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Set Oriented Approximation of Invariant Manifolds: Review of Concepts for Astrodynamical Problems

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Abstract. During the last decade set oriented methods have been developed for the approximation and analysis of complicated dynamical behavior. These techniques do not only allow the computation of invariant sets such as attractors or invariant manifolds. Also statistical quantities of the dynamics such as invariant measures, transition probabilities, or (finite-time) Lyapunov exponents, can be efficiently approximated. All these techniques have natural applications in the numerical treatment of problems in astrodynamics. In this contribution we will give an overview of the set oriented numerical methods and how they are successfully used for the solution of astrodynamical tasks. For the demonstration of our results we consider the (planar) circular restricted three body problem. In particular, we approximate invariant manifolds of periodic orbits about the L_1 and L_2 equilibrium points and show an extension to the application of a continuous control force. Moreover, we demonstrate that expansion rates (finite-time Lyapunov exponents), which so far have mainly been applied in fluid dynamics, can provide useful information on the qualitative behavior of trajectories in the context of astrodynamics. The set oriented numerical methods and their application to astrodynamical problems discussed in this contribution serve as further important steps towards understanding the pathways of comets or asteroids and the design of energy-efficient trajectories for spacecraft.

Keywords: space mission design, Hamiltonian systems, three body problem, set oriented numerics, invariant manifolds, finite-time Lyapunov exponents, reachable sets
PACS: 02.60.-x; 05.10.-a; 05.45.-a; 95.10.Ce

INTRODUCTION

In the last few years dynamical system techniques have been developed for the design of energy-efficient trajectories for space missions. These approaches are typically based on the (planar) circular restricted three body problem and exploit the structure of certain invariant sets such as periodic orbits and their associated invariant manifolds. Pioneers in using mathematical concepts in the context of trajectory design in celestial mechanics were already Poincaré [31], McGehee [28] and Conley [7]. Later their ideas were enhanced for direct applications to space missions, for instance by Belbruno and Miller [4] for the *Hiten Mission* and by Lo et al. [25] for the *Genesis Discovery Mission*. This led to the work of Koon, Lo, Marsden, and Ross [20, 22, 19, 21, 23, 27] with fundamental practical and theoretical results.

In this contribution we present numerical methods that enable an implementation of the dynamical system approach for the design of energy-efficient flight paths for spacecraft. In particular, we review the set oriented numerical techniques for the approximation of invariant manifolds in the three body problem and apply them to several relevant tasks in astrodynamics. The set oriented approach was introduced by Dellnitz and Hohmann [9, 10], the treatment of the special case of Hamiltonian systems is described in [18].

The present article is organized as follows: we begin by recalling the circular restricted three body problem (CRTBP) and the related planar system (PCRTBP) followed by a description of the set oriented numerical methods for the approximation of invariant manifolds. Here we focus on the special case of the computation of stable and unstable manifolds to periodic orbits [9, 10, 18]. We apply this approach to two relevant problems: first, we analyze the flight path of the comet Oterma. Secondly, we show how to obtain an initial guess for an energy-efficient trajectory in the Sun-Earth-spacecraft three body problem such as used for the *Genesis Discovery Mission* [12]. These ideas are extended for the problem of designing a low thrust trajectory for a mission to Venus. This includes the combination of several PCRTBPs as well as the application of continuous control forces [13]. Finally, we demonstrate how expansion rates (finite-time Lyapunov exponents), which, so far, have mainly been used in fluid dynamics (e.g. [17, 33, 36, 30]), can be applied in the context of astrodynamics. Such estimation of certain substructures on the manifolds can, for instance, be useful for the design of trajectories for spacecraft flying in a formation. We close this contribution with a short conclusion.

