Nonlinear Elasticity: Problem sheet 1

(This sheet will be discussed on Oct 21, 2019)

Exercise 1. *Convince yourself that* $\{e_i \otimes e_j \,:\, i = 1, \ldots, m, j = 1, \ldots, d\}$ *– the standard basis of* $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$ *– is indeed a basis.*

Exercise 2. *Convince yourself that*

- $\{e_i \otimes e_j \,:\, i = 1, \ldots, m, j = 1, \ldots, d\}$ *is an orthormal basis of* $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$*;*
- *for any* $A \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$*, we have*
$$A_{ij} = A \cdot (e_i \otimes e_j);$$
- *the scalar product are independent of the choice of the orthonormal basis of* $\mathbb{R}^d$ *(resp.* $\mathbb{R}^m$*): Let* $\bar{e}_1, \ldots, \bar{e}_n$ *and* $\tilde{e}_1, \ldots, \tilde{e}_m$ *denote orthonormal bases of* $\mathbb{R}^d$ *and* $\mathbb{R}^m$ *respectively; consider coordinate matrices* $A' = (A'_{ij}), B' = (B'_{ij}) \in \mathbb{R}^{m \times n}$ *and the linear mappings*
$$A := \sum_{ij} A'_{ij}(\bar{e}_i \otimes \tilde{e}_j), \qquad B := \sum_{ij} B'_{ij}(\bar{e}_i \otimes \tilde{e}_j);$$
note that in general $A_{ij} \neq A'_{ij}$ *and show that* $A \cdot B = \sum_{ij} A'_{ij} B'_{ij}$*.*

Exercise 3.

- *Consider* $f : \mathbb{R}^{n \times m} \to \mathbb{R}$*,* $f(A) := A_{11}^3 + A_{nm}^2$*. Compute* $\nabla f(A)$*.*
- *Consider* $f : \mathbb{R}^{n \times n} \to \mathbb{R}$*,* $f(A) := \det A$*. Compute* $\nabla f(Id)$*, where* Id *denotes the identity matrix.*
- *Let* $g_1 : \mathbb{R}^d \to \mathbb{R}^m, g_2 : \mathbb{R}^d \to \mathbb{R}^m$ *be differentiable, set* $f(x) := g_1(x) \cdot g_2(x)$ *and compute* $\nabla f(x)$*.*
- *Let* $g_1 : \mathbb{R}^d \to \mathbb{R}^m$*, and* $g_2 : \mathbb{R}^m \to \mathbb{R}$ *be continuously differentiable. Define* $f(x) := g_2(g_1(x))$ *and express* $Df(x)$ *(resp.* $\nabla f(x)$*) in terms* Dg_1*,* Dg_2 *(resp.* ∇g_2*) in a coordinate wise and coordinate-free format.*
$$\begin{aligned} Df(x)h &= Dg_2(g_1(x))[Dg_1(x)[h]] = \nabla g_2(g_1(x)) \cdot (Dg_1(x)h) \\ &= \partial_i g_2(g_1(x))(Dg_1(x))_{ij} h_j \\ \nabla f(x) &= Dg_1(x)^t \nabla g_2(g_1(x)). \end{aligned}$$
Plausibility test: $g_1(x) = Ax$ *with* $A \in \mathbb{R}^{m \times d}$*,* $g_2(y) = a \cdot y$ *with* $a \in \mathbb{R}^m$*,* $f(x) = a \cdot Ax = A^t a \cdot x$*, and thus* $\nabla f(x) = A^t a = (Dg_1)^t \nabla g_2$*.*

Exercise 4 (Outer unit normal)**.** *Let* Ω *be* C^1*-domain,* $x \in \partial\Omega$*. Let* (f, U) *be as in Definition* **??** *(b), and* (ϕ, V) *a parametrization at* x*. Show that the outer unit normal* $\nu(x)$ *at* x *is unique and given by* $\nu(x) = \frac{\nabla f(x)}{|\nabla f(x)|}$*. Show that* $\operatorname{span}(\nu(x)) = \{\, v \in \mathbb{R}^d \,:\, D\phi(y)^t v = 0 \,\}$ *where* $y = \phi^{-1}(x)$*.*

Exercise 5 (Divergence theorem)**.** *Let* $V \subset \mathbb{R}^{d-1}$ *and* $h \in C^1(\overline{V})$*. Consider* $\Omega := \{x = (x', x_d) \,:\, x' \in V,\, x_d < h(x')\}$ *and set* $\Gamma := \operatorname{graph}(h) \subset \partial\Omega$*.*

(a) Let $u : \Gamma \to \mathbb{R}$ *be continuous, bounded, and compactly supported. Show that*
$$\int_\Gamma u \, d\mathcal{H}^{d-1} = \int_V u(x', h(x'))\rho(x')\, dx',$$
for some $\rho : V \to \mathbb{R}$ *and identify* ρ*.*

(b) Compute the outer unit normal to $\Gamma \subset \partial\Omega$ *for* $x \in \Gamma$*.*

(c) Let $u \in C_c^1(\mathbb{R}^d; \mathbb{R}^d)$*. Prove that*
$$\int_\Omega \operatorname{div} u \, dx = \int_\Gamma u \cdot \nu \, d\mathcal{H}^{d-1}.$$

Exercise 6 (Orientation preserving implies local injectivity)**.** *Let* $\Omega \subset \mathbb{R}^d$ *open,* $\varphi \in C^1(\Omega; \mathbb{R}^d)$*.*

- *Show: If* $\det D\varphi(x) \neq 0$ *for some* $x \in \Omega$*, then there exists an open neighbourhood* $U \subset \Omega$ *of* x *and* $\varphi : U \to \varphi(U)$ *is a* C^1*-diffeomorphism.*
- *Find an example for a non-injective* $\varphi \in C^1(\Omega; \mathbb{R}^d)$ *satisfying* $\det D\varphi > 0$ *in* Ω*.*