Multiscale problems and relaxation in nonlinear elasticity

Technische Universität Dresden
Department of Mathematics
July 4 - 5, 2017

[1]Organizers: Antoine Gloria (ULB Bruxelles), Stefan Neukamm (TU Dresden)
Webpage: `https://tu-dresden.de/mn/math/wir/neukamm/die-professur/workshops/WorkshopElasticity2017`

1 General Information

Objectives

The workshop aims to bring together mathematicians working on problems in nonlinear elasticity that invoke different scales, e.g. homogenization, thin structures, discrete-to-continuum limits, relaxation, and related fields.

Organizers

Antoine Gloria (Universit libre de Bruxelles) and Stefan Neukamm (Technische Universität Dresden).

Location of the Lectures

The workshop will be hosted at TU Dresden. All lectures take place at

Departement of Mathematics
Willers-Bau (Zellescher Weg 12 - 14)
Room C207

`https://navigator.tu-dresden.de/etplan/wil/01/raum/219401.0220`

Dinner

The social dinner (registration required) takes place at "Villa Marie", see `http://www.villa-marie.com/`. You can get there by bus, see `https://goo.gl/maps/WdDY26fPXyD2`.

Support

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2 Program

Tuesday, July 4, 2017, 9:00 – 17:30

8:15 – 8:55		**Registration**
8:55 – 9:00		**Opening**
9:00 – 9.50	Gilles Francfort (Université Paris-Nord)	Periodic Homogenization in linear elasticity (revisited)
9:50 – 10:40	Mitia Duerinckx (ULB Bruxelles)	The Clausius-Mossotti formulas and beyond
10:40 – 11:00		**Coffee break**
11:00 – 11:40	Jean-Francois Babadjian (Université Paris 6)	Homogenization of brittle damage
11:40 – 12:30	Mathias Schäffner (TU Dresden)	Quantitative homogenization in nonlinear elasticity for small loads
12:30 – 14:30		**Lunch break (buffet)**
14:30 – 15:20	Julian Fischer (IST Austria)	An analysis of variance reduction methods in stochastic homogenization
15:20 – 16:10	Matthias Ruf (ULB Bruxelles)	Free energies on stochastic lattices
16:10 – 16:40		**Coffee break**
16:40 – 17:30	Marco Cicalese (TU Munich)	A variational approach to the generalised XY model: fractional vortices and string defects
ca. 19:00		*Social dinner*

Wednesday, July 5, 2017, 9:00–16:10

9:00 – 9.50	Sergio Conti (Universität Bonn)	Derivation of $F = F_e F_p$ from a microscopic single-slip model
9:50 – 10:40	Peter Bella (Universität Leipzig)	A variational viewpoint on wrinkling of thin elastic sheets
10:40 – 11:10		**Coffee break**
11:10 – 12:00	Heiner Olbermann (Universität Leipzig)	Rigidity theorems and integrability of the Brouwer degree
12:00 – 14:00		**Lunch break (buffet)**
14:00 – 14:50	Angkana Rüland (University of Oxford)	Microstructures in Shape-Memory Alloys: Rigidity vs Flexibility
14:50 – 15:20		**Coffee break**
15:20 – 16:10	Bernd Schmidt (Universität Augsburg)	Homogenization and the limit of vanishing hardening in Hencky plasticity with non-convex potentials

3 Abstracts

Periodic Homogenization in linear elasticity (revisited)

Gilles Francfort
Université Paris-Nord

In 1993, Giuseppe Geymonat, Stefan Müller and Nicolas Triantafyllidis demonstrated that, in the setting of linearized elasticity, a Gamma-convergence result holds for highly oscillating sequences of elastic energies whose functional coercivity constant over the whole space is zero while the corresponding coercivity constant on the torus remains positive. We find sufficient conditions for such a situation to occur through a rigorous revisiting of a laminate construction given by Gutierrez in 1999. We further demonstrate that isotropy prohibits such an occurrence. The results should apply to both the periodic and the stochastic setting. They were obtained in part with Marc Briane (Rennes), and in part with Antoine Gloria (Brussels).