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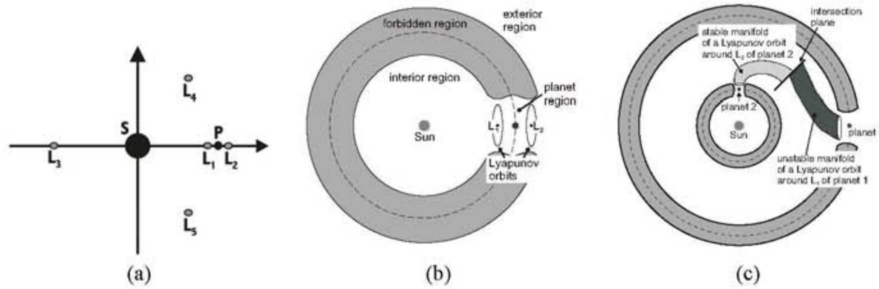


FIGURE 1. (a) Five equilibrium points of the circular restricted three body problem in the x_1x_2-plane. (b) Projection of an energy surface onto position space (schematic) for a value of the Jacobi integral for which the spacecraft is able to transit between the exterior and the interior region. (c) Sketch of the 'patched 3-body approach' (cf. [21, 23]). The idea is to travel within certain invariant manifold 'tubes' possibly including an impulsive maneuver at the intersection plane.

CIRCULAR RESTRICTED THREE BODY PROBLEM

We begin by recalling the equations of motion for the CRTBP, for more details see for instance [15, 16, 2, 34]. Here the Sun (denoted by S) and a Planet (denoted by P) are the two primaries with total mass normalized to one. The mass parameter μ is given by $\mu = \frac{m_P}{m_P+m_S}$, where m_P is the mass of the Planet and m_S the mass of the Sun, respectively. These two bodies rotate in the plane counterclockwise about their common center of mass with angular velocity normalized to one. The third body – either a spacecraft, a comet or an asteroid – is free to move in the three-dimensional space and its motion is assumed to not affect the primaries.

We choose a rotating coordinate system so that the origin is at the center of mass and the Sun and the Planet are fixed on the x-axis at $(-\mu,0,0)$ and $(1-\mu,0,0)$, respectively. We denote by (x_1,x_2,x_3) the position of the spacecraft in the rotating frame. The equations of motion of the third body can be written in second order form as

$$\ddot{x}_1 - 2\dot{x}_2 = \Omega_{x_1}, \quad \ddot{x}_2 + 2\dot{x}_1 = \Omega_{x_2}, \quad \ddot{x}_3 = \Omega_{x_3}, \tag{1}$$

where

$$\Omega(x_1,x_2,x_3) = \frac{x_1^2+x_2^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2},$$

$$r_1 = \sqrt{(x_1+\mu)^2 + x_2^2 + x_3^2}, \quad r_2 = \sqrt{(x_1-1+\mu)^2 + x_2^2 + x_3^2}$$

and Ω_{x_1}, Ω_{x_2} and Ω_{x_3} are the partial derivatives of Ω with respect to the variables x_1, x_2 and x_3. These are the equations of the CRTBP, see [1, 26, 34] for a more detailed description.

The system (1) has a first integral (also called Jacobi integral) which is given by

$$C(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3) = -(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) + 2\Omega(x_1,x_2,x_3). \tag{2}$$

In Figure 1 (a) the five equilibrium points of the system (1) in the x_1x_2-plane are shown. These equilibrium points are critical points of the function Ω. The level surfaces of the Jacobi integral, which are also energy surfaces, are invariant 5-dimensional manifolds. The projection of this surface onto position space is called Hill's region and its boundary is the zero velocity curve. The movement of the third body is confined to this region. The structure of Hill's region changes with energy. In this paper we restrict ourselves to the case shown in Figure 1(b).

For suitable values of the Jacobi integral (2) there exist periodic solutions (cf. Figure 1(b)), of (1) in the vicinity of the equilibrium points L_1 and L_2 that are unstable in both directions of time. Their unstable and stable manifolds W^u resp. W^s are (topologically) cylinders (also called '*tubes*') that locally partition the energy surface into two sets: 1. transit orbits that pass between the *interior region* and the *planet region* in the case of an L_1-periodic orbit or between the *exterior region* and the *planet region* in the case of L_2, and 2. non-transit orbits that stay either in the exterior or the interior region [16, 28, 20].

