}+\varepsilon A = R\sqrt{\mathbf{Id}+2\varepsilon \operatorname{sym} A + \varepsilon^2 A^t A}$. By Taylor expansion we have

$$\sqrt{\mathbf{Id}+2\varepsilon \operatorname{sym} A + \varepsilon^2 A^t A} = \mathbf{Id} + \varepsilon(\operatorname{sym} A + G_\varepsilon)$$

for some $G_\varepsilon \in \mathbb{R}^{d\times d}$ with $|G_\varepsilon| \to 0$ as $\varepsilon \downarrow 0$. Hence, by frame indifference we get

$$\varepsilon^{-2} W(\mathbf{Id}+\varepsilon A) = \varepsilon^{-2} W(\mathbf{Id}+\varepsilon(\operatorname{sym} A + G_\varepsilon)) = Q(\operatorname{sym} A + G_\varepsilon) \to Q(\operatorname{sym} A)$$

as $\varepsilon \downarrow 0$, and thus $Q(A) = Q(\operatorname{sym} A)$. In particular, $Q(\operatorname{skew} A) = Q(0) = 0$.
Argument for (4.6): Since $Q(-A) = (-1)^2 Q(A) = Q(A) = Q(\operatorname{sym} A)$, we may assume without loss of generality that A is symmetric and positive definite, and thus admits a spectral decomposition. Let $\lambda > 0$ denote the largest Eigenvalue of A and e_λ a normalized Eigenvector. We have

$$\begin{aligned}
\operatorname{dist}^2(\mathbf{Id}+\varepsilon A, SO(d)) &= \min_{R\in SO(d)} |\mathbf{Id}+\varepsilon A - R|^2 \\
&= \min_{R\in SO(d)} \max_{e\in\mathbb{R}^d : |e|=1} |e + \varepsilon A e - R e|^2 \\
&\geq \min_{R\in SO(d)} |e_\lambda + \varepsilon A e_\lambda - R e_\lambda|^2.
\end{aligned}$$

Note that

$$R e_\lambda = (R e_\lambda \cdot e_\lambda) e_\lambda + r_\lambda^\perp$$

defines an orthogonal decomposition. Since R is a rotation we deduce that $\beta := R e_\lambda \cdot e_\lambda \in [-1, 1]$. By $A e_\lambda = \lambda e_\lambda$ and Pythogaras' theorem we get

$$|e_\lambda + \varepsilon A e_\lambda - R e_\lambda|^2 = |(1+\varepsilon\lambda - \beta) e_\lambda|^2 + |r_\lambda^\perp|^2 \geq |(1+\varepsilon\lambda-\beta)e_\lambda|^2 \geq \varepsilon^2 \lambda^2,$$

and thus

$$\operatorname{dist}^2(\mathbf{Id}+\varepsilon A, SO(d)) \geq \varepsilon^2\lambda^2 = \varepsilon^2 |A|^2 = \varepsilon^2 |\operatorname{sym} A|^2.$$

We conclude

$$Q(A) = \lim_{\varepsilon\downarrow 0} \frac{W(\mathbf{Id}+\varepsilon A)}{\varepsilon^2} \geq c \lim_{\varepsilon\downarrow 0} \frac{\operatorname{dist}^2(\mathbf{Id}+\varepsilon A)}{\varepsilon^2} \geq c |\operatorname{sym} A|^2.$$

□

From Lemma 4.10 we learn the integrand in (4.2) is convex, yet it is not positive definite as required for the existence result Corollary 3.19. The integrand $Q(F)$ only controls the symmetric part of F. As we shall see, this enough to prove existence of a minimizer to (4.2).

Theorem 4.11 (Existence of minimizers for linearized elasticity)**.** *Let Q be as in Lemma 4.10 and suppose that $Q(A) \geq c|\operatorname{sym} A|^2$ for some $c > 0$. Then for all $f \in L^2(\Omega, \mathbb{R}^d)$ the functional $\mathcal{E}$ in* (4.2) *admits a unique minimizer in $W_0^{1,2}(\Omega, \mathbb{R}^d)$.*

Since $A \mapsto Q(A)$ is convex, $\mathcal{E}$ is lower semicontinuous w.r.t. weak convergence in $W^{1,2}(\Omega, \mathbb{R}^d)$. Hence, existence of a minimizer follows by the direct method (see Corollary 3.19 for details) provided we can show that

$$\|u\|_{W^{1,2}(\Omega,\mathbb{R}^d)} \leq C(\mathcal{E}(u) + 1) \tag{4.7}$$

for all $u \in W_0^{1,2}(\Omega, \mathbb{R}^d)$. For (4.7) we need the following estimate:

Lemma 4.12 (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be open. Then for all $u \in W_0^{1,2}(\Omega, \mathbb{R}^d)$ one has*

$$\int_\Omega |Du|^2 \,\mathrm{d}x \leq 2 \int_\Omega |\operatorname{sym} Du|^2 \,\mathrm{d}x.$$

Beweis. By an approximation argument, it suffices to consider $u \in C_c^\infty(\Omega; \mathbb{R}^d)$. We have

$$\int_\Omega |\operatorname{sym} Du|^2 \, dx = \frac{1}{4} \int_\Omega |Du + Du^t|^2 \, dx = \frac{1}{2} \int_\Omega |Du|^2 + Du \cdot Du^t \, dx.$$

Note that two integrations by parts yield

$$\int_\Omega Du \cdot Du^t = \sum_{i,j=1}^d \int_\Omega \partial_i u_j \partial_j u_i \, dx = \sum_{i,j=1}^d \int_\Omega \partial_j u_j \partial_i u_i \, dx = \int_\Omega (\operatorname{div} u)^2 \, dx.$$

Hence,

$$\int_\Omega |\operatorname{sym} Du|^2 \, dx = \frac{1}{2} \int_\Omega |Du|^2 + (\operatorname{div} u)^2 \, dx \geq \frac{1}{2} \int_\Omega |Du|^2 \, dx.$$

□

Proof of Theorem 4.11. The proof of existence is similar to the proof of Corollary 3.19. We only need to use Korn's inequality in Step 1.
Argument for uniqueness[7]: Since Q is a quadratic form, the parallelogramm identity yields

$$\int_\Omega Q(\frac{1}{2}(Du + Dv)) + Q(\frac{1}{2}(Du - Dv)) \, dx = \frac{1}{2} \int_\Omega Q(Du) + Q(Dv) \, dx. \tag{4.8}$$

Now, suppose that u and v are minimizers of $\mathcal{E}$. Since the functional $\mathcal{E}$ and the space $W_0^{1,2}(\Omega, \mathbb{R}^d)$ are convex, we deduce that

$$\mathcal{E}(\frac{u + v}{2}) \leq \frac{1}{2}(\mathcal{E}(u) + \mathcal{E}(v)) = \inf \mathcal{E},$$

[7] Not discussed in class

and thus $\mathcal{E}(\frac{u+v}{2}) = \mathcal{E}(u) = \mathcal{E}(v) = \inf \mathcal{E}$. Combined with (4.8) we get

$$\int_\Omega Q(\frac{1}{2}(Du - Dv))\, dx = \frac{1}{2}(\mathcal{E}(u) + \mathcal{E}(v)) - \mathcal{E}(\frac{u+v}{2}) = 0.$$

Hence, $\int_\Omega |\operatorname{sym}(Du - Dv)|^2\, dx = 0$. Since $u - v \in W_0^{1,2}(\Omega, \mathbb{R}^d)$, Korn's inequality yields $u - v = 0$. □

4.2.2 Rigorous derivation

In this section we give a rigorous derivation of linear elasticity from nonlinear elasticity. The content of this section is based on the two papers:

[FJM02] Friesecke G, James RD, Müller S: Ä Theorem on Geometric Rigidity and the Derivation of Nonlinear Plate Theory from Three-Dimensional Elasticity". Commun. Pure Appl. Math. 2002

[DNP02] Dal Maso, Gianni, Matteo Negri, and Danilo Percivale. "Linearized elasticity as Γ-limit of finite elasticity.SSet-Valued Analysis 10.2-3 (2002): 165-183.

General assumptions for this section are:
$W \in C(\mathbb{R}^{d\times d})$ satisfies

$$\forall F \in \mathbb{R}^{d\times d}, R \in SO(d) : \qquad W(RF) = W(F) \qquad \text{(frame-indifference)} \tag{A1}$$

$$W(Id) = \min W \geq 0 \qquad \text{(stress free reference)} \tag{A2}$$

$$W(F) \geq c \operatorname{dist}^2(F, SO(d)) \qquad \text{(non-degeneracy)} \tag{A3}$$

$$W \text{ is } C^3 \text{ in a neighbourhood of } SO(d) \tag{A4}$$

Q denotes the quadratic term in the Taylor expansion of W at identity:

$$Q(G) := \frac{1}{2}\frac{\delta^2 W(Id)}{\delta F \delta F}[G, G].$$

We consider the elastic energy functional

$$\mathcal{I}_\varepsilon : H^1(\Omega, \mathbb{R}^d) \to [0, +\infty], \qquad \mathcal{I}_\varepsilon(\varphi) := \begin{cases} \int_\Omega W(D\varphi(x)) - \int_\Omega f_\varepsilon \cdot \varphi\, dx & \text{if } \varphi - \mathrm{id} \in H_0^1(\Omega) \\ +\infty & \text{else.} \end{cases}$$

Above,

- $\Omega \subset \mathbb{R}^d$ denotes a Lipschitz domain,
- $\mathrm{id}(x) := x$ denotes the identity map,
- $(f_\varepsilon) \subset L^2(\Omega; \mathbb{R}^d)$ denotes a sequence of body

We make the assumption that the forces are moment-free, i.e.,

$$\int_\Omega f_\varepsilon(x) \cdot x = 0, \tag{A5}$$

and small in the sense that

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\|f_\varepsilon\|_{L^2(\Omega)} < \infty. \tag{A6}$$

The boundary condition (encoded in the definition of $\mathcal{I}_\varepsilon$) means that at the boundary the displacement is zero. Thus, since forces are small, we expect that for $0 < \varepsilon \ll 1$ the minimizer of $\mathcal{I}_\varepsilon$ is "close to the identity map". A first goal of our analysis is to make this statement rigorous and quantitative:

Theorem 4.13 (A priori estimate)**.** *There exists a constant C such that for all $\varepsilon > 0$ and (scaled) displacements $u \in H^1_0(\Omega;\mathbb{R}^d)$, we have*

$$\int_\Omega |Du|^2 \le C\Big(\frac{1}{\varepsilon^2}\mathcal{I}_\varepsilon(\mathrm{id}+\varepsilon u) + \frac{1}{\varepsilon^2}\int_\Omega |f_\varepsilon|^2\Big)$$

where $\mathrm{id}:\Omega\to\mathbb{R}^d$, $\mathrm{id}(x) := x$, *denotes the identity map.*

Motivated by this theorem we introduce the rescaled energy functional

$$\mathcal{E}_\varepsilon : H^1_0(\Omega,\mathbb{R}^d) \to [0,+\infty], \qquad \mathcal{E}_\varepsilon(u) := \frac{1}{\varepsilon^2}\mathcal{I}_\varepsilon(\mathrm{id}+\varepsilon u).$$

We show that for $0 < \varepsilon \ll 1$ the behavior of $\mathcal{E}_\varepsilon$ is captured by the energy functional of linear elasticity,

$$\mathcal{E}_0 : H^1_0(\Omega,\mathbb{R}^d) \to \mathbb{R}, \qquad \mathcal{E}_0(u) := \int_\Omega Q(\operatorname{sym} Du(x))\,dx - \int_\Omega f_0\cdot u.$$

Theorem 4.14. *Suppose that $\frac{1}{\varepsilon}f_\varepsilon \to f_0$ in $L^2(\Omega)$. Let (u_ε) denote a sequence of (almost) minimizers of $\mathcal{E}_\varepsilon$, i.e.,*

$$\mathcal{E}_\varepsilon(u_\varepsilon) \le \inf \mathcal{E}_\varepsilon + \varepsilon,$$

and let u_0 denote a minimizer of $\mathcal{E}_0$ (which exists and is unique thanks to Theorem 3.33). Then

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H^1_0(\Omega;\mathbb{R}^d),$$

and

$$\inf \mathcal{E}_\varepsilon \to \min \mathcal{E}_0.$$

The proof of these two theorems is divided in three sections on i) the a priori estimate, ii) a Γ-convergence statement and iii) the conclusion of Theorem 4.14

4.2.3 Geometric rigidity - Proof of Theorem 4.13

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The argument is based on non-trivial, non-linear version of Korn's inequality — the so called *geometric rigidity estimate* established in [FJM02]:

Theorem 4.15. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then there exists a constant $C = C(\Omega)$ such that for all $\varphi \in H^1(\Omega;\mathbb{R}^d)$ we have*

$$\min_{R\in SO(d)} \int_\Omega |\nabla\varphi(x) - R|^2\,dx \le C\int_\Omega \operatorname{dist}^2(\nabla\varphi(x), SO(d))\,dx.$$

For a proof see [FJM02].

