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The Radiosity Equation for Solving Global Heat Transfer in Industrial Furnaces

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Abstract—In this paper, a numerical method used in computer graphics to solve global illumination problems is applied to heat radiation. The radiosity equation is solved by an adaptive finite-element method and linked to heat conduction. The method is validated on an analytic benchmark problem. Due to the speed up compared with commonly used methods for heat radiation, this approach might be a way to solve complex three-dimensional problems in high temperature environments.

Keywords—Radiosity, Heat radiation, Hemicube method, Adaptivity.

1. INTRODUCTION

Radiation is the dominant mode of energy transport in most industrial furnaces. In engineering problems, radiative heat transfer calculations are very complex due to physical phenomena taking place within enclosures of complex shape. Because of its importance, an exact mathematical solution to radiation exchange problems in enclosures is desirable. It is, therefore, of primary importance to develop an efficient and accurate method to ease the computational effort involved in radiation and general heat transfer problems. In many applications, the exchange of radiative energy between surfaces is virtually unaffected by the medium that separates them. We shall also assume that all surfaces are gray and are diffuse emitters. Under these assumptions, the problem of global heat transfer can be formulated mathematically as a system of integrodifferential equations and might be solved by an operator-splitting method, decoupling heat diffusion, and radiation.

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A similar model to heat radiation is used in computer graphics to solve global illumination problems under the assumption of Lambertian diffuse reflectors. The simulation of light transfer is somewhat simpler as temperature plays no significant role in the reemission of light. The work in this field has developed fast algorithms which are used to render objects. In this paper, we will show that these developments can be applied to global heat radiation problems. In this article, we will concentrate on the numerical solution of the radiosity equation by a Galerkin method.

2. RADIOSITY EQUATION

The radiosity equation is a mathematical model for the global illumination problem in computer graphics under the assumption of Lambertian diffuse reflectors. The equation is

$$u(x) - \frac{\rho(x)}{\pi} \int_\Gamma u(\xi) G(x,\xi) V(x,\xi)\, d\xi = e(x), \qquad x \in \Gamma, \tag{1}$$

with $u(x)$ the radiosity at x, $\rho(x)$ the reflectivity at x, $e(x)$ the emissivity at x, and Γ a smooth surface. The function G is given by

$$G(x,\xi) = \frac{(\xi - x)\cdot \mathbf{n}_x (x-\xi)\cdot \mathbf{n}_\xi}{\|x-\xi\|^4} \tag{2}$$

$$= \frac{\cos\theta_x \cos\theta_\xi}{\|x-\xi\|^2} \tag{3}$$

with $\mathbf{n}_x$ and $\mathbf{n}_\xi$ the inner unit normals on Γ at x and ξ, θ_x the angle between $\mathbf{n}_x$ and $\xi - x$ and θ_ξ the angle between $\mathbf{n}_\xi$ and $x - \xi$, see Figure 1.

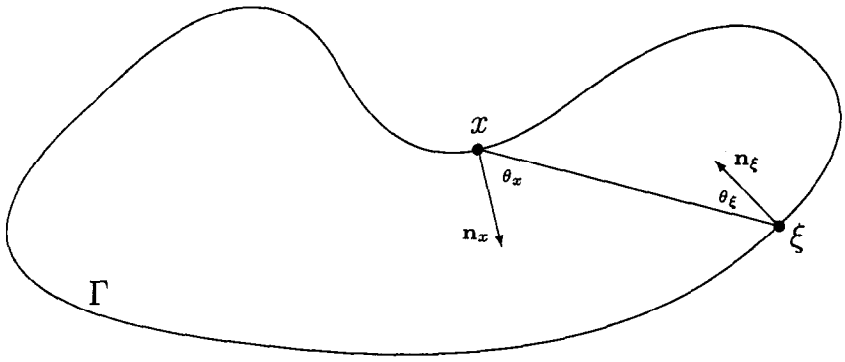

Figure 1. Illustration.

The function V is given by

$$V(x,\xi) = \begin{cases} 1, & x \text{ and } \xi \text{ see each other,} \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where "see each other" means, a straight line between x and ξ does not intersect Γ at any other point. Surfaces for which $V \equiv 1$ are called unoccluded.