In several applications it is sufficient to consider the PCRTBP because the orbital planes of the considered planets as well as their orbital eccentricities are very small. The equations of motion for this model are derived from (1) by simply setting $x_3 = \dot{x}_3 = \ddot{x}_3 = 0$. In this case the corresponding periodic orbits are called *Lyapunov orbits*.

Coupling Planar 3-Body Problems

A straightforward approach to construct a trajectory between several planets is to combine different PCRTBPs. By patching two PCRTBPs parts of the unstable manifold of a periodic orbit in one system may come close to the stable manifold of a Lyapunov orbit in the other system (where, for a moment, it may help to imagine that the two systems do not move relative to each other), cf. Figure 1(c). It may thus be possible for a spacecraft to 'bridge the gap' between two pieces of trajectories in the vicinity of these manifolds by exerting an impulsive maneuver [21, 23].

Close approaches of two such invariant manifolds can be detected by reducing the dimension of the problem: one computes the intersection of the two manifolds with a suitable intersection plane (cf. Figure 1(c)) and determines points of close approach in this surface – for example by inspecting projections onto suitable two-dimensional surfaces. This technique has in fact been used for a systematic construction of trajectories that follow prescribed itineraries around and between the Jovian moons [23].

SET ORIENTED APPROXIMATION OF INVARIANT MANIFOLDS

In this section we briefly outline the continuation method for the set oriented approximation of stable/unstable manifolds by [9, 10]. Our description focusses on a method for flows which is appropriate for the computation of invariant manifolds for periodic orbits in Hamiltonian systems. For a more detailed exposition we refer to [9, 10, 18].

Let $\varphi^t : \mathbb{R}^d \to \mathbb{R}^d$ be the flow, i.e. the solution operator for the equations of motion. Let $Q \subset \mathbb{R}^d$ be a compact set containing the periodic orbit p. We can parametrize this orbit $p(s)$ by time with $0 \le s < T$ where T is the period of the orbit.

A *partition* $\mathscr{P}$ of Q consists of finitely many subsets of Q such that

$$\bigcup_{B \in \mathscr{P}} B = Q \quad \text{and} \quad B \cap B' = \emptyset \quad \text{for all } B, B' \in \mathscr{P}, B \neq B'.$$

Let $\mathscr{P}_\ell$, $\ell \in \mathbb{N}$, be a nested sequence of successively finer partitions of Q, requiring that for all $B \in \mathscr{P}_\ell$ there exist $B_1, \ldots, B_m \in \mathscr{P}_{\ell+1}$ such that $B = \bigcup_i B_i$ and $\operatorname{diam}(B_i) \le \theta \operatorname{diam}(B)$ for some $0 < \theta < 1$.

Let $\mathscr{B}_0 \subset \mathscr{P}_\ell$ denote a subset of the partition elements containing the orbit p and which defines a neighborhood C of the orbit. For some $k \in \mathbb{N}$ let $\mathscr{B}_0^{(k)} \subset \mathscr{P}_{\ell+k}$ be a set oriented approximation of $W^u_{\mathrm{loc}}(p) \cap C$ as for instance obtained by an application of the subdivision algorithm from [9]. We call $\mathscr{B}_0^{(k)}$ a *covering* of $W^u_{\mathrm{loc}}(p) \cap C$.

The continuation algorithm for an approximation of the global unstable manifold $W^u(p)$ within Q now works as follows:

Continuation algorithm for flows [9, 18]: Consider a covering $\mathscr{B}_0^{(k)} \subset \mathscr{P}_{\ell+k}$ of $W^u_{\mathrm{loc}}(p) \cap C$ and choose $m \le \ell + k$, $m \in \mathbb{N}$. For $\tau \ge 0$ we define the new collection

$$\mathscr{B}_\tau^{(m)} = \{B \in \mathscr{P}_m | B \cap \varphi^t(\tilde{B}) \neq \emptyset \text{ for some } \tilde{B} \in \mathscr{B}_0^{(k)} \text{ and some } 0 \le t \le \tau\} \tag{3}$$

Convergence results for the subdivision and for the continuation algorithms can be found in [9, 10, 18].