Lemma 4.16 (Geometric rigidity with no-displacement boundary condition)**.** *Suppose that* $\varphi \in H^1(\Omega;\mathbb{R}^d)$ *and* $\varphi - \mathrm{id} \in H_0^1(\Omega;\mathbb{R}^d)$*. Then there exists* $C = C(\Omega)$ *s.t.*

$$\int_\Omega |\nabla\varphi(x) - Id|^2 \le C \int_\Omega \mathrm{dist}^2(\nabla\varphi(x), SO(d))\,dx.$$

Beweis. We write $A \lesssim B$ if $A \le CB$ for some constant C that (eventually) can be chosen only depending on Ω.
Schritt 1: We claim that

$$|F|_{\partial\Omega} := \min_{c\in\mathbb{R}^d}\Big(\int_{\partial\Omega} |Fx - c|^2\,dx\Big)^{\frac{1}{2}}$$

defines a norm on $\mathbb{R}^{d\times d}$.
It is easy to check that $|\cdot|_{\partial\Omega}$ defines a semi-norm on $\mathbb{R}^{d\times d}$ and that we have $|F|_{\partial\Omega} \lesssim |F|$. We only show that $|\cdot|_{\partial\Omega} \gtrsim |F|$. Suppose the opposite, then there exists a sequence F_k s.t.

$$|F_k| > k|F_k|_{\partial\Omega}.$$

W.l.o.g. we may assume that $|F_k| = 1$ and (since the unit ball in $\mathbb{R}^{d\times d}$ is compact) that $F_k \to F_0$ with $|F_0| = 1$. It is easy to check that

$$|F_k|_{\partial\Omega} = \int_{\partial\Omega} |F_kx - c_k|^2 \qquad \text{with } c_k := \fint_{\partial\Omega} F_kx\,d\mathcal{H}^{d-1}(x).$$

By continuity we have $F_kx - c_k \to F_0x - c_0$ strongly in $L^2(\partial\Omega)$, and thus

$$|F_0|_{\partial\Omega} = \lim_{k\to\infty} |F_k|_{\partial\Omega} \le \limsup_{k\to\infty} \frac{1}{k}|F_k| = 0.$$

Thus, $F_0x = c_0$ a.e. in $\partial\Omega$ and by continuity for all $x \in \partial\Omega$. Since $\partial\Omega$ is the boundary of a domain in $\mathbb{R}^d$, we have $\mathrm{span}(\partial\Omega) = \mathbb{R}^d$. Hence, $F_0x = c_0$ for all $x \in \mathbb{R}^d$. This is only possible, if $F_0 = 0$ and c_0. A contradiction to $|F_0| = 1$.
Schritt 2: We claim that for all $R \in \mathbb{R}^{d\times d}$ and $u \in H_0^1(\Omega;\mathbb{R}^d)$ we have

$$|Id - R|^2 \lesssim \int_\Omega |(Id + Du(x)) - R|^2.$$

Argument: Set $c := \fint_\Omega \mathrm{id}(x) + u(x) - Rx\,dx$. We have

$$\begin{aligned} |Id - R|^2 \lesssim |Id - R|^2_{\partial\Omega} &\overset{Step 1}{\le} \int_{\partial\Omega} |\mathrm{id}(x) + u(x) - Rx - c|^2 \\ &\overset{\text{trace-estimate}}{\lesssim} \int_\Omega |\mathrm{id} + u - R - c|^2 + \int_\Omega |(Id + Du) - R|^2 \\ &\overset{\text{Poincaré}}{\lesssim} \int_\Omega |(Id + Du) - R|^2. \end{aligned}$$

Schritt 3: Conclusion. Set $u(x) := \varphi(x) - x$ and note that $u \in H^1_0(\Omega;\mathbb{R}^d)$. By the geometric rigidity estimate there exists a rotation R such that

$$\int_\Omega |D\varphi - R|^2 \lesssim \int_\Omega \operatorname{dist}^2(D\varphi, SO(d)).$$

Since $D\varphi = Id + Du$, we obtain with Step 2:

$$\begin{aligned}\int_\Omega |D\varphi - I|^2 &\lesssim \int_\Omega |D\varphi - R|^2 + |Id - R|^2 \\ &\lesssim \int_\Omega |Id + Du - R|^2 \\ &= \int_\Omega |D\varphi - R|^2 \lesssim \int_\Omega \operatorname{dist}^2(D\varphi, SO(d)).\end{aligned}$$

□

Proof of Theorem 4.13. Set $\varphi(x) = x + \varepsilon u(x)$. Then $\varphi - \mathrm{id} \in H^1_0(\Omega)$, and the previous lemma yields

$$\begin{aligned}\varepsilon^2 \int_\Omega |Du|^2 &= \int_\Omega |D\varphi - Id|^2 \lesssim \int_\Omega \operatorname{dist}^2(D\varphi, SO(d)) \\ &\lesssim \int_\Omega W(D\varphi).\end{aligned}$$

Thus,

$$\int_\Omega |Du|^2 \lesssim \frac{1}{\varepsilon^2}\mathcal{I}_\varepsilon(\varphi) + \frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon \cdot \varphi \, dx.$$

With (A5) we find that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon \cdot \varphi \, dx &= \frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon \cdot (\varepsilon u) \, dx \\ &\leq \frac{1}{\varepsilon}\|f_\varepsilon\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\overset{\text{Poincare}}{\lesssim} \frac{1}{\varepsilon}\|f_\varepsilon\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)}.\end{aligned}$$

In combination,

$$\int_\Omega |Du|^2 \leq \frac{C}{\varepsilon^2}\mathcal{I}_\varepsilon(\varphi) + \frac{C}{\varepsilon}\|f_\varepsilon\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)}.$$

With Young's inequality, we may absorb the contribution of $\|Du\|_{L^2(\Omega)}$ in the second term into the left-hand side:

$$\int_\Omega |Du|^2 \lesssim \frac{1}{\varepsilon^2}(\mathcal{I}_\varepsilon(\varphi) + \|f_\varepsilon\|^2_{L^2(\Omega)}).$$

□

4.2.4 Γ-convergence of $\mathcal{E}_\varepsilon$

Lemma 4.17 (Taylor expansion)**.** *There exists a monotonically increasing function $r : [0,\infty) \to [0,+\infty]$ with $r(t) \to 0$ as $t \downarrow 0$, such that for all $G \in \mathbb{R}^{d\times d}$ we have*

$$|W(Id+G) - Q(G)| \leq |G|^2 r(|G|).$$

Beweis. By assumption there exists $\rho > 0$ such that W is C^3 in $\{Id+G\,:\,|G|<2\rho\}$. Fix $G \in \mathbb{R}^{d\times d}$ with $|G| = 1$ and set $f(s) := W(Id+sG)$. Note that $f \in C^3([-\rho,\rho])$. Thus, a Taylor expansion yields

$$\forall s \in [-\rho,\rho], \exists s' \in [-s,s] \;:, \qquad f(s) = f(0) + f'(0)s + \frac{1}{2}f''(0)s^2 + \frac{1}{6}f'''(s')s^3.$$

Since f is minimized at 0 (by (A2)), we have $f(0) = f'(0) = 0$. Further, by the chain rule, we have $\frac{1}{2}f''(0) = Q(G)$, and

$$\frac{1}{6}f'''(s') = \frac{1}{6}D^3W(Id+s'G)[G,G,G].$$

Note that since W is C^3, we have

$$C_W := \max\{\,|\frac{1}{6}f'''(s') = \frac{1}{6}D^3W(Id+s'G)[G,G,G]|\;:\;|s'|\leq\rho,\, G\in\mathbb{R}^{d\times d},\,|G|=1\,\} < \infty.$$

Now define

$$r : [0,\infty) \to [0,+\infty], \qquad r(s) := \begin{cases} C_W s & \text{if } 0 \leq s \leq \rho \\ +\infty & \text{else.}\end{cases}$$

Then $|f(s) - f''(0)s^2| \leq s^2 r(s)$, i.e.,

$$|W(Id+sG) - Q(sG)|^2 \leq s^2 r(|sG|).$$

Since r is independent of G, the estimate holds for all $G \in \mathbb{R}^{d\times d}$ with $|G| = 1$. For general $G \in \mathbb{R}^{d\times d}\setminus\{0\}$ apply the estimate with $s := |G|$ and $G \rightsquigarrow \frac{G}{|G|}$. □

Lemma 4.18 (Lower-bound)**.** *Suppose $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\Omega;\mathbb{R}^d)$. Then*

$$\liminf_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u_0).$$

Beweis. Schritt 1: (Truncation of peaks).
Define

$$\chi_\varepsilon(x) := \begin{cases} 1 & |Du_\varepsilon(x)| \leq \varepsilon^{-\frac{1}{2}} \\ 0 & \text{else.}\end{cases}$$

Claim

$$\chi_\varepsilon Du_\varepsilon \rightharpoonup Du_0 \text{ weakly in } L^2.$$

Argument: Since $|\chi_\varepsilon Du_\varepsilon| \leq |Du_\varepsilon|$, we deduce that $(\chi_\varepsilon Du_\varepsilon)$ is bounded in L^2, and thus it suffices to test with a dense set of test-function, and to show that

$$\forall\varphi \in C^\infty_c(\Omega)\;:\;\int_\Omega \chi_\varepsilon Du_\varepsilon\varphi \to \int_\Omega Du_0\varphi.$$

For the above, it suffices to show that $\|\chi_\varepsilon\varphi - \varphi\|_{L^2} \to 0$, since then the statement follows by the fact the we can pass to the limit in products with weakly $\times$ strongly convergent factors.

$$\begin{aligned}\int_\Omega |\chi_\varepsilon\varphi - \varphi|^2 &\leq \|\varphi\|_{L^\infty}^2 \int_\Omega |(\chi_\varepsilon - 1)|^2 = |\{|Du_\varepsilon(x)| > \varepsilon^{-\frac{1}{2}}\}| \\ &\leq \int_\Omega \frac{|Du_\varepsilon|^2}{\varepsilon^{-\frac{1}{2}}} \leq \sqrt{\varepsilon}\int_\Omega |Du_\varepsilon|^2.\end{aligned}$$

Since (Du_ε) weakly converges, it is bounded in $L^2(\Omega)$, and thus the right-hand side vanishes for $\varepsilon \downarrow 0$.
<u>Schritt 2:</u> (Taylor expansion).
We claim:

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon Du_\varepsilon(x))\,dx \geq \int_\Omega Q(\operatorname{sym} Du_0).$$

Argument: Since $W(Id) = 0 = \min W$, have

$$\frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon Du_\varepsilon(x))\,dx \geq \frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon\chi_\varepsilon(x)Du_\varepsilon(x))\,dx.$$

On the other hand for $x \in \Omega$ set

$$r_\varepsilon(x) := \frac{1}{\varepsilon^2}W(Id + \varepsilon\chi_\varepsilon(x)Du_\varepsilon(x)) - Q(\chi_\varepsilon(x)Du_\varepsilon(x)).$$

By Lemma 3.39 we have

$$|r_\varepsilon(x)| \leq \frac{1}{\varepsilon^2}|\varepsilon\chi_\varepsilon(x)Du_\varepsilon(x)|^2 r(\varepsilon\chi_\varepsilon(x)|Du_\varepsilon(x)|).$$

Since $|\varepsilon\chi_\varepsilon(x)Du_\varepsilon(x)| \leq \sqrt{\varepsilon} \to 0$, we have $r_\varepsilon(x) \to 0$ uniformly for a.e. $x \in \Omega$, and thus

$$\lim \int_\Omega |r_\varepsilon| = 0.$$

On the other hand, since Q is quadratic and convex, we get by weak lower-semicontinuity,

$$\liminf \int_\Omega Q(\chi_\varepsilon Du_\varepsilon) \geq \int_\Omega Q(Du_0).$$

Since $Q(F) = Q(\operatorname{sym} F)$, see Lemma 3.32, the statement follows.
<u>Schritt 3:</u> (Conclusion).

$$\begin{aligned}\frac{1}{\varepsilon^2}\mathcal{E}_\varepsilon(u_\varepsilon) &= \frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon Du_\varepsilon(x)) + \frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon \cdot (x + \varepsilon u_\varepsilon(x)) \\ &= \frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon Du_\varepsilon(x)) + \int_\Omega \varepsilon^{-1} f_\varepsilon \cdot u_\varepsilon(x).\end{aligned}$$

We may pass to the liminf by appealing to Step 2 for the first integral, and thanks to $\varepsilon^{-1}f_\varepsilon \to f_0$ and $u_\varepsilon \rightharpoonup u_0$ in the second integral. The claim follows.

$\square$

Lemma 4.19 (Recovery sequence). *Let $u_0 \in H^1_0(\Omega;\mathbb{R}^d)$. Then there exists a sequence $u_\varepsilon \to u_0$ strongly in $H^1_0(\Omega;\mathbb{R}^d)$ such that*

$$\lim_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u_0).$$

Beweis. Since $C^1(\overline{\Omega};\mathbb{R}^d) \subset W^{1,2}(\Omega;\mathbb{R}^d)$ is dense, for $0 < \delta \ll 1$ there exists $u_\delta \in C^1(\overline{\Omega};\mathbb{R}^d)$ such that

$$\|u_\delta - u_0\|_{W^{1,2}} \leq \delta.$$

Set $C_\delta := \|Du_\delta\|_{L^\infty(\Omega)}$. We have by Lemma 3.32

$$|\frac{1}{\varepsilon^2} W(Id + \varepsilon Du_\delta(x)) - Q(Du_\delta(x))| \leq C_\delta r(\varepsilon C_\delta),$$

and thus

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon Du_\delta(x)) = \lim_{\delta\downarrow 0}\int_\Omega Q(Du_\delta(x)) = \int_\Omega Q(Du_0(x)).$$

Set

$$m(\delta,\varepsilon) := |\frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon Du_\delta(x)) - \int_\Omega Q(Du_0(x))| + \|u_\delta - u_0\|_{W^{1,2}(\Omega)},$$

then

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} m(\delta,\varepsilon) = 0,$$

and thus there exists a diagonal sequence $\delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$[8], s.t.

$$\lim_{\varepsilon\downarrow 0} m(\delta(\varepsilon),\varepsilon) = 0.$$

Set $u_\varepsilon := u_{\delta(\varepsilon)}$. Since in particular, $u_\varepsilon \to u_0$ in L^2, we deduce that $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}_0(u_0)$, and thus we have found the recovery sequence. □

4.2.5 Proof of Theorem 3.36

The previous two-lemmas are often summarized by saying that $\mathcal{E}_\varepsilon$ Γ-converges (w.r.t. weak convergence in H^1) to $\mathcal{E}_0$. In combination with the a priori estimate of Theorem 3.35, it is a standard argument in the calculus of variations to conclude the proof of Theorem 3.36:
We first note that

$$\inf \mathcal{E}_\varepsilon \leq \mathcal{E}_\varepsilon(0) = 0,$$

and thus $\mathcal{E}_\varepsilon(u_\varepsilon) \leq 1$. From Theorem 3.35 we learn that

$$\int_\Omega |Du_\varepsilon|^2 \leq C\big(\mathcal{E}_\varepsilon(u_\varepsilon) + \frac{1}{\varepsilon^2}\int_\Omega |f_\varepsilon|^2\big) \leq C' < \infty,$$

where C' is independent of ε. Hence, by Poincaré's inequality (u_ε) is bounded in $H^1_0(\Omega)$ and we may pass to a subsequence (not relabeled) that weakly converges to some limit

[8] Attouch's Lemma

$u_0 \in H^1_0(\Omega)$. We claim that u_0 mimizes $\mathcal{E}_0$. For the argument let u_0^* denote a minimizer of $\mathcal{E}_0$. Then, Indeed, by the lower-bound, we have

$$\mathcal{E}_0(u_0^*) \leq \mathcal{E}_0(u_0) \leq \liminf_{\varepsilon \downarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon).$$

On the other hand, let u_ε^* denote a recovery sequence for u_0. Then (since u_ε is a almost minimizer of $\mathcal{E}_\varepsilon$),

$$\liminf_{\varepsilon} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon} (\mathcal{E}_\varepsilon(u_\varepsilon^*) + \varepsilon) = \lim_{\varepsilon \downarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon^*) = \mathcal{E}_0(u_0^*).$$

The combination of both estimates reads

$$\mathcal{E}_0(u_0^*) \leq \mathcal{E}_0(u_0) \leq \liminf_{\varepsilon \downarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathcal{E}_0(u_0^*),$$

and thus we conclude that u_0 is a minimizer for $\mathcal{E}_0$. By Theorem 3.33 $\mathcal{E}_0$ admits a unique minimizer, and thus $u_0 = u_0^*$. In particular, the limit u_0 does not depend on the subsequence of (u_ε) that we selected. In particular, from any subsequence of (u_ε) we can extract a subsequence that weakly converges to u_0 — and thus $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$ for the entire sequence. The above argument also shows that

$$\inf \mathcal{E}_\varepsilon \to \min \mathcal{E}_0.$$

4.3 Homogenization

4.4 Derivation of nonlinear elasticity from discrete models

A Functional Analysis, Lax-Milgram

Motivation: Consider Poisson's equation

$$-\triangle u = f$$

where u and f belong to function spaces X and Y, respectively. Then $-\triangle$ can be interpreted as a linear map from X to Y. E.g. $X = C^2\left(\mathbb{R}^d\right)$, $Y = C\left(\mathbb{R}^d\right)$, $TU := -\triangle u$, $T\colon X \to Y$ linear. Thus, to solve the Poisson equation means to invert T (which in fact is not possible for the X, Y defined as above). This is in analogy to the finite dimensional case: $Ax = y$, $A \in \mathbb{R}^{d\times d}$, $x, y \in \mathbb{R}^d$, $A\colon \mathbb{R}^d \to \mathbb{R}^d$.