Equation (1) can be written in the simpler form

$$u(x) - \int_\Gamma u(\xi) K(x,\xi)\, d\xi = e(x), \qquad x \in \Gamma, \tag{5}$$

or in operator form as

$$(\mathcal{I} - \mathcal{K})\, u = e. \tag{6}$$

It is a Fredholm integral equation of the second kind. If we define the bilinear form $a(\cdot,\cdot)$ and the linear form $b(\cdot)$, respectively, over $L^2(\Gamma) \times L^2(\Gamma)$ and $L^2(\Gamma)$ by

$$a(u,\phi) = (u,\phi)_{L^2(\Gamma)} - \int_\Gamma \int_\Gamma u(\xi)\phi(x)K(x,\xi)\,d\xi\,dx, \tag{7}$$

$$b(\phi) = (e,\phi)_{L^2(\Gamma)}, \tag{8}$$

the weak form of equation (5) reads: find $u \in L^2(\Gamma)$ such that

$$a(u,\phi) = b(\phi), \qquad \forall\, \phi \in L^2(\Gamma). \tag{9}$$

For smooth surfaces Γ the kernel K is weakly singular, i.e., it is defined and continuous for all $x, \xi \in \Gamma$, $x \neq \xi$, and there exist a positive constant c such that

$$\|K(x,\xi)\| \leq c, \qquad x,\xi \in \Gamma, \quad x \neq \xi. \tag{10}$$

This was shown for convex surfaces in [1]. For nonconvex surfaces, all critical points cancel out due to the definition of V. Furthermore, K is symmetric, nonnegative, square-integrable, and

$$\int_\Gamma K(x,\xi)\,d\xi = \rho(x), \qquad x \in \Gamma, \tag{11}$$

see [2]. Therefore, $\mathcal{K}$ is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$. We now assume $\|\rho\|_\infty < 1$. With these statements, the continuity of $a(\cdot,\cdot)$ and $b(\cdot)$ as well as the coercivity of $a(\cdot,\cdot)$ follows and by the theorem of Lax-Milgram, the existence of a unique solution $u \in L^2(\Gamma)$ of equation (9) can be concluded.

If e is a continuous function, the properties of the weakly singular kernel K guarantee the solution u to be continuous.

3. DISCRETIZATION

The Galerkin formulation of (9) reads: find $u_h \in V_h$ such that

$$a(u_h,\phi) = b(\phi), \qquad \forall\, \phi \in V_h, \tag{12}$$

with $V_h \subset L^2(\Gamma)$ a finite-dimensional subspace. The general theory for integral equations of the second kind shows the existence of a Galerkin solution and their convergence, see [3]. Assume that $V_h = \operatorname{span}\{\phi_1,\ldots,\phi_n\}$. Then we express u_h as a linear combination $u_h = \sum_{j=1}^n u_j\phi_j$ and solve

$$\sum_{j=1}^n u_j a(\phi_j,\phi_i) = b(\phi_i), \qquad i = 1,\ldots,n, \tag{13}$$

or equivalently the linear system

$$(M - K)u_h = B, \tag{14}$$

with matrices $M = (M_{ij})$, $K = (K_{ij})$ and the right-hand side $B = (B_i)$ defined as follows:

$$M_{ij} = \int_\Gamma \phi_j(x)\phi_i(x)\,dx, \tag{15}$$

$$K_{ij} = \int_\Gamma \int_\Gamma \phi_j(\xi)\phi_i(x)K(x,\xi)\,d\xi\,dx, \tag{16}$$

$$B_i = \int_\Gamma \phi_i(x)e(x)\,dx. \tag{17}$$

For piecewise constant basis functions, this reduces to

$$M_{ij} = \delta_{ij}, \tag{18}$$

$$K_{ij} = \rho_i F_{ij}, \tag{19}$$

$$B_i = e_i \tag{20}$$

with ρ_i the average reflectance for element i, e_i the average emission for element i, and F_{ij} the well-known form factor

$$F_{ij} = \frac{1}{|A_i|} \int_{A_i} \int_{A_j} \frac{\cos\theta_x \cos\theta_\xi}{\|x-\xi\|^2} V(x,\xi)\, dA_j\, dA_i \tag{21}$$

with $x \in A_i$, $\xi \in A_j$ and A_i and A_j the area of element i and j, respectively. The form factor has a simple physical interpretation: F_{ij} is the proportion of the total power leaving the element i that is received by element j.