In this paper we obtain initial conditions (e.g. a starting set for the continuation algorithm) by choosing a fine approximation $p(n \cdot h)$, $h = T/n$, $n \in \mathbb{N}$, of the periodic orbit and by moving these points slightly within the unstable eigenspace of the periodic orbit. The unstable directions are obtained by propagating the respective eigenvector of the monodromy matrix for one period along the orbit [29].

In the continuation step it is sufficient to consider the boxes that are hit by the respective orbits, a complete covering of the global unstable manifold is ensured by interpolation schemes. The stable manifold is obtained by considering the flow under time reversal. The set oriented algorithms are implemented in the software package GAIO [8].

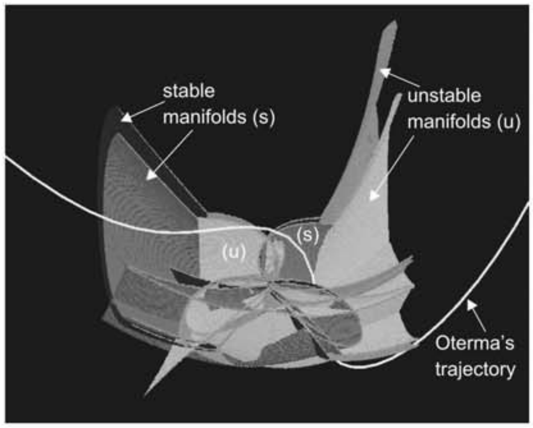


FIGURE 2. Invariant manifolds of periodic orbits around L_1 and around L_2 computed by using the continuation algorithm and visualized by GRAPE [32]. For better visibility only slices of the manifolds are shown. The solid line represents part of Oterma's trajectory as observed in the 1930s. In the vicinity of Jupiter the Oterma's trajectory lies inside the invariant manifold tubes confirming the results in [22, 19].

Invariant Manifolds and Oterma's Trajectory

In order to illustrate our methods we first consider the CRTBP with the Sun and Jupiter as primaries, i.e. $\mu = 9.5368 \times 10^{-4}$. Figure 2 shows the result of using the continuation algorithm described above for the computation of invariant manifolds of periodic orbits around L_1 and L_2. We choose values of the Jacobi integral of $C_1 = 3.020$ and $C_2 = 3.025$. The solid line depicts part of Oterma's trajectory crossing Jupiter's region in the 1930s. The comet Oterma belongs to a family of comets which make a rapid transition from heliocentric orbits outside the orbit of Jupiter to orbits inside that of Jupiter and vice versa. During this transition the orbit passes close to the equilibrium points L_1 and L_2.

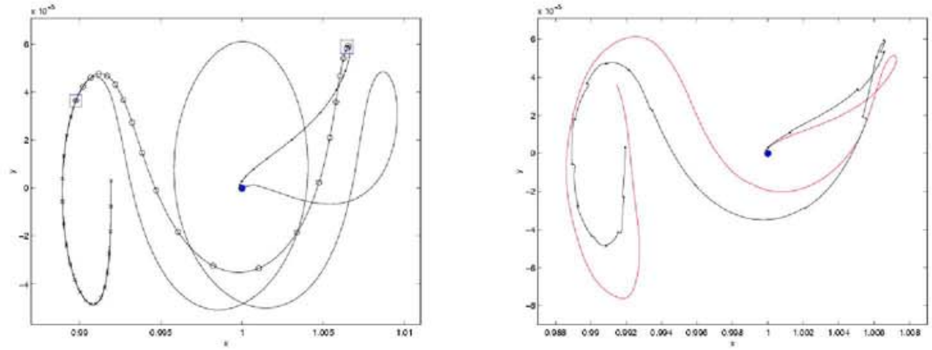
Belbruno and B. Marsden [3] (see also [2]) considered comet transitions using the 'weak stability boundary' concept. Lo and Ross [24] as well as Koon et al. [22, 19] used the PCRTBP as the underlying model to explain resonance transition and related the transition to invariant manifolds. Results of McGehee [28] and Koon et al. [20] showed that the comet closely follows the invariant manifolds of the corresponding periodic orbits. That means for a transition from the exterior to the interior region a comet's trajectory like Oterma's uses a 'celestial highway' built from a concatenation of parts of the stable and unstable manifolds of periodic orbits in the vicinity of L_1 and L_2.