The finite dimensional case. If X is a finite dimensional (normed) vector space, then

- X is isomorphic to $\mathbb{R}^d$ for some dimension d.
- For $A\colon \mathbb{R}^d \to \mathbb{R}^d$ linear (i.e. $A \in \mathbb{R}^{d\times d}$) we have the following:

$$\begin{aligned} A \text{ injective} \iff A \text{ surjective} &\iff A \text{ bijective} \\ &\iff \forall y \in \mathbb{R}^d \exists! x \in \mathbb{R}^d \colon Ax = y. \end{aligned}$$

- If $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are two normed vector spaces, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in the sense, that
$$\exists c > 0\colon \frac{1}{c}\|\cdot\|_2 \leq \|\cdot\|_1 \leq c \cdot \|\cdot\|_2.$$
- If $A \subset X$ is bounded and closed, then A is compact. This is called the *Heine-Borel-Property.*

The infinite dimensional case. If X is an infinite dimensional (normed) vector space, the situation is richer. In the following examples we consider sequence spaces, in particular

$$\ell^p(\mathbb{N}) := \left\{(a_k) \mid a\colon \mathbb{N} \to \mathbb{R}, \|(a_k)\|_{\ell^p(\mathbb{N})} < \infty\right\} \text{ with norm}$$
$$\|(a_k)\|_{\ell^p(\mathbb{N})} := \begin{cases} \left(\sum_{k\in\mathbb{N}} |a_k|^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_k |a_k|, & p = \infty. \end{cases}$$

and

$$c_c(\mathbb{N}) := \{(a_k) \mid a\colon \mathbb{N} \to \mathbb{R}, a_k = 0 \text{ for all but finitely many } k \in \mathbb{N}\} \text{ with norm}$$
$$\|(a_k)\|_{c_0} := \|(a_k)\|_{\ell^\infty(\mathbb{N})}.$$

- Let $T\colon X \to X$ be a linear map. Injectivity of T (“T is one-to-one”) in general does not imply surjectivity of T (“T is onto”) and vice versa. Examples: right-shift on $c_c(\mathbb{N})$ and left-shift $c_c(\mathbb{N})$, respectively.
- $Tx = y$ solvable for all $y \in X \iff T$ is onto.
- $Tx = y$ has at most on solution for all $y \in X$? $\iff T$ is one-to-one.
- Infinite dimensional spaces can be endowed with differen topologies. Example: $(c_c(\mathbb{N}), \|\cdot\|_{\ell^2(\mathbb{N})})$ and $(c_c(\mathbb{N}), \|\cdot\|_{\ell^\infty(\mathbb{N})})$ are not isomorph.
- Boundedness and closedness of $A \subset X$ does not imply compactness. For example $\overline{B(0,1)} \subset X$ compact $\iff \dim X < \infty$.

Functional analysis is about extending and developing structures for infinite dimensional vector spaces.

A.1 Hilbert spaces and Banach spaces

All spaces are $\mathbb{R}$-valued. Basic objects and keywords you should be familiar with:

- (X, d) metric space, cauchy sequences, completeness,
- norm $|\cdot|$ and induced metric $d(x, y) = \|x - y\|$,
- inner product $(\cdot, \cdot)$ and induced norm $\|x\| = \sqrt{(x, x)}$.

Definition A.1 (Banach space, Hilbert space)**.** *A normed vector space $(X, \|\cdot\|)$ is called Banach, if it is complete. An inner product space $(X, (\cdot, \cdot))$ is called Hilbert, if it is complete.*

Example A.2.

- *Consider $X = \mathbb{R}^{\mathbb{N}}$. We define the Fréchet metric on X by*

$$d(\{a\},\{b\}) = \sum_{k\in\mathbb{N}} 2^{-k}\frac{|a_k - b_k|}{1+|a_k-b_k|}$$

- *Euclidean space $\mathbb{R}^d$ with standard scalar product $(a,b) = a\cdot b$ and Euclidean norm $\|a\| = \sqrt{(a,a)}$.*

- *Consider $(a,b)_A := Aa\cdot b$ for $A\in\mathbb{R}^{d\times d}$ and $a,b\in\mathbb{R}^d$. Then $\big(\mathbb{R}^d,(\cdot,\cdot)_A\big)$ is Hilbert $\iff$ $A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is positive definit.*

- *$(\ell^p(\mathbb{N}),\|\cdot\|_{\ell^p(\mathbb{N})})$ are Banach spaces. $(\ell^2(\mathbb{N})$ with $((a_k),(b_k))_{\ell^2(\mathbb{N})} := \sum_{k\in\mathbb{N}} a_k b_k$ is a Hilbert space.*

Exercise A.3. *Show that $(c_c(\mathbb{N}),\|\cdot\|_{\ell^2(\mathbb{N})})$ is not complete.*

Now we take a look on function spaces.

Example A.4. *Let $K\subset\mathbb{R}^d$ be compact.*

- $C^0(K) = \{u\colon K\to\mathbb{R} \mid u \text{ continuous in } K\}$. *We equip $C^0(K)$ with the norm $\|u\|_{C^0(K)} = \sup_{x\in K}|u(x)|$. Then $\big(C^0(K),\|\cdot\|_{C^0(K)}\big)$ is a Banach space.*

- *$C^k(K)$ defined as the set of all $u\colon K\to\mathbb{R}$, which are k times continuously differentiable with $D^\alpha u\in C^0(K)$ for all $|\alpha|\le k$, equipped with the norm $\|u\|_{C^k(K)} = \sum_{0\le|\alpha|\le k}\|D^\alpha u\|_{C^0(K)}$ is a Banach space.*

- *$C_c(\Omega) := \{u\in C(\Omega) \mid \operatorname{supp} u \subset\subset \Omega\}$, $\Omega\subset\mathbb{R}^d$ open. Then $(C_c(\Omega),\|\cdot\|_{C_0(\Omega)})$ is not a Banach space.*

- *The Lebesgue spaces*

$$L^p(\Omega) = \left\{u\colon\Omega\to\mathbb{R} \text{ measurable } \mid \|u\|_{L^p(\Omega)}<\infty\right\},$$

where $\Omega\subset\mathbb{R}^d$ open, $1\le p\le\infty$ and

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u|^p\right)^{\frac{1}{p}} & ,1\le p<\infty \\ \operatorname{esssup}_{x\in\Omega}|u(x)| & ,p=\infty. \end{cases}$$

are Banach spaces. For $p=2$, i.e. $(L^2(\Omega)$ with inner product $(u,v)_{L^2(\Omega)} := \int_\Omega uv\,dx$ is a Hilbert space.

Remark A.5. *If $(X,\|\cdot\|)$ is a Banach space and $d(x,y) := \|x-y\|$, then (X,d) is a complete metric space and $u_j\to u \iff d(u_j,u)\to 0$.*

Hilbert spaces have more structure.

Definition A.6 (Orthogonality). *Let $(H,(\cdot,\cdot,))$ be a Hilbert space, $A,B\subset H$ and $a,b\in H$. We say*

- *$a \perp b$, if $(a,b)=0$.*
- *$b \perp A$, if $a \perp b$ for all $a \in A$.*
- *$A \perp B$, if $a \perp b$ for all $a \in A$, $b \in B$.*

Example A.7. $H = L^2((0,1))$, $A = \{u \in H \mid u \text{ constant}\}$, $B = \{u \in H \mid \int_0^1 u = 0\}$. *Then $A \perp B$.*

Lemma A.8 (Important inequalities)**.** *Let $(H,(\cdot,\cdot))$ be Hilbert. Then for $a,b \in H$*

a) $|(a,b)| \leq \|a\| \cdot \|b\|$ Cauchy-Schwarz.

b) $\|a+b\|^2 = \|a\|^2 + \|b\|^2 + 2(a,b)$ Pythagoras.

c) $|(a,b)| \leq \frac{1}{2}\|a\|^2 + \frac{1}{2}\|b\|^2$.

Beweis. We give a proof only for a). Let $a,b \in H$, $\alpha,\beta > 0$.

$$\begin{aligned} 0 \leq \left\|\frac{a}{\alpha} - \frac{b}{\beta}\right\|^2 &= \left(\frac{\|a\|}{\alpha}\right)^2 + \left(\frac{\|b\|}{\beta}\right)^2 - \frac{2}{\alpha\beta}\langle a,b\rangle \\ \implies 2\langle a,b\rangle &\leq \frac{\beta}{\alpha}\|a\|^2 + \frac{\alpha}{\beta}\|b\|^2. \end{aligned}$$

Set $\alpha = \|a\| + \varepsilon$, $\beta = \|b\| + \varepsilon$. Then

$$2\langle a,b\rangle \leq (\|b\|+\varepsilon)\frac{\|a\|}{\|a\|+\varepsilon} + \|a\| + (\|a\|+\varepsilon)\frac{\|b\|}{\|b\|+\varepsilon}\|b\|.$$

For $\varepsilon \downarrow 0$, the RHS converges to $2\|b\| \cdot \|a\|$. □

Definition A.9 (Orthogonal complement)**.** *Let $(H,(\cdot,\cdot))$ be Hilbert and $A \subset H$. Then*

$$A^\perp := \{b \in H \mid b \perp A\}.$$

Lemma A.10. *$(H,(\cdot,\cdot))$ Hilbert, $A \subset H$. Then $A^\perp$ is a closed, linear subspace of H.*

Exercise A.11. *Give a proof for lemma A.10.*

Definition A.12 (Dense subset)**.** *(X,d) metric space, $A \subset X$. A is called dense in X, if*

$$\forall x \in X \exists \left(x_j\right)_{j\in\mathbb{N}} \subset A \colon x_j \xrightarrow{j\to\infty} x \text{ in } (X,d).$$

Example A.13.

- *$\Omega \subset \mathbb{R}^d$ open, $1 \leq p < \infty$, $C_c^\infty(\Omega) \subset L^p(\Omega)$ dense.*
- *$C(\Omega) \cap L^\infty(\Omega) \subset L^\infty(\Omega)$ but not dense.*
- *$A = \operatorname{span}\left\{\chi_Q \mid Q = (a,b) \subset \mathbb{R}^d, a.b \in \mathbb{Q}^d\right\} \subset L^p\left(\mathbb{R}^d\right)$ dense, if $1 \leq p < \infty$.*
- *$\mathbb{Q}^d \subset \mathbb{R}^d$ dense.*

- (X,d) *metric space,* $A \subset X$ *subset,*
$$\overline{A} := \left\{ x \in X \mid \exists (x_j)_j \subset A \text{ s.t. } x_j \xrightarrow{j\to\infty} x \right\},$$
then $A \subset \overline{A}$ *dense.*

Definition A.14. (X,d) *metric space is called separable, if there exists a countable subset* $A \subset X$*, which is dense in* X*.*

Example A.15.

- $\mathbb{R}^d$ *separable (since* $\mathbb{Q}^d \subset \mathbb{R}^d$ *dense).*
- *Consider* $L^p\left(\mathbb{R}^d\right)$ *for* $1 \le p < \infty$*.*
  $A = \operatorname{span}\left\{ \mathbf{1}_Q \mid Q \subset \mathbb{R}^d \text{ where } Q \text{ is a cube with rational vertices } \right\}$.
- $L^\infty\left(\mathbb{R}^d\right)$ *is not separable.*

A.2 Completion of a metric space

Motivation: Let (X,d) be complete; consider sequences $(x_j)_j$, $(y_j)_j \subset X$ that converge in (X,d). Then:

- $(x_j)_j$, $(y_j)_j$ are Cauchy.
- Both sequences have the same limit, if and only if $d(x_j,y_j) \to 0$.

Definition A.16 (Completion)**.** *Let* (X,d) *be a metric space and consider* $\tilde{X} := \{(x_j)_j \subset X \mid (x_j)_j \text{ cauchy }\}$*. Let* $\sim$ *denote the equivalence relation on* X *defined by*
$$(x_j)_j \sim (y_j)_j :\iff d(x_j,y_j) \xrightarrow{j\to\infty} 0.$$
$\left(\tilde{X}/\sim, \tilde{d}\right)$ *is a complete metric space with the metric*
$$\tilde{d}\left((x_j)_j,(y_j)_j\right) := \lim_{j\to\infty} d(x_j,y_j).$$

Note that the definition of the metric is well-posed, since it is independent of the representing sequence: If $(\tilde{x}_j) \sim (x_j)$ and $(\tilde{y}_j) \sim (y_j)$, then
$$\begin{aligned} &\tilde{d}\left((\tilde{x}_j),(\tilde{y}_j)\right) = \lim d\left(\tilde{x}_y,\tilde{y}_j\right) \\ &\qquad d\left(\tilde{x}_y,\tilde{y}_j\right) \le d(x_j,y_j) + d\left(\tilde{x}_j,x_j\right) + d\left(\tilde{y}_j,y_j\right) \le \tilde{d}((x_j),(y_j)) \le \tilde{d}\left((\tilde{x}_j),(\tilde{y}_j)\right) \to 0, \end{aligned}$$
and thus $\tilde{d}((\tilde{x}_j),(\tilde{y}_j)) = \tilde{d}((x_j),(y_j))$.
The map $J\colon (X,d) \to \left(\tilde{X}/\sim,\tilde{d}\right)$, $x \mapsto (x_j)$ *with* $x_j = x$ *for all* j *is one-to-one and isometric* ($d(x,y) = \tilde{d}(J(x),J(y))$ *(i.e. "*J *isometrically embeds* X *into* $\tilde{X}/\sim$*"). Therefore we call* $\left(\tilde{X}/\sim,\tilde{d}\right)$ *the completion of* (X,d)*. We usually just write* $\tilde{X}$ *for convenience.*

Example A.17.