The Clausius-Mossotti formulas and beyond

Mitia Duerinckx
Université libre de Bruxelles

We study the behavior of the homogenized coefficients associated with some ergodic stationary random medium under a Bernoulli perturbation. More precisely, a stationary family of possible inclusions is considered, and each inclusion is chosen independently according to a Bernoulli process, thus yielding perturbed inclusions in a given reference medium. Introducing a new family of energy

estimates that combine probability and physical spaces, we prove the analyticity of the perturbed homogenized coefficients with respect to the Bernoulli parameter. Our approach holds under the minimal assumptions of stationarity and ergodicity, both in the scalar and vector cases, and it leads to semi-explicit formulas for each derivative that essentially coincide with the so-called cluster expansions used by physicists. In particular, the first term in this expansion yields the celebrated (electric and elastic) Clausius-Mossotti formulas for isotropic spherical random inclusions in an isotropic reference medium. This joint work with Antoine Gloria constitutes the first general proof of these formulas in the case of random inclusions.

Homogenization of brittle damage

Jean-François Babadjian
Université Pierre et Marie Curie, Laboratoire Jacques-Louis Lions

This talk is concerned with a quasistatic evolution model for a continuum which undergoes damage and possibly fracture. In both cases, the model appears to be ill posed so that it is necessary to introduce a relaxed variational evolution preserving the irreversibility of the process, the minimality at each time, and the energy balance. From a mechanical point of view, it turns out that the material prefers to form microstructures through the creation of fine mixtures between the damaged and healthy parts of the medium. The brutal character of the damage process is then replaced by a progressive one, where the original damage internal variable, i.e. the characteristic function of the damaged part, is replaced by the local volume fraction. The analysis rests on a locality property for mixtures which enables to use an alternative formula for the lower semicontinuous envelope of the elastic energy in terms of the G-closure set.

[1] J.-F. Babadjian: A quasistatic evolution model for the interaction between fracture and damage, *Arch. Rational Mech. Anal.* **200**, no. 3, (2011), 945–1002.

Quantitative homogenization in nonlinear elasticity for small loads

Mathias Schäffner
TU Dresden

We consider a nonlinear elastic composite with a periodic micro-structure described by the non-convex integral functional

$$I_\varepsilon(u) = \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) - f(x)\cdot u(x)\,dx,$$

where $u:\Omega\to\mathbb{R}^d$ is the deformation, $f:\Omega\to\mathbb{R}^d$ is an external force, $\varepsilon>0$ denotes the size of the micro-structure, and $W(y,F)$ is a stored energy function which is periodic in y. As it is well-known, under suitable growth conditions, I_ε Γ-converges to a functional with a homogenized energy density $W_{\mathsf{hom}}(F)$, which is given by an *infinite-cell formula.* Under appropriate assumptions on W (namely, $p\geq d$-growth from below, frame indifference, minimality at identity, non-degeneracy and smoothness in a neighborhood close to the set of rotations) and on the microstructure, we show that in a neighbourhood of rotations the homogenized stored energy function W_{hom} is of class C^2

and characterized by a *single-cell homogenization formula*. Moreover, for small data, we establish an estimate on the homogenization error, which measures the distance between (almost) minimizers u_ε of I_ε and the minimizer of the homogenized problem. More precisely, we prove that the L^2-error as well as the H^1-error of the associated two-scale expansion decays with the rate $\sqrt{\varepsilon}$. This is joint work with S. Neukamm.

[1] S. Neukamm and M. Schäffner, Quantitative homogenization in nonlinear elasticity for small loads, *arXiv:1703.07947*

An analysis of variance reduction methods in stochastic homogenization

Julian Fischer
IST Austria

In stochastic homogenization of linear elliptic PDEs, the effective coefficients are given in terms of an infinite volume limit of a cell formula, similarly to the case of periodic homogenization of nonconvex integral functionals. For practical approximations of the effective coefficient, a large but finite sample volume of the random medium must be chosen, say, a cube with side length L; using the cell formula on this sample volume, one may obtain an approximation for the exact effective coefficient. The resulting approximation is a random quantity, as it depends on the sample of the random medium. It turns out that the leading-order contribution to the error of this approximation is actually caused by the random fluctuations of the approximation around its expected value. To increase the accuracy of approximations, it is therefore desirable to reduce the variance. We provide a rigorous analysis of the variance reduction concepts in stochastic homogenization introduced by Le Bris, Legoll, and Minvielle, including a – rather particular – counterexample for which the variance reduction methods fail and a wide class of random coefficient fields for which they succeed.