We picked up this concept and used the set oriented continuation algorithm to compute stable and unstable invariant manifolds of the periodic orbits around L_1 and L_2 in the CRTBP. During Oterma's transition from the external to the internal region the comet changed its energy which influenced the value of the Jacobi integral. For this reason we computed the invariant manifolds for two different Jacobi constants $C_1 = 3.020$ and $C_2 = 3.025$, which are representative for Oterma's orbit in the vicinity of Jupiter. Figure 2 shows an $x_1x_2x_3$-projection of slices of the manifolds visualized by GRAPE [32]. In the neighborhood of Jupiter the trajectory of the comet lies inside the invariant manifold tubes which confirms the results of Koon et al. [22, 19].

Application to the Genesis Mission

In the last few years the design of energy-efficient trajectories has received considerable interest, in particular in view of long interplanetary missions. A problem that often arises in this context is to find initial guesses for such fuel-efficient trajectories.

In order to attack this problem we have developed efficient methods to search for trajectories which connect given start and end points based on a box covering of the invariant manifolds [11, 12, 13]. For instance, in the Genesis mission [25] a trajectory from a periodic orbit around L_1 to the Earth under certain landing constraints was sought.



(a) (b)

FIGURE 3. Trajectories in the Sun-Earth-spacecraft three body problem projected onto the x_1x_2-plane. (a) A possible return trajectory for the Genesis spacecraft with loop (without markers) and a sequence of three patched trajectories corresponding to one pseudo-orbit computed by a modified continuation algorithm. The two discontinuities are marked by boxes. (b) Pseudo-orbit as in (a) (shown in dark) and resulting trajectory after local optimization by LTool.

In Figure 3(a) the trajectory without markers depicts a possible Genesis return trajectory. To remove the undesired loop (and thus reduce the flight time by several months) Dellnitz et al. [12] used a modified version of the continuation algorithm for the computation of pseudo-trajectories. A pseudo-trajectory or pseudo-orbit is, roughly speaking, a finite sequence of short trajectories which connect a given initial point to a given end point while allowing discontinuities between the individual trajectories. Such pseudo-trajectories might serve as suitable initial guesses for the solution of more complex optimal control problems.

More precisely, in the Genesis mission the CRTBP with Sun and Earth as primaries was used, i.e. $\mu = 3.04042 \times 10^{-6}$. In Figure 3(a) the loop-free trajectory is a possible pseudo-orbit with two jumps (marked by boxes) which was computed using the modified continuation algorithm mentioned above. This pseudo-orbit provides a good initial guess for a local solver to obtain the final trajectory in the full model. Figure 3(b) shows the optimized trajectory (light line without markers) which was computed using JPL's LTool [25].

PATCHED THREE BODY PROBLEMS WITH CONTROL

In current mission concepts, like for the ESA interplanetary mission *BepiColombo* to Mercury, ion propulsion systems are being investigated that continuously exert a small force on the spacecraft ('low-thrust propulsion'). Thus, it is necessary to include a low thrust control into the PCRTBP. The idea of intersecting manifolds of periodic orbits as mentioned earlier will therefore be enhanced in order to compute suitable reachable sets [6]. As an example we present some numerical results for an Earth-Venus transfer. A more detailed description of the method and this special example can be found in [13].

A Controlled Planar Three Body Problem

To model ion propulsion systems that continuously exert a small force on the spacecraft ('low-thrust propulsion') the PCRTBP needs to be enhanced by a suitably defined control term. Here we will restrict our considerations to the special case of a control force whose direction is defined by the spacecraft's velocity. This is motivated by the fact that the acceleration and velocity vectors are parallel for the force in order to have a maximum instantaneous impact onto the kinetic energy of the spacecraft (see [14]). Therefore, the control term which is to be included into the model is parametrized by a single real value u, determining the magnitude of the control acceleration. This heuristic approach aims at reducing the number of control functions under consideration. It is not globally optimal (see [35]) but here the focus is on computing trajectories as initial guesses for optimal control algorithms.