- *$X = \mathbb{Q} \implies \tilde{X} \cong \mathbb{R}$.*
- *(X, d) complete, $A \subset X$. Then (A, d) is a metric space with completion $\cong \overline{A}$.*

Example A.18.

- *Let $\Omega \subset \mathbb{R}^d$ be open, $X = C_c^\infty(\Omega)$,*

$$\|u\|_X := \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

  Then

  a) $(X, \|\cdot\|_X)$ is a normed space.

  b) The completion $\tilde{X} \cong L^p(\Omega)$.

- *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $X := C^1\left(\bar{\Omega}\right)$,*

$$\|u\|_X := \left(\int_\Omega |u|^2 + |\nabla u|^2 \right)^{\frac{1}{2}}.$$

  Then $(X, \|\cdot\|_X)$ is an incomplete normed space.

- *However, if u_ε was convergent in X and since $X \subset L^2(\Omega)$, $\|\cdot\|_{L^2(\Omega)} \le \|\cdot\|_X$, it converges in $L^2(\Omega)$ to the same limit. But $u_\varepsilon \to u_0$ in $L^2(\Omega)$, $u_0(x) = |x|$ and $u_0 \in X$.*

A.3 Linear Operators

Lemma A.19. *Let X and Y be normed vector spaces and $T\colon X \to Y$ linear. TFAE:*

a) T is continuous.

b) T is continuous at $x_0 \in X$.

c) T is bounded, i.e.

$$\sup_{\|x\|_X \le 1} \|Tx\|_Y < \infty.$$

d) $\exists C > 0 \forall x \in X\colon \|Tx\|_Y \le C\|x\|_X$.

Proof for b) $\implies$ c). $B_1(Tx_0)$ is open in Y $\implies$ $\exists \varepsilon > 0\colon \overline{B_\varepsilon(x_0)} \subset T^{-1}(B_1(Tx_0))$. Therefore $x_0 + \varepsilon x \in \overline{B_\varepsilon(x_0)}$ for all $\|x\| \le 1$. By linearity of T:

$$Tx_0 + \varepsilon Tx = T(x_0 + \varepsilon x) \subset B_1(Tx_0) \implies Tx \in B_{\varepsilon^{-1}}(0)$$

for alle $x \in X$ with $\|x\| \le 1$. Hence, T is bounded. □

Definition A.20. $\mathcal{L}(X;Y) = \{T\colon X \to Y \mid T$ *linear and continuous* $\}$. $\mathcal{L}(X) := \mathcal{L}(X;X)$. $\|T\|_{\mathcal{L}(X;Y)} := \sup_{\|x\|_X \le 1} \|Tx\|_Y$ *(operator norm).*

We define addition of two elements $T, S \in \mathcal{L}(X;Y)$ as well as multiplication of T with a scalar $\lambda \in \mathbb{R}$ pointwise, i.e. for all $x \in X$: $(\lambda T + S)(x) = \lambda Tx + Sx$. This turns $\mathcal{L}(X;Y)$ into a vector space.

Theorem A.21. *Let Y be Banach. Then $(\mathcal{L}(X;Y), \|\cdot\|_{\mathcal{L}(X;Y)})$ is Banach.*

Beweis. Let $(T_j) \subset \mathcal{L}(X;Y)$ be Cauchy.

1. For all $x \in X$, $y_j := Tx_j$ defines a cauchy sequence in Y. Let $\delta > 0$. Since (T_j) is cauchy, there exists $j_0 \in \mathbb{N}$ such that

$$\|T_j - T_k\|_{\mathcal{L}(X;Y)} \leq \frac{\delta}{\|x\| + 1}$$

for all $j, k \geq j_0$. We have

$$\|y_j - y_k\| = \|T_j x - T_k x\| = \|(T_j - T_k)x\| \leq \|T_j - T_k\| \cdot \|x\|$$

2. Since Y is Banach and $(T_j x)$ are cauchy, the limit $\lim_{j\to\infty} T_j x =: Tx$ exists in Y. This defines a map $T\colon X \to Y$.

3. T is linear: $T(x + \lambda x') = \lim_{j\to\infty} T_j(x + \lambda x') = \lim_{j\to\infty} T_j x + \lambda T_j x' = Tx + \lambda Tx'$.

4. $T \in \mathcal{L}(X;Y)$: $\forall \delta > 0 \forall x \in B_1(0) \subset X \exists j\colon \|Tx - T_j x\| \leq \delta$. Therefore we have that

$$\|Tx\| \leq \|T_j x\| + \|Tx - T_j x\| \leq \|T_j\|_{\mathcal{L}(X;Y)} + \delta \leq \underbrace{\sup_k \|T_k\|_{\mathcal{L}(X;Y)}}_{<\infty} + \delta < \infty.$$

Hence $\sup_{\|x\|\leq 1} \|Tx\| < \infty$.

5. $T_j \to T$ in $\mathcal{L}(X;Y)$. Let $x \in X$ with $\|x\| \leq 1$.

$$\|(T - T_j)x\| = \lim_{k\to\infty} \underbrace{\|(T_k - T_j)x\|}_{\leq \|T_k - T_j\|_{\mathcal{L}(X;Y)}\|x\|} \leq \delta\|x\|$$

for all $k, j \geq j_\delta$. Therefore $\|T - T_j\|_{\mathcal{L}(X;Y)} \leq \delta$ for all $j \geq j_\delta$ and $T_j \to T$ in $\mathcal{L}(X;Y)$. $\square$

Definition A.22. *Let $T\colon X \to Y$ be linear. Then we define*

- $\mathcal{N}(T) := \{x \in X \mid Tx = 0\}$ *kernel (or null space) of T.*
- $\mathcal{R}(T) := \{Tx \mid x \in X\}$ *range of T.*
- $\operatorname{graph}(T) = \{(x, Tx) \in X \times Y \mid x \in X\}$ *graph of T.*

Exercise A.23. *Let X, Y be Banach spaces. Show that*

a) $T \in \mathcal{L}(X;Y) \implies \mathcal{N}(T) \subset X$ *closed linear subspace.*

b) Set $S := [0,1]$ *and consider for* $f \in C^0(S)$

$$(Tf)(x) = \int_0^x f(y)\ dy.$$

Show $T \in \mathcal{L}(C^0(S), C^0(S))$, $T \in \mathcal{L}(C^0(S), C^1(S))$ *and check if* $\mathcal{R}(T)$ *is closed in* $C^0(S)$ *or* $C^1(S)$.

c) $T \in \mathcal{L}(X;Y)$. *Suppose* $\exists C > 0$ *such that* $\forall x \in X \colon \|x\| \leq C\|Tx\|$. *Then* $\mathcal{R}(T)$ *is closed and* $T\colon X \to \mathcal{R}(T)$ *is an isomorphism, i.e.* T *and* T^{-1} *are bounded and linear.*

Theorem A.24 (Open mapping theorem)**.** *Let* X, Y *be Banach spaces and* $T \in \mathcal{L}(X;Y)$. T *is open, i.e.* $\forall U \subset X$ *open* : $T(U) \subset Y$ *open, if and only if* T *is surjective.*

Theorem A.25 (Inverse mapping theorem)**.** *Let* X, Y *be Banach and* $T \in \mathcal{L}(X;Y)$. *If* T *is a bijection, then* $T^{-1} \in \mathcal{L}(X;Y)$.

Theorem A.26 (Closed graph theorem)**.** *Let* X, Y *be Banach and* $T\colon X \to Y$ *linear. Then* T *is bounded if and only if its graph is closed in* $X \times Y$.

Proof for $\implies$. Recall, that $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$ turns $X \times Y$ into a Banach space. $Z := \operatorname{graph}(T)$ closed, linear $\implies$ Banach space.

$$P_x\colon Z \to X,\ (x,y) \mapsto x, \qquad P_y\colon Z \to Y,\ (x,y) \mapsto y$$

are linear and bounded. And in particular P_x is a Bijection. By the inverse mapping theorem: P_x^{-1} is linear and bounded. $Tx = P_y P_x^{-1} x \implies T$ is linear and bounded as a concatenation of linear and bounded operators. □

Typical situations: T defined on a dense, linear subset.

Lemma A.27. *Let* X, Y *be normed vector spaces,* Y *Banach,* $D \subset X$ *linear and dense,* $T\colon D \to Y$ *linear and bounded,* $\sup_{\|x\|\leq 1} \|Tx\| \leq C < \infty$. *Then there exists a unique* $\tilde{T} \in \mathcal{L}(X;Y)$ *with* $\tilde{T}x = Tx$ *for all* $x \in D$ *and* $\|\tilde{T}\|_{\mathcal{L}(X;Y)} \leq C$. *We call* $\tilde{T}$ *the* extension *of* T *to* X.

Beweis. $\tilde{T}(x) := \lim_{j\to\infty} Tx_j$ where $(x_j) \subset D$ with $x_j \to x$ in X.

- $\tilde{T}$ is well-defined:
  - $x_j \to x \implies (x_j)$ cauchy $\implies (Tx_j)$ cauchy.
  - Let $x_j \to x$, $\tilde{x}_j \to x$, then

$$\begin{aligned} \|Tx_j - T\tilde{x}_j\| &\leq C\|x_j - \tilde{x}_j\| \leq C(\|x_j - x\| + \|x - \tilde{x}_j\|) \\ \lim_{j\to\infty} Tx_j &= \lim_{j\to\infty} T\tilde{x}_j. \end{aligned}$$

- Linearity: For $D \supset (x_j) \to x \in X$, $D \supset (y_j) \to y \in X$, $\lambda \in \mathbb{R}$ we have

$$\tilde{T}(x + \lambda y) = \lim_{j\to\infty} T(x_j + \lambda y_j) = \lim_{j\to\infty} (Tx_j + \lambda T y_j) = \tilde{T}x + \lambda \tilde{T}y$$

- Boundedness: $\|\tilde{T}x\| = \|\lim_{j\to\infty} Tx_j\| \leq \limsup_{j\to\infty} \|Tx_j\| \leq C\|x\|$. Therefore $\|\tilde{T}\| \leq C$. $\square$

A variant of this lemma would be: Let X be a normed vector space, Y Banach, $\tilde{X}$ the completion of X and $T \in \mathcal{L}(X;Y)$. Then there exists a unique $\tilde{T} \in \mathcal{L}(\tilde{X};Y)$, such that for all $x \in X$ we have $Tx = \tilde{T}Jx$, where J is the canonical injection of X into $\tilde{X}$.

Exercise A.28. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $X = C^1(\bar{\Omega})$, $T\colon X \to L^2\left(\Omega, \mathbb{R}^{d+1}\right)$, $Tu := (u, \partial_1 u, \ldots, \partial_d u)$. Denote by $\|u\|_X := \|Tu\|_{L^2\left(\Omega,\mathbb{R}^{d+1}\right)}$ a norm in X. This turns X into a normed vector space. Denote by $\tilde{X}$ the completion of $(X, \|\cdot\|_X)$ and by $\tilde{T}$ the extension of T to $\tilde{X}$. Set $Z := \mathcal{R}(\tilde{T})$. Show that*

a) Z is closed in $L^2\left(\Omega, \mathbb{R}^{d+1}\right)$.

b) Z is isometric isomorph to $\tilde{X}$ (= space of example A.18).

c) Let $P\colon Z \to L^2(\Omega)$, $P(u, v_1, \ldots, v_d) := u$. Check if $\mathcal{R}(P)$ is open.

A.4 Dual spaces, Riesz-Fréchet, Lax-Milgram

Motivation: for any euclidean vector space element $x \in \mathbb{R}^d$ are equivalent:

- $x = 0_{\mathbb{R}^d}$.
- $\forall T \in \mathcal{L}\left(\mathbb{R}^d, \mathbb{R}\right) : Tx = 0_{\mathbb{R}}$.

The concept of dual spaces extends this to infinite dimensional spaces.

Definition A.29 (Dual space)**.** *Let X be a Banach space. Then $X^* := \mathcal{L}(X;\mathbb{R})$ is called* dual space *of X. If we equip X^* with the norm $\|\cdot\|_{X^*} := \|\cdot\|_{\mathcal{L}(X;\mathbb{R})}$, then $(X^*, \|\cdot\|_{X^*})$ is a Banach space. The space $X^{**} := (X^*)^* = \mathcal{L}\left(X^*;\mathbb{R}\right)$ is called* bidual space *of X.*

Remark A.30. *Evaluations of elements of a dual space $T \in X^*$ at $x \in X$ is symbolized by duality brackets: $\langle T, x\rangle = T(x)$ (as opposed to $(\cdot,\cdot)$ which is reserved for inner products).*

Lemma A.31 (Consequences of Hahn-Banach)**.** *Let X be Banach and $x \in X$. Then*

$$x = 0 \iff \forall T \in X^*\colon \langle T, x\rangle = 0.$$

For finite dimensional space, say $\mathbb{R}^d$:

- Any linear functional is continuous. $\left(\mathbb{R}^d\right)^* = \{T\colon \mathbb{R}^d \to \mathbb{R} \text{ linear }\} \cong \mathbb{R}^d$.

$$\left(\mathbb{R}^d\right)^* \ni T \mapsto \sum_{i=1}^{d} (T, e_i) e_i \in \mathbb{R}^d$$

$$\langle T, x\rangle = \langle T, \sum_{i=1}^{d} x_i e_i\rangle = \sum_{i=1}^{d} \left(\langle T, e_i\rangle \left(e_i \cdot x\right)\right)$$

$$= y \cdot x \text{ with } y = \sum_{i=1}^{d} \langle T, e_i\rangle e_i \in \mathbb{R}^d$$

- We can identify the bidual space with $\mathbb{R}^d$ itself: $\left(\mathbb{R}^d\right)^{**} \cong \mathbb{R}^d$.