Free energies on stochastic lattices

Matthias Ruf
Université libre de Bruxelles

We study the asymptotic behavior of large volume Gibbs measures associated to discrete Hamiltonians that are defined on deformations of a stationary stochastic lattice. Assuming polynomial growth and finite range interactions we prove a large deviation principle with a continuum nonlinear elasticity-type rate functional. We then investigate this functional in the small temperature regime. Under suitable continuity assumptions on the discrete interaction energy, we show that there exists a zero temperature limit and that it coincides with the Γ-limit of the rescaled discrete Hamiltonians. This is joint work with Antoine Gloria and Marco Cicalese.

A variational approach to the generalised XY model: fractional vortices and string defects

Marco Cicalese
TU München

We analyse a generalised two dimensional XY model, whose interaction potential has n weighted wells, describing corresponding symmetries of the system. As the lattice spacing vanishes, we derive by Γ-convergence the discrete-to-continuum limit of this model. In the energy regime we deal with, the asymptotic ground states exhibit fractional vortices, connected by string defects. The Γ-limit takes into account both contributions, through a renormalised energy, depending on the configuration of fractional vortices, and a surface energy, proportional to the length of the strings. Our model describes in a simple way several topological singularities arising in Materials Science. This is a joint paper with R. Badal, L De Luca and M. Ponsiglione.

[1] R. Badal, M. Cicalese, L. De Luca and M. Ponsiglione, Γ-convergence analysis of a generalised XY model: fractional vortices and string defects *arXiv:1612.03128*

Derivation of $F = F_e F_p$ from a microscopic single-slip model

Sergio Conti
Universität Bonn

The geometrically linear theory of plasticity is based on the additive decomposition of the strain $e(v) = (\nabla v + \nabla v^T)/2$ into an elastic and a plastic part, $e(v) = \beta_e + \beta_p$; the plastic part can be easily understood as the homogenized limit of singular slip over crystallographically determined slip planes. Whereas this decomposition is universally accepted, in a finite setting there is not yet a similar agreement on a kinematic expression, and no such clear relation to a micromechanical interpretation. I shall discuss a micromechanical model with individual slips and its mathematical analysis, which shows that, at least in two dimensions and in the setting of our model, $Du = F_e F_p$ naturally arises as a macroscopic limit of local elastic deformation and slip over slip lines. The key mathematical difficulties reside in passing to the limit in the product of the weakly converging sequences F_e^ε, F_p^ε as well as in the determinants $\det F_p^\varepsilon$.

[1] *Kinematic description of crystal plasticity in the finite kinematic framework: a micromechanical understanding of $F = F^e F^p$*, C. Reina and S. Conti, J. Mech. Phys. Solids **67** (2014), 40–61.

[2] *Derivation of $F = F_e F_p$ as the continuum limit of crystalline slip*, C. Reina, A. Schlömerkemper, and S. Conti, J. Mech. Phys. Solids **89** (2016), 231–254.

[3] *Isochoric inelasticity as an essential ingredient for the multiplicative decomposition in finite kinematics*, C. Reina and S. Conti, preprint (2017).

A variational viewpoint on wrinkling of thin elastic sheets

Peter Bella
Universität Leipzig

Wrinkling of thin elastic sheets can be viewed as a way how to avoid compressive stresses. While the question of where the wrinkles appear is (mostly) well-understood, understanding properties of wrinkling is not trivial. Considering a variational viewpoint, the problem amounts to minimization of an elastic energy, which can be viewed as a non-convex membrane energy singularly perturbed by a higher-order bending term. To understand the global minimizer (ground state), the first step is to identify its energy, in particular how it depends on the small physical parameter (thickness). I will discuss few problems where the optimal energy scaling law was identified, and also what we learn from it.