The velocity vector of the spacecraft has to be viewed with respect to the inertial instead of the rotating coordinate system. This leads to the following control system, modeling the motion of the spacecraft under the influence of its low thrust propulsion system in rotating coordinates:

$$\ddot{x} + 2\dot{x}^{\perp} = \nabla\Omega(x) + u\,\frac{\dot{x} + \omega x^{\perp}}{\|\dot{x} + \omega x^{\perp}\|}. \tag{4}$$

Here, $u = u(t) \in [u_{min}, u_{max}] \subset \mathbb{R}$ denotes the magnitude of the control force, $x = (x_1, x_2)$, $x^{\perp} = (-x_2, x_1)$ and ω is the common angular velocity of the primaries.

Coupling Controlled 3-Body Problems

Obviously, every solution of the uncontrolled PCRTBP is also a solution of (4) for the control function $u \equiv 0$. We are going to exploit this fact in order to generalize the patched 3-body approach as described in the beginning to the case of controlled 3-body problems. We are still going to use the L_1- and L_2-Lyapunov orbits as 'gateways' for the transition between the interior, the planet and the exterior regions. However, instead of computing the relevant invariant manifolds of these periodic orbits, we compute certain *reachable sets* (see e.g. [6]), i.e. sets in phase space that can be accessed by the spacecraft when employing a certain set of control functions.

Reachable sets. We denote by $\phi(t,z,u)$ the solution of the control system (4) for a given initial point $z = (x, \dot{x})$ in phase space at $t_0 = 0$ and a given admissible control function $u \in \mathscr{U} = \{u : \mathbb{R} \to [u_{min}, u_{max}], u \text{ admissible}\}$. Here $u_{min}, u_{max} \in \mathbb{R}$ are predetermined bounds on the magnitude of the control force, and the attribute 'admissible' alludes to the fact that only a certain subset of functions is allowed. Both the bounds and the set of admissible control functions will be determined by the design of the thrusters.

For a set S in phase space Z and a given function $\tau : S \times \mathscr{U} \to \mathbb{R}$, we call

$$\mathscr{R}(S,\tau) = \{\phi(\tau(z,u),z,u) \mid u \in \mathscr{U}, z \in S\}$$

the set which is (τ-)*reachable from* S. We choose $\tau(x,u)$ in such a way that the reachable sets are contained in the intersection plane.

Patched controlled 3-body systems. The idea is, roughly speaking, to mimic the patched 3-body approach while replacing the invariant manifolds of the Lyapunov orbits by certain reachable sets. We describe the approach by considering a mission from an outer planet (e.g. Earth) to an inner planet (e.g. Venus).

For two suitable sets $\mathscr{O}_1$ and $\mathscr{O}_2$ (in the vicinity of an L_1-Lyapunov orbit of Earth and an L_2-Lyapunov orbit of Venus, respectively) one computes associated reachable sets $\mathscr{R}(\mathscr{O}_1, \tau_1) \subset \Sigma_1$ and $\mathscr{R}(\mathscr{O}_2, \tau_2) \subset \Sigma_2$ within suitably chosen intersection planes Σ_1 and Σ_2 in each system, respectively. After a transformation of one of these reachable sets into the other rotating system (e.g. transforming $\mathscr{R}(\mathscr{O}_1, \tau_1)$ into $\hat{\mathscr{R}}(\mathscr{O}_1, \tau_1) \subset \Sigma_2$), the intersection $\hat{\mathscr{R}}(\mathscr{O}_1, \tau_1) \cap \mathscr{R}(\mathscr{O}_2, \tau_2)$ is determined. Set oriented methods allow to efficiently compute an outer covering of this intersection. By construction, for each point in this intersection there exists a controlled trajectory that provides a transit from $\mathscr{O}_1$ (Earth) to $\mathscr{O}_2$ (Venus).