Let H be a Hilbert space. It is easy to define linear functionals: Let $a \in H$, then

$$\langle T, x\rangle := (a, x) \quad \forall x \in H$$

defines a linear and bounded functional $T \in H^*$ with $\|T\|_{H^*} = \|a\|$ by the Cauchy-Schwarz inequality. Moreover, the map

$$J\colon H \to H^*,\ \langle Ja, \cdot\rangle := (a, \cdot).$$

is a linear isometry. Indeed, by bilinearity of the inner product, J is linear; and J is an isometry, because

$$\|T\|_{H^*} := \sup_{\|x\|\leq 1} \langle T, x\rangle \leq \|a\|, \ \left(T, \frac{a}{\|a\|}\right) = \frac{(a,a)}{\|a\|} = \|a\| \leq \|T\|_{H^*}.$$

Theorem A.32 (Riesz-Fréchet)**.** *Let H be Hilbert. Then $J\colon H \to H^*$, $a \mapsto (a, \cdot)$ is an isometric isomorphism, i.e. $H \cong H^*$.*

Beweis. See Brezis, Alt. □

J is called the canonical injection of H into H^*. If $T \in H^*$, then $\langle T, \cdot\rangle = (a, \cdot)$ for the unique element $a = J^{-1}T \in H$. We call a the *Riesz-representative* of T.

Example A.33.

- *$H = \mathbb{R}^d$, $(x, y)_A := Ax \cdot y$ with $A \in \mathbb{R}^{d\times d}$ symmetric, positive definit.*
- *$a \in \mathbb{R}^d$, $T\colon \mathbb{R}^d \to \mathbb{R}$, $Tx := \sum_{i=1}^d a_i x_i$.*

How to determine the Riesz-representative of T in $(H, (\cdot,\cdot)_A)$? We seek for $y \in H$ such that $\langle T, x\rangle = (y, x)_A$ for all $x \in H$. By definition of T, we have $\langle T, x\rangle = a\cdot x = AA^{-1}a\cdot x = \left(A^{-1}a, x\right)_A$, so $y = A^{-1}a$ is the Riesz-representative of T. In this case we observe: The Riesz-representative y of $x \mapsto a \cdot x$ in $\left(\mathbb{R}^d, (\cdot,\cdot)_A\right)$ solves the linear equation $Ay = a$.

Definition A.34 (Bilinear form)**.** *Let H be Hilbert, $a\colon H \times H \to \mathbb{R}$.*

a) a is called bilinear*, if $\forall u, u', v, v' \in H \forall \alpha, \beta \in \mathbb{R}$:*

$$a(u + \alpha u', v) = a(u, v) + \alpha a(u', v), \quad a(u, v + \beta v') = a(u, v) + \beta a(u, v').$$

b) a is called bounded*, if $\exists \Lambda < \infty$, such that*

$$|a(u, v)| \leq \Lambda \|u\| \cdot \|v\| \quad \forall u, v \in H.$$

c) a is called coercive*, if $\exists \lambda > 0$, such that*

$$a(u, u) \geq \lambda \|u\|^2 \quad \forall u \in H.$$

Example A.35. *Let $A\colon H \to H$ linear, $a(u, v) = (Au, v)$.*

- *a is bilinear.*
- *a is bounded $\iff$ $A \in \mathcal{L}(H)$.*

Lemma A.36 (Bilinear form versus linear operator)**.** *Let H be Hilbert and a blinear and bounded with boundedness constant Λ as in definition A.34. Then there exists a unique $A \in \mathcal{L}(H)$ s.t.*

$$a(u,v) = (Au, v) \quad \forall u, v \in H.$$

Moreover: $\|A\|_{\mathcal{L}(H)} \leq \Lambda$.

Beweis.

1. Consider $Tu = a(u, \cdot) \colon H \to \mathbb{R}$. Tu is a linear and bounded functional, i.e. $Tu \in H^*$. By the theorem A.32 of Riesz-Fréchet, we obtain $(Tu, \cdot) = (J^{-1}Tu, \cdot)$, where $J \colon H \to H^*$ is the canonical injection of the Riesz-Fréchet theorem A.32. Therefore set $Au := J^{-1}Tu \in H$. This defines a map $A \colon H \to H$.

2. A is linear and bounded with $\|A\|_{\mathcal{L}(H)} \leq \Lambda$. Since $J^{-1} \colon H^* \to H$ is a linear isometric isomorphism, it suffices to show that $H \ni u \mapsto T_u \in H^*$ is linear and bounded with $\|Tu\|_{H^*} \leq \Lambda \|u\|$.

   - Linearity: Let $u, v \in H$ and $\alpha \in \mathbb{R}$.

   $$\begin{aligned} (T(u + \alpha v), \cdot) &= a(u + \alpha v, \cdot) = a(u, \cdot) + \alpha a(v, \cdot) \\ &= (Tu, \cdot) + \alpha (Tv, \cdot) = (Tu + \alpha Tv, \cdot) \\ \implies T(u + \alpha v) &= Tu + \alpha Tv. \end{aligned}$$

   - Boundedness: $\|Tu\|_{H^*} \leq \cdot \Lambda \|u\|$ and $|(Tu, v)| = |a(u, v)| \leq \Lambda \|u\| \|v\|$.
   - Uniqueness of A: If $A' \in \mathcal{L}(H)$ satisfies $a(u, v) = (Au, v)$ for all $u, v \in H$, then

   $$0 = a(u, v) - a(u, v) = (Au, v) - (A'u, v) = ((A'_A)u, v.$$

   Set $v = (A - A')u$. Then $0 = \|(A - A')u\|^2 \iff Au = A'u$. Since $u \in H$ is arbitrary, we get $A = A'$. ☐

Lemma A.37 (Lax-Milgram)**.** *Let H be Hilbert, a bilinear, bounded and coercive. Then there exists a unique isomorphism $A \in \mathcal{L}(H)$ with $\|A\|_{\mathcal{L}(H)} \leq \Lambda$ and $\|A^{-1}\|_{\mathcal{L}(H)} \leq \lambda^{-1}$ such that $a(u, v) = (Au, v)$ for all $u, v \in H$.*

Beweis. Let $A \in \mathcal{L}(H)$ be as in lemma A.36.

1. A-priori-estimate:

   $$\lambda \|u\|^2 \leq a(u, u) = (Au, v) \leq \|Au\| \|u\|.$$

   After dividing by $\|u\|$, we get $\lambda \|u\| \leq \|Au\|$.

2. A is injective: $Au = Av \iff A(u - v) = 0$. Then $0 = \|A(u, v)\| \geq \lambda \|u - v\|$. Therefore $\|u - v\| = 0 \implies u = v$.

3. A is surjective, i.e.

$$\mathcal{R}(A) = H \overset{\text{see lemma below}}{\iff} \begin{cases} \mathcal{R}(A) \text{ closed in } H \\ \mathcal{R}(A)^\perp = \{0\} \end{cases}.$$

$(v_j)_j \subset R(A)$ with $v_j \to v$ in H. $(v_j)_j = (Au_j)_j$ for some $(u_j)_j \subset H$. Claim: By $\lambda\|u\| \leq \|Au\|$, we have $(u_j)_j$ is Cauchy.

$$\lambda\|u_j - u_k\| \leq \|A(u_j - u_k)\| = \|Au_j - Au_k\| = \|v_j - v_k\|.$$

Hence $(v_j)_j$ is Cauchy $\implies$ $(u_j)_j$ is Cauchy. H is complete, therefore $u_j \to u$ in H. A is continuous $\implies$ $v_j = Au_j \to Au = v \implies v \in R(A)$. Let $v_0 \in R(A)^\perp \iff (v_0, v) = 0$ for all $v \in R(A)$. In particular $v = Av_0 \implies 0 = (Av_0, v_0) \geq \lambda\|v_0\|^2 \implies v_0 = 0 \implies R(A)^\perp = \{0\}$. Hence, A is a linear and bounded bijection $\implies A^{-1} \in \mathcal{L}(H)$. Set $u = A^{-1}v$. Then

$$\lambda\|A^{-1}v\| \leq \|AA^{-1}v\| = \|v\| \implies \|A^{-1}v\| \leq \frac{1}{\lambda}\|v\|.$$

We get $\|A^{-1}\|_{\mathcal{L}(H)} \leq \lambda^{-1}$. □

Lemma A.38. *Let H be Hilbert and $A \subset H$ a closed, linear subspace. Then $H = A \oplus A^\perp$, i.e. $\forall u \in H \exists_1 v \in A, w \in A^\perp \colon u = v + w$.*

This decomposition follows from properties of the *orthogonal projection*: Let $A \subset H$ a closed, linear subspace. Then $\exists_1 p \in \mathcal{L}(H) \colon \mathcal{R}(p) = A$ and $\|x - Px\| = \inf_{a \in A} \|x - a\|$.

Corollary A.39. *Let H be Hilbert and a bilinear, bounded and coercive. Let $f \in H$. Then $\exists_1 u \in H \colon a(u, \phi) = (f, \phi)$ for all $\phi \in H$. This equation is called* variational equation.

B Sobolev Spaces

B.1 Basic definition and properties

Motivation: Let $u \colon \mathbb{R}^d \to \mathbb{R}$ describe a quantity of interest, e.g. temperature.

- pointwise: $u(x)$, $x \in \mathbb{R}^d$.
- averaged: $\int_{\mathbb{R}^d} u(x)\phi(x)\, dx$ for an weight function $\phi \in C_c^\infty\left(\mathbb{R}^d\right)$.

The connection between both perspectives is the statement of the following lemma.

Lemma B.1 (Fundamental thm of the Calculus of Variations)**.** *Let $u \in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$. TFAE:*

- *$u = 0$ almost everywhere in $\mathbb{R}^d$.*
- *$\int_{\mathbb{R}^d} u\phi = 0$ for all $\phi \in C_c^\infty\left(\mathbb{R}^d\right)$.*

Beweis. Sketch for $\Leftarrow$:

1. $u \in C\left(\mathbb{R}^d\right)$. (Then $\int u\phi = 0$ for $\phi \in C_c^\infty\left(\mathbb{R}^d\right) \implies u = 0$ everywhere in $\mathbb{R}^d$.)

2. $C_c\left(\mathbb{R}^d\right) \subset L^1\left(\mathbb{R}^d\right)$ dense.

3. Let η_i be a partition of unity, i.e. $\eta_i \in C_c^\infty\left(\mathbb{R}^d\right)$ with $0 \le \eta_i \le 1$ and $\sum_{i=1}^\infty \eta_i \equiv 1$ in $\mathbb{R}^d$. Let $u \in L^1_{\text{loc}}$ with $\int u\phi = 0$ for all $\phi \in C_c^\infty\left(\mathbb{R}^d\right)$. Then $\sum_{i=1}^\infty \int u\eta_i\phi = 0$ for all ϕ. We have $u\eta_i\phi \in L^1\left(\mathbb{R}^d\right)$. Apply Step 2. □

Let $u \in C^1\left(\mathbb{R}^d\right)$, then by integration by parts:

$$\forall \phi \in C_c^\infty\left(\mathbb{R}^d\right): \int \partial_i u\phi = -\int u\partial_i\phi.$$

The right-hand-side expression $-\int u\partial_i\phi$ is well-defined for any $u \in L^1_{\text{loc}}\left(\mathbb{R}^d\right) \supsetneq C^1\left(\mathbb{R}^d\right)$.

Definition B.2 (weak derivative)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and suppose $u, v \in L^1_{\text{loc}}(\Omega)$. We say that v is the* α-th weak partial derivative *of u, and write $v = D^\alpha u$, if*

$$\forall \phi \in C_c^\infty(\Omega)\colon \int_\Omega v\phi = (-1)^{|\alpha|}\int_\Omega uD^\alpha\phi.$$

Lemma B.3. *Let $\Omega \subset \mathbb{R}^d$ be open.*

a) $u \in C^k(\Omega)$, $|\alpha| \le k$. Then the classical $D^\alpha u$ equals the weak $D^\alpha u$ a.e. in Ω.

b) The weak derivative of a $L^1_{\text{loc}}(\Omega)$-function is unique a.e. .

Beweis.

b) Assume $u, v, \tilde{v} \in L^1_{\text{loc}}$ satisfy

$$\forall \phi \in C_c^\infty(\Omega)\colon \int uD^\alpha\phi = (-1)^{|\alpha|}\int v\phi = (-1)^{|\alpha|}\int \tilde{v}\phi$$
$$\implies \int (v - \tilde{v})\phi = 0.$$

By lemma B.1: $v = \tilde{v}$ a.e. in Ω.

a) By integration by parts. □

Remark B.4. *The weak derivative is* local, *i.e.: If $u \in L^1_{\text{loc}}(\Omega)$ has weak derivative $D^\alpha u \in L^1_{\text{loc}}$ and $B \subset \Omega$ open, then $D^\alpha u|_B$ is a weak derivative of $u|_B$, since $C_c^\infty(B) \subset C_c^\infty(\Omega)$.*

Example B.5 (one-dimensional)**.** *Consider*

$$u(x) = \begin{cases} x & , x \le 1, \\ 1 & , x > 1. \end{cases}$$

Then u is smooth in $(-\infty, 1) \cup (1, \infty)$. If u is weakly differentiable, then by the previous remark and lemma B.3, the weak derivative must coincide with the classical one in $(-\infty, 1) \cup (1, \infty)$.

$$v(x) = u'(x) = \begin{cases} 1 & , x < 1, \\ 0 & , x > 1. \end{cases}$$

Then $v = u'$ a.e. in $\mathbb{R}$. Claim: $v(x)$ as defined above is the weak derivative of u. Let $\phi \in C_c^\infty(\mathbb{R})$. W.l.o.g. we may assume, that $\operatorname{supp}\phi \subset (-x_0, x_0)$ *for some $x_0 > 1$. Then*

$$\begin{aligned}\int u\phi' = \int_{-x_0}^{x_0} u\phi' &= \int_{-x_0}^{1} x\phi' + \int_1^{x_0} 1\phi' \\ &= -\int_{-x_0}^{1} \phi + [x\phi]_{-x_0}^{1} + [\phi]_1^{x_0} \\ &= -\int v\phi + \phi(1) - 0 + 0 - \phi(1) = -\int v\phi.\end{aligned}$$

Recall: For $u \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^d$ open and connected, we have: $\nabla u \equiv 0$ in $\Omega \implies u$ is constant. Argument: Let $x, y \in \Omega$ and γ a curve in Ω with $x = \gamma(0)$ and $y = \gamma(1)$. Then

$$u(x) - u(y) = u(\gamma(0)) - u(\gamma(1)) = \int_0^1 (u \circ \gamma)'(t)\ dt = \int_0^1 \nabla u(\gamma(t)) \cdot \gamma'(t)\ dt = 0.$$

Therefore $u(x) = u(y)$. Since x and y are arbitrary, u is constant.