Rigidity theorems and integrability of the Brouwer degree

Heiner Olbermann
Universität Leipzig

In the theory of geometrically nonlinear elastic plates and shells, it is important to understand the properties of isometric immersions of two-dimensional Riemannian manifolds into three-dimensional Euclidean space. In particular, the question of uniqueness of isometric immersions is relevant for phenomena such as crumpling. We reconsider the proof of uniqueness of isometric immersions of spheres with positive Gauss curvature, with derivatives in a certain Hlder class. We observe that an understanding of the integrability properties of the Brouwer degree is crucial to extend the range of validity for the uniqueness statement. We take this as a motivation to state and prove a theorem about the integrability of the Brouwer degree with irregular arguments.

Microstructures in Shape-Memory Alloys – Rigidity and Flexibility

Angkana Rüland
University of Oxford

Shape-memory materials undergo a first order, diffusionless phase transformation, in which symmetry is lost. Mathematically, they can be modelled by non-convex, multi-well energies within the framework of the calculus of variations. Minimizers of these energies are often subject to a fascinating dichotomy: While solutions with high regularity are often quite rigid, solutions with low regularity are in many cases very flexible. I will discuss first quantitative results on this dichotomy which are based on joint work with C. Zillinger and B. Zwicknagl.

[1] A. Rüland, C. Zillinger and B. Zwicknagl, Higher Sobolev Regularity of Convex Integration Solutions in Elasticity, *arXiv:1610.02529*

Homogenization and the limit of vanishing hardening in Hencky plasticity with non-convex potentials

Bernd Schmidt
Universität Augsburg

I will report on recent results with M. Jesenko (Univ. Augbsurg), cf. [1]. We consider integral functionals in Hencky plasticity of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f\big(\tfrac{x}{\varepsilon}, \mathcal{E}u(x)\big)\,dx \quad \text{and} \quad \mathcal{F}_\varepsilon^{(\delta)}(u) = \int_\Omega f\big(\tfrac{x}{\varepsilon}, \mathcal{E}u(x)\big) + \delta|\mathcal{E}u(x)|^2\,dx,$$

where $f : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a non-convex Carathéodory function that is $(0,1)^n$-periodic in the first variable and satisfies the growth condition of Hencky plasticity

$$\alpha(|X_{\mathrm{dev}}| + (\operatorname{tr} X)^2) \le f(x,X) \le \beta(|X_{\mathrm{dev}}| + (\operatorname{tr} X)^2 + 1),$$

$X_{\mathrm{dev}} = X - \frac{\operatorname{tr} X}{n} I$. Generalizing previous work on convex functionals, see [2], we first prove a homogenization result for $\mathcal{F}_\varepsilon$.

Theorem A. Under suitable assumptions, $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}_{\mathrm{hom}}$ with limiting functional of the form

$$\mathcal{F}_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}\big(\mathcal{E}u(x)\big)\,dx + \int_\Omega (f_{\mathrm{hom}})^{\#}\big(\tfrac{dE^s u}{d|E^s u|}(x)\big)\ d|E^s u|(x)$$

on BD-functions with square-integable divergence.

Our second results shows that homogenization and taking the vanishing hardening limit commute.

Theorem B. Under suitable assumptions,

$$\Gamma\text{-}\lim_{\delta\to 0}\ \Gamma\text{-}\lim_{\varepsilon\to 0}\ \mathcal{F}_\varepsilon^{(\delta)} = \Gamma\text{-}\lim_{\varepsilon\to 0}\ \Gamma\text{-}\lim_{\delta\to 0}\ \mathcal{F}_\varepsilon^{(\delta)} = \mathcal{F}_{\mathrm{hom}}.$$

[1] M. Jesenko and B. Schmidt, Homogenization and the limit of vanishing hardening in Hencky plasticity with non-convex potentials, *arXiv:1703.09443*

[2] F. Demengel, T. Qi, Convex function of a measure obtained by homogenization, *SIAM J. Math. Anal.* **21** (1990), no. 2, 409–435.