Application to a Mission to Venus

We applied the method to a mission to Venus and got the following results (for more details see [13]): We choose a trajectory with control force $u_1 = -651\,\text{mN}$ for the transfer from Earth's gateway set $\mathscr{O}_1$ to the intersection plane, $u_2 = -96\,\text{mN}$ for the trajectory from that plane to the gateway set $\mathscr{O}_2$ of Venus. Linking also the gateway sets to a closer neighborhood of the planets, we finally end up with a flight time of roughly 1.8 years and a corresponding ΔV of slightly less than 4000 m/s for the complete journey from Earth to Venus. Again, note that these are rough estimates and that the trajectory that we constructed should be viewed as an initial guess for an appropriate software that uses a more detailed model of the solar system.

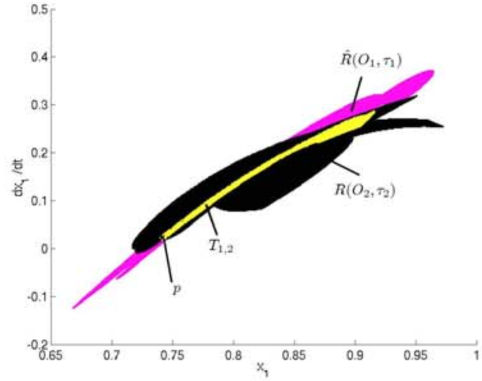


FIGURE 4. Intersection $\mathscr{T}_{1,2}$ (light grey, 9387 boxes) of two reachable sets in a common intersection plane. $\hat{\mathscr{R}}(\mathscr{O}_1,\tau_1)$ (dark grey, 121075 boxes): reachable set of the gateway set of Earth, $\mathscr{R}(\mathscr{O}_2,\tau_2)$ (black, 171579 boxes): reachable set of the gateway set of Venus. The figure shows a projection of the covering in 3-space onto the $(x_1,\dot{x}_1)$-plane (normalized units). The computation of these sets took 2.8h on a 3.2 GHz Xeon processor.

EXPANSION RATES ON INVARIANT MANIFOLDS

As demonstrated above invariant manifolds serve as an appropriate basis for the design of energy-efficient trajectories for space missions. In this section we will investigate these geometrical structures in more detail and analyze the qualitative behavior of orbits on the manifolds with respect to small perturbations in the initial states. For such an analysis we use finite-time Lyapunov exponents, a concept which has been successfully used in the analysis of geophysical fluid flows, see e.g. [17, 33, 36, 30]. Finite-time Lyapunov exponents measure the exponential growth of infinitesimal perturbations in the initial conditions. Ridges in the finite-time Lyapunov fields correspond to repelling objects, which under some further assumptions, are related to stable manifolds of hyperbolic trajectories. Analogous results hold for unstable manifolds when the system is considered under time reversal.

However, unlike the usual finite-time Lyapunov exponent approaches where the maximum expansion with respect to all possible directions is measured here we are only interested in the exponential growth rates for perturbations *within* the stable and unstable manifolds of periodic orbits. Hence, in our computational method we will only allow for perturbations tangential to the manifolds. For this we propose the following approach. We consider a box covering of the unstable manifold of a periodic orbit as discussed above. In each box we only take the center point to get an approximate *tangential expansion rate* as follows:

$$\delta(T,B_i) := \log\left(\max_j \frac{1}{T}\frac{|\varphi^T(c_i)-\varphi^T(n_j(c_i))|}{|n_j(c_i)-c_i|}\right),$$

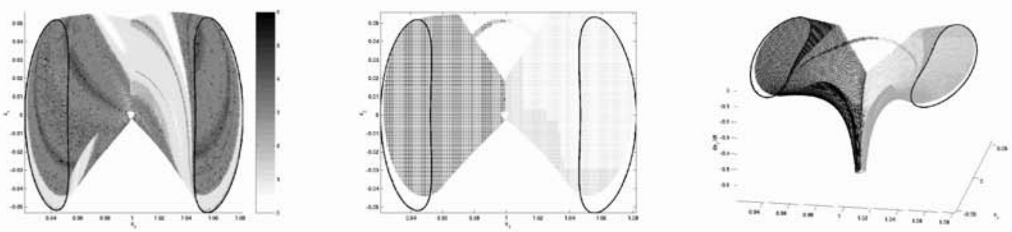
where c_i denotes the center point of the box B_i, $n_j(c_i)$ the center point of the j-th neighboring box of B_i, φ the flow and $T\in\mathbb{R}$ the final time of the trajectory integration. This idea is very much in spirit of the relative dispersion approaches as discussed e.g. in [5, 37] which use finite initial perturbations for an approximation of finite-time Lyapunov exponents. So each box gets a value which measures the relative growth rate of initial perturbations on the basis of all neighboring boxes. This approach ensures that - at least approximately - only perturbations with respect to the manifold are considered.