Lemma B.6 (vanishing weak gradient implies constancy)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and connected. Suppose $u \in L^1_{\text{loc}}$ has α-th weak derivative for $|\alpha| \le 1$, then $\nabla u = 0$ a.e. in Ω implies $\exists C \in \mathbb{R}\colon u = C$ a.e. in Ω.*

Beweis. It suffices to show, that u is constant in any ball $B = B(x_0, r)$ with the property $B(x_0, 2r) \subset \Omega$. Let η_ε denote a standard (Friedrichs') mollifier (in particular: $\eta_\varepsilon \in C_c^\infty\left(\mathbb{R}^d\right)$ and $\operatorname{supp}\eta_\varepsilon = \overline{B(0;\varepsilon)}$). For $x \in B$, $0 < \varepsilon < r$ consider

$$u_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x - y) u(y)\ dy.$$

Then $u_\varepsilon \in C^1(B)$ and

$$\int_B |u_\varepsilon - u| \xrightarrow{\varepsilon \downarrow 0} 0 \tag{*}$$

by lemma **??** and theorem **??**. Claim: $\nabla u_\varepsilon = 0$ (classical) in B.

$$\begin{aligned}\partial_i u_\varepsilon(x) &= \int (\partial_i \eta_\varepsilon)(x - y) u(y)\ dy \\ &= -\int \partial_i \eta_{x,\varepsilon}(y) u(y)\ dy \\ &= \int \eta_{x,\varepsilon}(y) \partial_i u(y)\ dy = 0\end{aligned}$$

with $\eta_{x,\varepsilon}(y) := \eta_\varepsilon(x - y) \in C_c^\infty(B(x,\varepsilon)) \subset C_c^\infty(\Omega)$. We get $\nabla u_\varepsilon = 0$ in B. This implies $u_\varepsilon = ⨍_B u_\varepsilon$ and constant in B.

$$\begin{aligned}⨍_B \left| u - ⨍_B u \right| &\le ⨍_B |u - u_\varepsilon| + ⨍_B \left| u_\varepsilon - ⨍ u \right|, \qquad u_\varepsilon = ⨍_B u_\varepsilon \\ &\le ⨍_B |u - u_\varepsilon| + ⨍_B |u - u_\varepsilon| \xrightarrow[(*)]{\varepsilon \downarrow 0} 0.\end{aligned}$$

Therefore $u = ⨍_B u$ a.e. in B. $\square$

Lemma B.7 (Additivity)**.** *Let $\Omega \subset \mathbb{R}^d$ open, $\alpha \in \mathbb{N}_0^d$, $u, v \in L^1_{\text{loc}}(\Omega)$ with weak α-th derivatives $D^\alpha u, D^\alpha v$. Then for all $\lambda \in \mathbb{R}$ $u + \lambda v$ has weak α-th derivative, namely*

$$D^\alpha(u + \lambda v) = D^\alpha u + \lambda D^\alpha v.$$

Beweis. Let $\phi \in C_c^\infty(\Omega)$.

$$\begin{aligned}(-1)^{|\alpha|} \int_\Omega (u + \lambda v) D^\alpha \phi &= (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi + \lambda (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \\ &= \int_\Omega D^\alpha u \phi + \lambda \int_\Omega D^\alpha v \phi \\ &= \int_\Omega (D^\alpha u + \lambda D^\alpha v)\,.\end{aligned}$$ □

Example B.8.

$$u \colon \mathbb{R} \to \mathbb{R},\ u(x) = \begin{cases} x & ,x \le 1 \\ 1 & ,x > 1 \end{cases}, \quad u'(x) = \begin{cases} 1 & ,x \le 1 \\ 0 & ,x > 1 \end{cases}, \quad w(x) = \begin{cases} x & ,x \le 1 \\ 2 & ,x > 1. \end{cases}$$

Assume w has weak derivative $w' = u'$, since w is smooth in $(-\infty, 1) \cup (1, \infty)$. Then $v = u - w$ is weakly differentiable by lemma B.7 and $v' = u' - w' = 0$ a.e. $\implies \exists C \in \mathbb{R} \colon v = c$ a.e. in $\mathbb{R}$ by lemma B.6. Contradiction to

$$r(x) = \begin{cases} 0 & ,x \le 1 \\ -1 & ,x \ge 1. \end{cases}$$

Definition B.9 (Sobolev space)**.** *Let $\Omega \subset \mathbb{R}^d$ open, $k \in \mathbb{N}_0$, $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of all $u \in L^1_{\text{loc}}(\Omega)$ s.t. for all $\alpha \in N_0^d$ with $|\alpha| \le k$ the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$.*

Remark B.10. *In the case $p = 2$, $H^k(\Omega) := W^{k,2}(\Omega)$ is a standard notation.*

Note: $W^{0,p}(\Omega) = L^p(\Omega)$. Elements of $W^{k,p}(\Omega)$ are equivalence classes of functions that agree a.e. in Ω. In particular, $f = g$ in $W^{k,p}(\Omega) \iff D^\alpha f = D^\alpha g$ a.e. in Ω for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$. For $k' < k$ we have $W^{k,p} \subset W^{k',p}$. Further $W^{\infty,p}(\Omega) = \cap_{k=1}^\infty W^{k,p}(\Omega) = C^\infty(\Omega) \cap W^{k,p}(\Omega)$.

Definition B.11. *For $u \in W^{k,p}(\Omega)$ we define its norm:*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & ,1 \le p < \infty \\ \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(\Omega)} & ,p = \infty. \end{cases}$$

Theorem B.12. $\left(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}\right)$ *is a Banach space.*

Beweis. First note that $W^{k,p}(\Omega)$ is a vector space.

1. $\|\cdot\| := \|\cdot\|_{W^{k,p}(\Omega)}$ is a norm:
   - $\|\lambda u\| = |\lambda| \|u\|$ ✓.

- $\|u\| = 0 \implies \|u\|_{L^p(\Omega)} = 0 \implies u = 0$ a.e. in Ω, therefore $u = 0$ in $W^{k,p}(\Omega)$.
- triangle inequality: $p = \infty$ (follows from triangle inequality for $\|\cdot\|_{L^\infty(\Omega)}$.)

$$
\begin{aligned}
\|u+v\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha|\le k} \|D^\alpha (u+v)\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \\
&\le \left(\sum_{|\alpha|\le k} (\|D^\alpha u\|_{L^p} + \|D^\alpha v\|_{L^p})^p \right)^{\frac{1}{p}} \\
&\le \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\le k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}
\end{aligned}
$$

$= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)}$.

2. Completeness: Let (u_j) be cauchy in $W^{k,p}(\Omega)$. For all $|\alpha| \le k$, $(D^\alpha u_j)$ is cauchy in $L^p(\Omega)$. $L^p(\Omega)$ is Banach, therefore $\exists u_\alpha \in L^p(\Omega)\colon D^\alpha u_j \to u_\alpha$ in $L^p(\Omega)$ for all $|\alpha| \le k$. In particular: $u_j = D^0 u_j \to u_0 =: u$ in $L^p(\Omega)$. Claim: u_α is α-th weak derivative of u. For all $\phi \in C_c^\infty(\Omega)$:

$$
\begin{aligned}
\int_\Omega u_\alpha \phi = \lim_{j\to\infty} \int_\Omega D^\alpha u_j \phi &= \lim_{j\to\infty} (-1)^{|\alpha|} \int_\Omega u_j D^\alpha \phi \\
&= (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi.
\end{aligned}
$$

Thus $u \in W^{k,p}(\Omega)$ and $D^\alpha u_j \to D^\alpha u$ in $L^p(\Omega)$ for all $|\alpha| \le k$. Therefore $\|u_j - u\|_{W^{k,p}(\Omega)}$ □.

Theorem B.13. *$W^{k,p}(\Omega)$ with the inner product*

$$
(u,v)_{W^{k,p}(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega D^\alpha u D^\alpha v
$$

is a Hilbert space.

Beweis. The proof is left as an exercise. □

Note: $C_c^\infty(\Omega) \subset W^{k,p}(\Omega)$.

Definition B.14. *Let $\Omega \subset \mathbb{R}^d$ be open. We denote by $W_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

Lemma B.15 (Poincaré's Inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ open and bounded, $1 \le p < \infty$. Then $\forall u \in W_0^{1,p}(\Omega)$:*

$$
\|u\|_{L^p(\Omega)} \le \operatorname{diam}(\Omega) \|\nabla u\|_{L^p(\Omega)}.
$$

Beweis.

1. The case $\Omega = (0,1)^d$, $u \in C_c^\infty(\Omega)$. Write $x = (x_1, x')$, $x_1 \in (0,1)$, $x' \in (0,1)^{d-1}$. Then
$$u(x) = \int_0^{x_1} \partial_1 u(t, x')\ dt$$
for all $x \in (0,1)^d = \Omega$. Then by Jensen's inequality:
$$|u(x)|^p \le \int_0^1 |\partial_1 u(t,x')|^p\ dt$$
Integrate over $\Omega = (0,1)^d$:
$$\begin{aligned}\int_\Omega |u(x)|^p\ dx &\le \int_\Omega \int_0^1 |\partial_1 u(t,x')|^p\ dt\ d(x_1,x') \\ &= \int_0^1 \int_{(0,1)^{d-1}} \int_0^1 |\partial_1 u(t,x')|^p\ dt\ dx'\ dx_1 \\ &= \int_0^1 \int_{(0,1)^d} |\partial_1 u|^p\ dy\ dx_1, \qquad y = (t,x') \\ &= \int_\Omega |\partial_1 u|^p\ dy \le \int_\Omega |\nabla u|^p\ dy.\end{aligned}$$

2. Let Ω be open and bounded, $u \in C_c^\infty(\Omega)$. There exists a transformation $T\colon \mathbb{R}^d \to \mathbb{R}^d$, $Tx = (Qx - x_0)r^{-1}$ for some $Q \in \mathcal{O}(d)$, $x_0 \in \mathbb{R}^d$, $r = \operatorname{diam}(\Omega)$, s.t. $T\Omega =: \Omega' \subset (0,1)^d$. Set $w(x) = u\left(T^{-1}x\right) = u(Q^T(rx + x_0))$. Since $u \in C_c^\infty(\Omega)$, $w \in C_c^\infty(\Omega') \subset C_c^\infty((0,1)^d)$. By the 1st step:
$$\int_{\Omega'} |w|^p\ dx \le \int_{\Omega'} |\nabla w|^p\ dx.$$
We have
$$\fint_\Omega |u|^p = \fint_{\Omega'}, \quad \nabla w(x) = rQ^T \nabla u\left(T^{-1}x\right)$$
$$\fint_{\Omega'} |\nabla w(x)|^p\ dx = r^p \fint_{\Omega'} |\nabla u\left(T^{-1}x\right)|^p = r^p \fint_\Omega |\nabla u|^p\ dx.$$
$$\implies \fint_\Omega |u|^p \le r^p \fint_\Omega |\nabla u|^p$$
$$\implies \fint_\Omega |u|^p \le (\operatorname{diam}\Omega)^p \int_\Omega |\nabla u|^p$$
This argument is called a *scaling argument*.

3. Proof of the general case. Let $u \in W_0^{1,p}(\Omega)$. Since $C_c^\infty(\Omega) \subset W_0^{1,p}(\Omega)$ dense:
$$\exists (u_j) \subset C_c^\infty(\Omega)\colon u_j \to u \text{ in } W^{1,p}(\Omega)$$
By the 2nd step: For all j and taking the limit $j \to \infty$:
$$\int_\Omega |u_j|^p \le \operatorname{diam}(\Omega)^p \int_\Omega |\nabla u_j|^p$$
$$\int_\Omega |u|^p \le \operatorname{diam}(\Omega)^p \int_\Omega |\nabla|^p$$
This is called a *density argument*. $\square$

Exercise B.16. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $A\colon \Omega \to \mathbb{R}^{d\times d}$ measurable and suppose $\exists 0 < \lambda < \Lambda < \infty$:*

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2, \quad |A(x)\xi| \leq \Lambda|\xi|$$

for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$. Consider a bilinear form $a\colon W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) := \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\ dx.$$

1. *Show that a is bounded and coercive.*

2. *Show that for all $f \in L^2(\Omega)$, $F \in L^2(\Omega, \mathbb{R}^d)$ there exists a unique function $u \in W_0^{1,2}(\Omega)$ s.t.*
$$a(u,\phi) = \int_\Omega f\phi + \int_\Omega F \cdot \nabla\phi\ dx.$$
for all $\phi \in W_0^{1,2}(\Omega)$.

3. *Show that the solution of 2) satisfies*
$$\|u\|^2_{W_0^{1,2}(\Omega)} \leq C\left(\|f\|^2_{L^2(\Omega} + \|F\|^2_{L^2(\Omega)}\right)$$
for some $C = C(\mathrm{diam}(\Omega), \lambda, \Lambda)$.

4. *Suppose Ω is a C^1-domain, $A \in C^\infty\left(\bar{\Omega}, R^{d\times d}\right)$, $u \in C_0^2\left(\bar{\Omega}\right)$, $f \in C\left(\bar{\Omega}\right)$, $F \in C^1\left(\bar{\Omega}, \mathbb{R}^d\right)$. Then u is a classical solution to*
$$\begin{cases} -\nabla \cdot (A\nabla u) & = f - \nabla \cdot F \text{ in } \Omega \\ u & = 0 \text{ on } \partial\Omega \end{cases}$$

5. *Show that any $T \in \left(W_0^{1,2}(\Omega)\right)^*$ can be written as $\langle T, \phi\rangle = \int_\Omega F \cdot \nabla\phi\ dx$ for a unique $F \in L^2(\Omega, \mathbb{R}^d)$.*

$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \mid \forall|\alpha| \leq k\colon D^\alpha u \in L^p(\Omega)\}$.

Example B.17 (Sobolev functions can be (locally) unbounded)**.** *Let $d > 1$ and $\Omega = B(0,1) \subset \mathbb{R}^d$. Define $u(x) := |x|^{-\alpha}$ for $x \neq 0$ and $\alpha > 0$.*

1. *$u \in L^p(B)$? Posing this question is equivalent to investigating*
$$\lim_{\rho\downarrow 0} \int_{B\setminus B(0,\rho)} |u|^p < \infty$$
We compute the integral.
$$\begin{aligned} \int_{B\setminus B(0,\rho)} |u|^p &= \int_\rho^1 \int_{\partial B(0,r)} |u(\xi)|^p\ dS(\xi)\ dr \\ &= |\partial B(0,1)| \int_\rho^1 r^{-\alpha p} r^{d-1}\ dr \xrightarrow{\rho\downarrow 0} \text{something } < \infty \end{aligned}$$
which is equivalent to $-\alpha p + d - 1 > -1 \iff \alpha < dp^{-1}$.

