MKMECH: Problem sheet 4

Exercise 1 Let $d \geq 2$ and $r > 0$. Consider the stored energy function

$$W(F) := \frac{1}{2}\,\mathrm{trace}(F^t F) + \frac{1}{r}(\det F)^{-r} - (\frac{d}{2} + \frac{1}{r})$$

(a) Show that W is frame-indifferent and isotropic.

(b) Show that W is minimized at $SO(d)$.

(c) Let $d = 3$. Determine the bulk modulus K of W. (Recall that K is defined through the identity $\frac{1}{3K}\mathbb{L}\mathbf{Id} = \mathbf{Id}$, where $\mathbb{L} = D\hat{S}(\mathbf{Id})$).

Exercise 2 Let $g(\xi) := (1 - \xi^2)^2$. Show that

- $I_1(u) := \int_{-1}^{1} g(u')\,dx$ has a minimizer in $W_0^{1,4}((-1,1))$, but no in $X = \{u \in C^1((-1,1)) \cap C([-1,1]) : u(-1) = u(1) = 0\}$.
- $I_2(u) := \int_{-1}^{1} g(u') + u^2(x)\,dx$ has no minimizer in $W_0^{1,4}((-1,1))$.

Exercise 3 Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $f \in C^1(\mathbb{R}^{d\times d})$ and suppose that

$$|\nabla f(F)| \leq C(1 + |F|^{q-1})$$

for some $C > 0$ and $q \geq 1$. Further, let $\varphi \in W^{1,q}(\Omega, \mathbb{R}^d)$ be a minimizer of

$$I(\varphi) := \int_\Omega f(D\varphi(x))\,\mathrm{d}x$$

w.r.t. its own boundary conditions, i.e. $I(\varphi + \theta) \geq I(\varphi)$ for all $\theta \in W_0^{1,q}(\Omega, \mathbb{R}^d)$. Then one has

$$\int_\Omega \nabla f(D\varphi(x)) \cdot D\theta(x)\,\mathrm{d}x = 0$$

for all $\theta \in W_0^{1,q}(\Omega, R^d)$.

Exercise 4 Let $F \in \mathbb{R}^{d\times d}$ and let $m(F)$ denote a minor of F. More precisely, suppose that $m(F) = \det \tilde{F}$, where $\tilde{F}$ denotes the $(d-\ell)\times(d-\ell)$-matrix obtained by deleting the rows $i_1, \dots i_\ell$ and columns $j_1, \dots j_\ell$. Show that

$$m(F) = \pm \frac{\partial^\ell \det(F)}{\partial F_{i_1,j_1} \cdots \partial F_{i_\ell,j_\ell}}.$$

Hint: First treat the case when $m(F)$ is the minor associated to the upper left $(d-\ell)\times(d-\ell)$-matrix. You may use induction and Leibniz formula for the determinant.

Exercise 5

- Show that there exists a constant $C > 0$ s.t.

$$|\det(F+G) - \det(F)| \leq C(|F+G|^{d-1} + |F|^{d-1})|G|$$

- Let $m(F)$ denote a $\ell \times \ell$-minor of F. Show that there exists a constant $C > 0$ s.t.

$$|m(F+G) - m(F)| \leq C(|F+G|^{\ell-1} + |F|^{\ell-1})|G|$$

Remark: A function f with the property

$$|f(A) - f(B)| \leq C(1 + |A|^{p-1} + |B|^{p-1})|A - B|$$

is called *p-Lipschitz continuous.*