In our context it is interesting to analyze the distribution of expansion rates for two main reasons: first, the knowledge of substructures of the manifold that behave qualitatively like stable manifolds of hyperbolic objects may help to design efficient control laws for the control of single spacecraft. Secondly, regions that are characterized by almost vanishing expansion may be useful for the design of trajectories for several spacecraft flying in a formation. In this case the natural dynamics is particularly favorable for the stability of the formation.

Application to the PCRTBP

We consider the PCRTBP with Sun and Jupiter as primaries. Using the method described above we compute a set oriented approximation of parts of the stable manifold of a Lyapunov orbit about L_1 as well as the unstable manifold of a periodic orbit around the equilibrium point L_2, corresponding to a choice of the Jacobi integral of $C = 3.03$. The resulting box covering consists of 680046 boxes of diameters $d_x = 0.0004$, $d_y = 0.0002$ and $d_{\dot{x}} = d_{\dot{y}} = 0.0027$. For the unstable manifold we estimate the expansion rates as described above using a integration time of $T = 3$, for the stable manifold we compute the expansion rates in backward time, hence choosing $T = -3$.

In Figure 5(a) the result of this computation projected onto the x_1x_2-plane is shown. Light colored regions are characterized by low expansion and hence, in these areas the natural dynamics may support, for instance, keeping a formation of spacecraft. Dark regions correspond to expansive behavior, i.e. small perturbations in the initial conditions will grow considerably under the flow. This is particularly true for orbits that pass in the vicinity of Jupiter where the dynamics behaves very sensitive to small perturbations. Therefore, large parts of the manifolds are characterized by considerable expansiveness. To visualize these regions we only show the center points of boxes with a particularly large expansion rate (> 2.5), see Figure 5(b). Apart from the large connected regions related to Jupiter orbits a one-dimensional structure is picked up, see also Figure 5(c) for a projection into the $x_1x_2\dot{x}_1$-space. We were able to numerically verify that the stable and unstable manifolds have a nonempty intersection in the intersection plane defined by $x_1 = 1 - \mu$ exactly where the two parts of the one-dimensional structure meet. Hence, we have numerically detected a heteroclinic connection between the two Lyapunov orbits as described in [20].



(a) (b) (c)

FIGURE 5. (a) Expansion rates for parts of the stable manifold of the periodic orbit near L_1 (left) and of the unstable manifold of the Lyapunov orbit near L_2 (right); projection onto the x_1x_2-plane. Light colored regions are characterized by low expansion, dark regions by large expansion. (b) Center points of boxes with large expansion rates projected onto x_1x_2-plane. (c) Center points of boxes with large expansion rates projected into $x_1x_2\dot{x}_1$-space.

CONCLUSION

Set oriented numerical methods provide a robust framework for the approximation of invariant manifolds in complex systems such as the three body problem and enable a reliable detection of energy-efficient trajectories. The methods are not only applicable to the numerical analysis of the dynamics induced by gravitational forces but also when a continuous control force is considered such as realized in low thrust propulsion systems. The strength of the numerical techniques discussed in this article is that they provide reliable initial guesses for energy-efficient trajectories for space missions which then can be improved by optimal control algorithms. Moreover, the set oriented approach allows further investigation of the dynamics on the approximated manifolds, for instance the sensitivity of trajectories with respect to small perturbations in the initial condition.

Future work will include the investigation of further interplanetary missions with much longer flight times and with several spacecraft flying in a formation. This will probably necessitate the development and application of efficient methods for the solution of related multiobjective optimization and optimal control problems.

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