4. FORM FACTOR

Evaluating the terms in F is the major computational bottleneck of the radiosity method. Both closed form analytic and numerical methods have been applied to solving the form factor integral, see [4]. A fast but approximate method is the hemicube algorithm. If two surface patches are distant from each other, relative to their size, the inner integral in (21) varies little across the surface of A_i. In such a case, the form factor can be computed as that from a point to a finite area. If we project an element radially onto a hemisphere centered at that point and then project it orthogonally down onto the base, the fraction of the base area covered by this projection is equal to the form factor. This is the so-called Nusselt analog [5] and it illustrates the fact, that for an element projected radially onto any intermediate surface, the projection will have the same form factor than the element itself. This observation forms the basis of the hemicube algorithm, in which elements are projected onto a hemicube centered at the given point. The hemicube sides are subdivided into grid cells, see Figure 2.

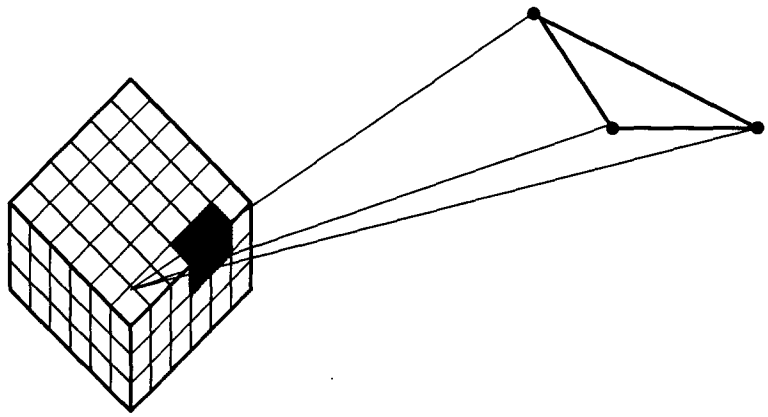

Figure 2. Hemicube.

For each grid cell, a delta-form factor is precomputed and stored in a lookup table. The form factor to an element is then approximated by projecting the element onto the sides of the hemicube and summing the delta form factors of the grid cells covered by the projection.

5. *A POSTERIORI* ERROR ESTIMATE AND ADAPTIVITY

The error and the residual are defined by $e_h = u - u_h$ and $r_h = (M-K)e_h$, respectively. The residual can be computed given u_h by

$$r_h = B - (M-K)u_h. \tag{22}$$

The proposed global error estimate is, therefore, taken to be the norm of the residual

$$\eta_h = \|r_h\|_{L^2(\Gamma)}. \tag{23}$$

The local error indicators are taken to be the element based quantities

$$\eta_{h,A_i} = \|r_h\|_{L^2(A_i)}. \tag{24}$$

Due to the additivity of the L^2-norm, η_h can be computed by

$$\eta_h = \left(\sum_i \eta_{h,A_i}^2 \right)^{1/2}. \tag{25}$$

The following lemma will show that this estimator is equivalent to the discretisation error.

LEMMA. *Let $K : L^2(\Gamma) \to L^2(\Gamma)$ be compact and $(I-K) : L^2(\Gamma) \to L^2(\Gamma)$ injective. There exist constants c_1 and c_2 independent of the grid size h with*

$$c_1 \eta_h \le \|e_h\|_{L^2(\Gamma)} \le c_2 \eta_h. \tag{26}$$

PROOF. Due to (18) and (22), the residual can be written as $r_h = (I-K)e_h$. Since K is compact, and therefore, bounded,

$$\|r_h\|_{L^2(\Gamma)} \le \|I-K\| \cdot \|e_h\|_{L^2(\Gamma)}, \tag{27}$$

and the lower bound holds with $c_1 = \|I-K\|^{-1}$. Since $I-K$ is injective, $(I-K)^{-1}$ exist and is bounded by the Fredholm alternative. Therefore,

$$\|e_h\|_{L^2(\Gamma)} \le \left\|(I-K)^{-1}\right\| \cdot \|r_h\|_{L^2(\Gamma)}, \tag{28}$$

and the upper bound holds with $c_2 = \|(I-K)^{-1}\|$. ∎

This result was also shown in [6]. Under the assumption of a global refinement, a stronger result of asymptotic exactness can be shown, see [7]. The adaptive strategy based on the L^2-residual estimate and the equidistribution principle now looks like the following:

- solve the radiosity equation for a given grid $\mathcal{A}$ on the surface Γ,
- for each element A_i, compute the error indicator η_{h,A_i},
- refine all elements A_i with $\eta_{h,A_i} \ge \text{tol}$,
- solve the system on the refined grid.

6. NUMERICAL RESULTS

A simple three-dimensional problem with two concentric isothermal black spheres serves as a test problem. A coupled heat radiation-diffusion problem is solved with given temperature values at the boundary. In the solid parts, a linear heat equation is solved defining the temperature at the radiative boundaries. By the Stefan-Boltzmann law and Kirchhoff's law, these temperatures are related to the right-hand side of the radiosity equation by $e = \sigma\rho T^4$, where σ is the Stefan-Boltzmann constant. Only two surfaces are present. All radiation from the inner sphere with radius $r_1 = 1$ travels to the outer sphere with radius $r_2 = 2$. We have $F_{12} = 1$ and, by reciprocity, $F_{21} = A_1/A_2$, with $A_i = 4\pi r_i^2$ the area of the sphere. The analytic solution for given temperatures T_1 and T_2 at the spheres and a constant reflectivity $\rho = 0.5$ reads

$$u_1 = \frac{1}{9}\left(5\sigma T_1^4 + 4\sigma T_2^4\right),$$

$$u_2 = \frac{1}{9}\left(\sigma T_1^4 + 8\sigma T_2^4\right).$$

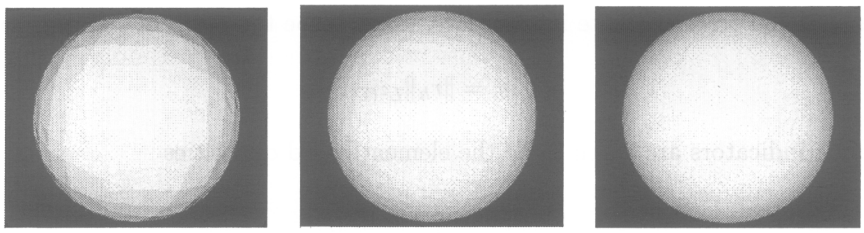

Figure 3. Surface elements for used refinement levels.

Table 1. Number of surface elements, error e_h and maximal difference in u_h on each sphere in percent.

$\#\Gamma_1$	$\#\Gamma_2$	$\|e_h\|_{L^2(\Gamma_1)}$	$\|e_h\|_{L^2(\Gamma_2)}$	$\max \Delta u_h\|_{\Gamma_1}$	$\max \Delta u_h\|_{\Gamma_2}$
192	206	0.128	1.107	0.172	2.880
768	810	0.207	1.519	0.007	0.543
3072	3270	0.245	1.733	0.018	0.352

The toolbox ALBERT, see [8], is used to define a numerical grid consisting of tetrahedrons. The grid is constructed out of a coarse macrotriangulation by refinement and projection onto the surface, Figure 3 shows only the outer sphere.

For $T_1 = 2000K$ and $T_2 = 1000K$, the calculations are performed. The L^2 norm of the error e_h as well as the maximal difference of u_h on each sphere is shown in Table 1.

The linear system (14) was solved with a BiCGStab-solver. The calculations demonstrate that the hemicube method can be applied to heat radiation problems and produces results in an acceptable tolerance by decreasing the computational cost, which will be most relevant for problems with moving boundaries. Further developments on the hemicube method will be applied to heat radiation problems and they might be the way to solve complex three-dimensional problems in high temperature environments, such as crystal growth furnaces.

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