2. $u \in W^{1,p}(B)$? *Assume* $u \in W^{1,p}(B)$. *Then* ∇u *(weak)* $= \nabla u$ *(classical) in* $B \setminus \{\cdot\}$. *We make a computation:*

$$\partial_i u = -\alpha \frac{x_i}{|x|^{\alpha+2}} \implies |\nabla u(x)| = \frac{\alpha}{|x|^{\alpha+1}}.$$

$|\nabla u| \in L^p(B) \iff \alpha + 1 < dp^{-1} \iff \alpha < (d-p)p^{-1}$. *That is a necessary condition. Let* $\alpha < (d-p)p^{-1}$. *We claim:* $\partial_i u$ *as above is a weak derivative of* u. *Let* $\phi \in C_c^\infty(B)$. *We have to show*

$$\int_B \left(-\alpha \frac{x_i}{|x|^{\alpha+2}}\right) \phi(x)\ dx = -\int_B u \partial_i \phi.$$

For the left-hand-side we have

$$\int_{B\setminus B(0,\rho)} \left(-\alpha \frac{x_i}{|x|^{\alpha+2}}\right) \phi(x)\ dx \xrightarrow{\rho\to 0} \int_B \left(-\alpha \frac{x_i}{|x|^{\alpha+2}}\right) \phi(x)\ dx,$$

because the integrand is a $L^1(B)$*-function, since* $\alpha < (d-p)p^{-1}$. *For the right-hand-side we have*

$$-\int_{B\setminus B(0,\rho)} u\partial_i\phi \xrightarrow{\rho\to 0} -\int_B u\partial_i\phi.$$

We apply Gauss's theorem (integration by parts)

$$\int_{B\setminus B(0,\rho)} \partial_i u\phi = -\int_{B\setminus B(0,\rho)} u\partial_i\phi + \int_{\partial B(0,\rho)} u\phi\nu\cdot e_i\ dS.$$

Note:

$$\begin{aligned} \left|\int_{\partial B(0,\rho)} u\phi\nu\cdot e_i\ dS\right| &\le \int_{\partial B(0,\rho)} |u|\cdot|\phi|\ dS \\ &\le \|\phi\|_\infty \rho^{-\alpha}|\partial B(0,\rho)| \\ &\le \|\phi\|_\infty \rho^{-\alpha}\rho^{d-1}|\partial B(0,1)| \\ &\le C(\phi,d)\rho^{d-1-\alpha}. \end{aligned}$$

Since $\alpha < (d-p)p^{-1}$, *we have* $d-1-\alpha > 0$ *and*

$$\left|\int_{\partial B(0,\rho)} u\phi\nu\cdot e_i\ dS\right| \xrightarrow{\rho\downarrow 0} 0.$$

Any function $u \in L^p(B)$ *satisfies*

$$\lim_{\rho\downarrow 0} \int_{B\setminus B(0,\rho)} |u|^p < \infty$$

and vice versa. We apply Fubini's theorem:

$$\begin{aligned} \int_{B\setminus B(0,\rho)} |u|^p &= \int_\rho^1 \int_{\partial B(0,r)} |u(\xi)|^p\ dS(\xi)\ dr \\ &= |\partial B(0,1)| \int_\rho^1 r^{-\alpha p} r^{d-1}\ dr \\ &\xrightarrow{\rho\downarrow 0} \text{ something } < \infty \end{aligned}$$

if and only if $-\alpha p + d - 1 > -1$ *or equivalently* $a < dp^{-1}$ *and* $u \in W^{1,p}(B)$.

Exercise B.18. *Let $\Omega \subset \mathbb{R}^d$ be open, $1 \le p < \infty$ and $k = 1$. Show that $\exists u \in W^{1,p}(\Omega)$ s.t. u is unbounded in any open $U \subset \Omega$.*

Exercise B.19. *$B := B(0,1) \subset \mathbb{R}^d$, $d \ge 2$. Claim: $\exists u \in W^{1,1}(B)$ that is unbounded on every open subset of B, i.e.*

$$\forall A \subset B \text{ open } \forall 0 < R < \infty\colon\ |\{x \in A\colon |u(x)| > R\}| > 0.$$

Proposition B.20 (Calculus rules)**.** *$u, v \in W^{k,p}(\Omega)$, $|\alpha| \le k$.*

1. *$\forall |\beta| + |\alpha| \le k\colon D^\alpha D^\beta u = D^\beta D^\alpha u = D^{\alpha+\beta} u$.*
2. *$V \subset \Omega$ open $\implies u \in W^{k,p}(V)$.*
3. *$\eta \in C_c^\infty(\Omega) \implies \eta u \in W^{k,p}(\Omega)$. Leibniz rule:*

$$D^\alpha(\eta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\eta} D^\beta \eta D^{\alpha-\beta} u$$

with $\beta \le \alpha :\iff \forall i = 1, \dots, d\colon \beta_i \le \alpha_i$ and the binomial coefficient

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \qquad \alpha! = \alpha_1 \cdot \ldots \cdot \alpha_d.$$

Beweis. 1. Let $\phi \in C_c^\infty(\Omega)$, $D^\beta \phi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned} \int D^\alpha u D^\beta \phi &= (-1)^{|\alpha|} \int u \underbrace{D^\alpha D^\beta \phi}_{=D^{\alpha+\beta}\phi} \\ &= \underbrace{(-1)^{|\alpha|} \cdot (-1)^{|\alpha+\beta|}}_{=(-1)^{|\beta|}} \int D^{\alpha+\beta} u \phi \\ &= (-1)^{|\beta|} \int D^{\alpha+\beta} u \phi. \end{aligned}$$

Using the definition of the weak derivative we get

$$D^\beta D^\alpha u = D^{\alpha+\beta} u = D^{\beta+\alpha} u = D^\alpha D^\beta u,$$

the so-called *Schwarz rule.*

2. This follows directly from the locality property of weak derivative.

3. Proof by induction: First consider $|\alpha| = 1$. Let $\phi \in C_c^\infty(\Omega)$.

$$\begin{aligned} \int u \eta D^\alpha \phi &= \int u \left(D^\alpha(\underbrace{\eta\phi}_{\in C_c^\infty(\Omega)}) - \eta D^\alpha \phi \right) \\ &= -\int (D^\alpha u)\, \eta\phi + u D^\alpha \eta \phi \\ &= -\int \underbrace{((D^\alpha u)\, \eta + u D^\alpha \eta)}_{=D^\alpha(u\eta)} \phi. \end{aligned}$$

Next, $\alpha_1 \rightsquigarrow \alpha_1 + \alpha_0 =: \alpha$ with $|\alpha_0| = 1$.

$$\begin{aligned}
\int u\eta D^\alpha \phi &= \int u\eta D^{\alpha_1} D^{\alpha_0}\phi \\
&= (-1)^{|\alpha_1|} \int D^{\alpha_1}(u\eta) D^{\alpha_0}\phi \\
&= (-1)^{|\alpha_1|} \sum_{\beta_1 \le \alpha_1} \binom{\alpha_1}{\beta_1} \int D^{\beta_1}\eta D^{\alpha_1-\beta_1} u D^{\alpha_0}\phi \\
&= (-1)^{|\alpha_1|} \sum_{\beta_1 \le \alpha_1} \binom{\alpha_1}{\beta_1} \left(- \int \left(D^{\beta_1+\alpha_0}\eta D^{\alpha_1-\beta_1} u + D^{\beta_1} D^{\alpha_1+\alpha_0-\beta_1} u\phi \right) \right) \\
&= I + II
\end{aligned}$$

$\beta = \beta_1 + \alpha_0$, $\alpha = \alpha_1 + \alpha_0$. We compute I and II:

$$\begin{aligned}
I &= (-1)^{|\alpha|} \sum_{\alpha_0 \le \beta \le \alpha} \binom{\alpha-\alpha_0}{\beta-\alpha_0} \int D^\beta \eta D^{\alpha-\beta} u\phi \\
&= (-1)^{|\alpha|} \binom{\alpha-\alpha_0}{\alpha-\alpha_0} \int D^\alpha \eta u\phi + (-1)^{|\alpha|} \sum_{\alpha_0 \le \beta \le \alpha_1} \binom{\alpha-\alpha_0}{\beta-\alpha_0} \int D^\beta \eta D^{\alpha-\beta} u\phi \\
II &= (-1)^{|\alpha|} \binom{\alpha_1}{0} \int \eta D^\alpha u\phi + (-1)^{|\alpha|} \sum_{\alpha_0 \le \beta \le \alpha_1} \binom{\alpha_1}{\beta} \int D^\beta \eta D^{\alpha-\beta} u\phi
\end{aligned}$$

Adding up both summands:

$$\begin{aligned}
I + II = (-1)^{|\alpha|} \binom{\alpha}{\alpha} \int D^\alpha \eta u\phi &+ (-1)^{|\alpha|} \sum_{\alpha_0 \le \beta \le \alpha_1} \binom{\alpha}{\beta} \int D^\beta \eta D^{\alpha-\beta} u\phi \\
&+ (-1)^{|\alpha|} \binom{\alpha}{0} \int \eta D^\alpha u\phi
\end{aligned}$$

which equals the RHS of claimed identity. □

B.2 Approximation

Motivation: In an exercise we observed that the completion $C^1(\bar{\Omega})$ with respect to $\|\cdot\|_{W^{1,2}}$ is a (closed) linear subspace of $W^{1,2}(\Omega)$ (up to a isometric isomorphism). We show that any $u \in W^{k,p}(\Omega)$ is a limit of a sequence in $C^\infty(\Omega) \cap W^{k,p}(\Omega) = \{u \in C^\infty(\Omega) \mid D^\alpha u \in L^p(\Omega)\ \forall |\alpha| \le k\}$.

Remark B.21 (Notation). $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Theorem B.22 (local approximation by smooth function). *Let $u \in W^{k,p}(\Omega)$, $1 \le p < \infty$ and $u_\varepsilon := u \star \eta_\varepsilon$, where η_ε is a standard mollifier. Then*

1. *$u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and*
2. *$u_\varepsilon \to u$ in $W^{k,p}(V)$ for all $V \subset\subset \Omega$.*

Beweis. We identify u with its trivial extension to $\mathbb{R}^d$, i.e.

$$x \mapsto \begin{cases} u(x) & , x \in \Omega, \\ 0 & , x \notin \Omega. \end{cases}$$

$u \in L^p\left(\mathbb{R}^d\right)$.

1. Since $\eta_\varepsilon \in C_c^\infty(\Omega) \implies u_\varepsilon \in C^\infty\left(\mathbb{R}^d\right)$ by the differentiation lemma for convolutions **??**. Note that $u_\varepsilon(x)$ for $x \in \Omega_\varepsilon$ only depends on $u|_\Omega$, since $\operatorname{supp} \eta_\varepsilon \subset \overline{B(0,\varepsilon)}$.

2. Claim: $D^\alpha u_\varepsilon = \eta_\varepsilon \star D^\alpha u$ in Ω_ε for $|\alpha| \leq k$. Argument: For $\phi \in C_c^\infty(\Omega_\varepsilon)$ we have

$$\begin{aligned}
\int_\Omega D^\alpha u_\varepsilon \phi &= (-1)^{|\alpha|} \int_\Omega u_\varepsilon D^\alpha \phi \\
&= (-1)^{|\alpha|} \int_\Omega \int_\Omega \eta_\varepsilon(x-y) u(y) D^\alpha \phi(x) \; dy \; dx \\
&= (-1)^{|\alpha|} \int_\Omega u(y) \int_\Omega \eta_\varepsilon(x-y) D^\alpha \phi(x) \; dx \; dy \\
&= (-1)^{|\alpha|} \int_\Omega u(y) \int_\Omega \underbrace{D_x^\alpha \eta_\varepsilon(x-y)}_{=(-1)^{|\alpha|} D_y^\alpha \eta_\varepsilon(x-y)} \phi(x) \; dx \; dy \\
&= (-1)^{|\alpha|} \int_\Omega \int_\Omega u(y) \left(D_y^\alpha \eta_\varepsilon\right)(x-y) \phi(x) \; dy \; dx.
\end{aligned}$$

$\Psi(y) = \eta_\varepsilon(x-y)$ defines a function $\Psi \in C_c^\infty(\Omega)$.

$$\begin{aligned}
D_y^\alpha \eta_\varepsilon(x-y) &= D^\alpha \Psi(y) \\
&= \int_\Omega \int_\Omega D^\alpha u(y) \eta_\varepsilon(x-y) \; dy \Psi(x) \; dx \\
&= \int_\Omega D^\alpha u \star \eta_\varepsilon \phi \\
\implies D^\alpha u_\varepsilon &= D^\alpha u \star \eta_\varepsilon \text{ in } \Omega_\varepsilon.
\end{aligned}$$

Let $V \subset\subset \Omega$ open, $\varepsilon > 0$ small enough, s.t. $V \in \Omega_\varepsilon$. $D^\alpha u_\varepsilon = D^\alpha u \star \eta_\varepsilon$ and $D^\alpha u \in L^p(\Omega) \implies D^\alpha u_\varepsilon \to D^\alpha u$ in $L^p(\Omega_\varepsilon)$ and since $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$ arbitrary, we conclude

$$\|u_\varepsilon - u\|_{W^{1,p}(\Omega)} \to 0.$$

□

Theorem B.23 (global approximation by smooth functions)**.** *Let $\Omega \subset \mathbb{R}^d$ open and bounded, $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ dense.*

Beweis. $u \in W^{k,p}(\Omega)$. We need to show $\exists (v_j)_j \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$, s.t. $v_j \to u$ in $W^{k,p}(\Omega)$.

1. Construction of a covering of Ω.

$$\Omega = \bigcup_{k=1}^\infty \Omega_k, \; \Omega_k := \left\{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{k} \right\}.$$

Define $A_k := \Omega_{k+4} \setminus \bar{\Omega}_{k+1}$. $\{A_k\}_{k=1}^{\infty}$ covers Ω locally uniform and finite. Choose $A_0 \subset\subset \Omega$, s.t.

$$\Omega = \bigcup_{k=0}^{\infty} A_k.$$

Let $(\eta_k)_k$ be a partition of unity, subnormal to $(A_k)_k$, i.e.

$$\eta_k \in C_c^{\infty}(A_k),\ 0 \le \eta_k \le 1 \wedge \sum_{k \in \mathbb{N}_0} \eta_k = 1.$$

We get $u \in W^{k,p}(\Omega)$, $u_k = \eta_k u \in W^{k,p}(\Omega)$ and $\sum_k u_k = u$. $u_k \equiv 0$ a.e. in $\Omega \setminus A_k$. Fix $\delta > 0$. By theorem B.22 we find $0 < \varepsilon_k << 1$ s.t. $v_k = u_k \star \eta_{\varepsilon_k}$, where η_{ε_k} is the mollifier of thm B.22.

2. $v_k \in C^{\infty}(\Omega)$, $\operatorname{supp} v_k$ is supported in $\Omega_{k+4} \setminus \bar{\Omega}_k$.
3. $\|u_k - v_k\|_{W^{k,p}(\Omega)} \le \delta 2^{-(j+1)}$. Set $v_\delta = \sum_{k \in \mathbb{N}} v_k$. In any open ball $B \subset\subset \Omega$, we have $v_j = 0$ in B for all but finitely many j's and $v_\delta \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$.

$$\begin{aligned}
\|v_\delta - u\|_{W^{k,p}(\Omega)} &\le \sum_k \|v_k - u_k\|_{W^{k,p}(\Omega)} \\
&\le \delta \sum_k 2^{-(k+1)} \le \delta
\end{aligned}$$

Therefore $C^{\infty}(\Omega) \cap W^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ dense. □

Typical application: Lift a property known for smooth functions to Sobolev functions.

Example B.24. *For $u, v \in C^{\infty}(\Omega)$, $\alpha \in \mathbb{N}_0^d$, we have*

$$D^{\alpha}(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^{\beta} u D^{\alpha-\beta} v.$$

Then this identity survives for Sobolev functions as can be seen using an approximation arguemtn based on theorem B.23, i.e. this identity holds for $u \in W^{k,p}(\Omega)$ and $|\alpha| \le k$.

Lemma B.25. *$u \in W^{k,p}(\Omega)$, $\varphi \in C_c^{\infty}(\Omega)$, $|\alpha| \le k$:*

$$D^{\alpha}(u\varphi) = \sum_{|\beta| \le |\alpha|} \binom{\alpha}{\beta} D^{\beta} u D^{\alpha-\beta} \varphi. \tag{B.1}$$

Beweis. By Analysis 1 and 2 we know (B.1) for u, $\varphi \in C^{\infty}(\Omega)$. By density of $C^{\infty} \cap W^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ we can find $(u_k) \subset C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$ s.t. $u_k \to u$ in $W^{k,p}(\Omega)$, so

$$D^{\alpha}(u_k\varphi) = \sum_{|\beta| \le |\alpha|} \binom{\alpha}{\beta} D^{\beta} u_k D^{\alpha-\beta} \varphi =: F_k.$$

We claim:

$$F_k \to F_0 := \sum_{|\beta| \le |\alpha|} \binom{\alpha}{\beta} D^{\beta} u D^{\alpha-\beta} \varphi$$

in $L^1(\Omega)$ as $k \uparrow \infty$.

$$\begin{aligned}\int_\Omega |F_k - F_0| &\le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_\Omega \left|D^\beta u_k - D^\beta u\right| \left|D^{\alpha-\beta}\varphi\right| \\ &\le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left\|D^\beta u_k - D^\beta u\right\|_{L^p(\Omega)} \left\|D^{\alpha-\beta}\varphi\right\|_{L^q(\Omega)}\end{aligned}$$

by Hölder's inequality. From $u_k \to u$ in $W^{k,p}(\Omega)$ we learn that

$$\left\|D^\beta u_k - D^\beta u\right\|_{L^p(\Omega)} \to 0$$

and therefore $RHS \to 0$. Conclusion: Let $\eta \in C_c^\infty(\Omega)$.

$$\int D^\alpha(u_k\varphi)\eta = (-1)^{|\alpha|} \int u_k \varphi D^\alpha \eta \to (-1)^{|\alpha|} \int u\varphi D^\alpha \eta.$$

So F_0 is α-th weak derivative of $u\varphi$. $\square$

Theorem B.26. *Let Ω be open, bounded and $\partial\Omega$ be C^1. Let $1 \le p < \infty$. Then $C^\infty\left(\bar\Omega\right) \subset W^{k,p}(\Omega)$ dense.*

Beweis.

1. Step: Fix $x_0 \in \partial\Omega$. Since $\partial\Omega$ is C^1, $\exists \nu$ outer unit normal vector at x_0. W.l.o.g. $\nu = -e_d$. Further $\exists r > 0, \gamma\colon \mathbb{R}^{d-1} \to \mathbb{R}$ C^1 s.t. $\partial\Omega$ locally at x_0 is the graph of γ. $\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \mid \gamma(x) < x_d\}$. Set $V = \Omega \cap B(x_0, \frac{r}{2})$. Define transformation
$$\varphi_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d,\ \varphi_\varepsilon(x) = x + \lambda\varepsilon e_d$$
with $\lambda > 1$ fixed. For all $\varepsilon > 0$ sufficiently small, have $B(\varphi_\varepsilon(x), \varepsilon) \subset \Omega \cap B(x_0, r)$ for all $x \in V$. Define
$$u_\varepsilon\colon V \to \mathbb{R},\ u_\varepsilon(x) = u(\varphi_\varepsilon(x))$$
translation of u by $\varepsilon\lambda e_d$. Define $v_\varepsilon = \eta_\varepsilon \star u_\varepsilon$. By construction $v_\varepsilon \in C^\infty\left(\bar V\right)$. Claim: $v_\varepsilon \to u$ in $W^{k,p}(V)$. Fix $|\alpha| \le k$.
$$\|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)} \le \|D^\alpha v_\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} + \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)}.$$
The second term on the RHS vanishes for $\varepsilon \downarrow 0$ since $D^\alpha u_\varepsilon(x) = (D^\alpha u)(\varphi_\varepsilon(x))$ and translation is a continuous operation on L^p. The first expression vanishes for $\varepsilon \downarrow 0$ by an argument similar to the one in theorem B.12. To conclude the proof argue by a partition of unity (sketch):
    - $\exists u_0, u_1, u_2, \ldots \in W^{k,p}(\Omega)$ s.t.
        - u_0 is supported in $A \subset\subset \Omega$.
        - u_i is supported in $B(x_i, \frac{r_i}{2})$ with $x_i \in \partial\Omega$ and r_i so small s.t. the above argument applies.
        - $u = \sum_{i\in\mathbb{N}} u_i$ where is locally finite.

For u_0, get proxy by $u_{0,\varepsilon} = \eta_\varepsilon \star u_0$ with $0 < \varepsilon < \operatorname{dist}(A, \partial\Omega)$. For u_i use argument above.

For detailed proof see book by Evans. $\square$

Example B.27 (counter)**.** *$B = B(0,1)$, $\Omega = \{x \in B \mid x_1 \neq 0\}$. Claim: $C^\infty(\bar{\Omega}) \subset W^{k,p}(\Omega)$ but not dense. Argument: Suppose opposite: $C^\infty(\bar{\Omega}) = C^\infty(\overline{B(0,1)}) \subset W^{k,p}(\Omega)$. The function*

$$u(x) = \begin{cases} 1 & , x_1 > 0 \\ 0 & , x_1 < 0 \end{cases}$$

is contained in $W^{k,p}(\Omega)$ and can be approximated by $C^\infty(\bar{\Omega})$ functions. Let $(u_k) \subset C^\infty(\bar{\Omega})$ s.t. $u_k \to u$ in $W^{k,p}(\Omega)$. Note: Since $u_k \in C^\infty \subset W^{k,p}(B(0,1))$ and $|B(0,1) \setminus \Omega| = 0$

$$\int_{B(0,1)} |u_k|^p + \sum_\alpha |D^\alpha u_k|^p = \int_\Omega |u_k|^p + \sum_\alpha |D^\alpha u_k|^p \implies \|u_k\|_{W^{k,p}(B(0,1))} = \|u_k\|_{W^{k,p}(\Omega)}.$$

$u_k \to u$ in $W^{k,p}(\Omega)$. Therefore (u_k) is cauchy in $W^{k,p}(B(0,1))$. There exists $\bar{u} \in W^{k,p}(B(0,1))$: $u_k \to \bar{u}$ in $W^{k,p}(B(0,1))$. Since $\Omega \subset B(0,1)$:

- *$\bar{u} = u$ a.e. in $B(0,1)$.*
- *$\nabla \bar{u} = \nabla u = 0$ a.e. in $B(0,1)$ (piecewise constant).*

Since $B(0,1)$ is connected, $\bar{u} = u$ is constant. This is the desired contradiction.

B.3 Trace

Consider

$$-\nabla \cdot A \nabla u = 0 \text{ in } \Omega \subset \mathbb{R}^d,$$
$$u = g \text{ on } \partial\Omega.$$

Existence theory based on Lax-Milgram uses Sobolev space $W^{1,2}(\Omega)$. Elements in $W^{1,2}(\Omega)$ are only determined up to dx-null sets. But typically $|\partial\Omega| = 0$. So, it is not clear of how to understand $u = g$ on $\partial\Omega$. Assume, that Ω is a C^1-domain. For $u \in C(\bar{\Omega})$, the boundary condition makes sense and we know $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Theorem B.28 (Trace operator)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded C^1-domain. There exists a unique, linear and bounded operator*

$$T \colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that $\forall u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})\colon Tu = u|_{\partial\Omega)}$.

Beweis.

- $T \colon W^{1,p} \cap C(\bar{\Omega}) \to L^p(\partial\Omega)$, $Tu := u|_{\partial\Omega}$. Claim: T is well-defined (since u is bounded on $\bar{\Omega}$, we have $u|_{\partial\Omega} \in L^p(\partial\Omega)$) and linear (since the restriction is linear). We only need to show that T is bounded in the sense that $\exists C$ such that:

$$\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \tag{B.2}$$

  for all $u \in C^\infty(\bar{\Omega})$.

- Note: from (B.2) and the first step, the statement follows. Indeed, since $C^\infty(\bar\Omega) \subset W^{1,p}(\Omega) \cap C(\bar\Omega)$ is dense in $W^{1,p}(\Omega)$, (B.2) implies T is densely defined and bounded. Thus $\exists_1$ linear and bounded operator that extends T. By this procedure, if suffices to analyze the following situation: $x_0 \in \partial\Omega$, $\Gamma = \partial\Omega \cap B(x_0, 2r) \subset \{x \mid x_d = 0\}$. Let ξ be a cut-off function for $\hat B := B(x_0, r)$ in $B = B(x_0, 2r)$, i.e. $0 \le \xi \le 1$, $\xi = 1$ in $\hat B$, $\operatorname{supp} \xi \subset B$.

$$\begin{aligned}\int_{\Gamma\cap\hat B} |u|^p = \int_{\Gamma\cap\hat B} |u|^p \xi &\le \int_{\{x_d=0\}\cap B} |u(x',0)|^p \xi(x',0)dx' \\ &= \int_{\partial B_+} |u|^p \xi \; dS = -\int_{B_+} \partial_d \left(|u|^p \xi\right) \\ &= -\int_{B_+} |u|^p \partial_d \xi + p|u|^{p-1} \operatorname{sgn}(u) \partial_d u \xi\end{aligned}$$

where $x' = (x_1, \dots, x_{d-1})$ and $B_+ = B \cap \{x_d > 0\}$. Use that $|\partial_d \xi|_\infty + |\xi|_\infty \le C < \infty$ with a constant C independent of u. We get

$$\begin{aligned}\dots &= c\left(\int_{B_+} |u|^p + p\int_{B_+} |u|^{p-1} |\partial_d u|\right) \\ &\le C(p)\left(\int |u|^p + \int |\nabla u|^p\right)\end{aligned}$$

as a result of Hölder's and Young's inequalities. In conclusion $\exists C = C(p, \xi)$ such that

$$\int_{\Gamma\cap\hat B} |u|^p \le C\left(\int_{B_+} |u|^p + \int_{B_+} |\nabla u|^p\right). \qquad \square$$

Theorem B.29. *Let Ω be a C^1-domain.*

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) \mid Tu = 0 \text{ in } L^2(\partial\Omega)\right\}.$$

Beweis. Only easy direction: $\subseteq$. Let $u \in W_0^{1,p}(\Omega)$. $\exists (u_k) \subset C_c^\infty(\Omega)$ s.t. $u_k \to u$ in $W^{1,p}(\Omega) \implies Tu_k \to Tu$ in $L^p(\partial\Omega)$. Since $Tu_k = u_k|_{\partial\Omega} = 0 \implies Tu = 0$. For $\supseteq$, see Evans. $\square$

Lemma B.30 (Integration by parts)**.** *Let Ω be a C^1-domain. Then for all $u \in W^{1,1}(\Omega)$, $\varphi \in C^1(\bar\Omega)$:*

$$\int_\Omega u \partial_i \phi = -\int_\Omega \partial_i u \varphi + \int_{\partial\Omega} (Tu)\varphi \nu \cdot e_i \; dS.$$

Beweis. Let $(u_k) \subset C^\infty(\bar\Omega)$ s.t. $u_k \to u$ in $W^{1,1}(\Omega)$. Then

$$\int_\Omega u_k \partial_i \varphi = -\int_\Omega \partial_i u_k \varphi + \int_{\partial\Omega} T(u_k)\varphi \nu \cdot e_i \; dS.$$

As $k \to \infty$:

$$\begin{aligned}u_k &\to u \text{ in } L^1(\Omega) \\ \nabla u_k &\to \nabla u \text{ in } L^1(\Omega) \\ Tu_k &\to Tu \text{ in } L^1(\partial\Omega)\end{aligned}$$

since T is bounded and linear and thus continuous. $\square$

Lemma B.31 (Interface). *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Consider a partition $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, where $\Omega_1 \cap \Omega_2 \neq \emptyset$ and Ω_i C^1-domains. Define $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \cap \Omega$. Consider $u\colon \Omega \to \mathbb{R}$, $u_i\colon \Omega_i \to \mathbb{R}$, $u_i = u|_{\Omega_i}$ for $i = 1, 2$. TFAE:*

- *$u \in W^{1,1}(\Omega)$.*
- *$u_i \in W^{1,1}(\Omega_i)$ and $T_{\partial\Omega_1} u_1 = T_{\partial\Omega_2} u_2$ on Γ.*