Calcolo (2018) 55:50
https://doi.org/10.1007/s10092-018-0293-0

SDFEM for an elliptic singularly perturbed problem with two parameters

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Received: 10 July 2018 / Accepted: 8 November 2018 / Published online: 13 November 2018
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Abstract

A singularly perturbed problem with two small parameters in two dimensions is investigated. Using its discretization by a streamline-diffusion finite element method with piecewise bilinear elements on a Shishkin mesh, we analyze the superconvergence property of the method and suggest the choice of stabilization parameters to attain optimal error estimate in the corresponding streamline-diffusion norm. Numerical tests confirm our theoretical results.

Keywords Singularly perturbed problem · Two small parameters · Streamline-diffusion method · Superconvergence · Stabilization parameter

Mathematics Subject Classification 65N12 · 65N15 · 65N30 · 65N50

This paper was written during a visit by Mirjana Brdar to the Technical University of Dresden in December 2017–February 2018 supported by DAAD Grant No. 57314019. The work was also partially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, Projects III44006 and 174030.

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1 Introduction

Mathematical models of certain physical processes often involve parameter-dependent differential equations. In some cases when parameters have very small values, those problems are singularly perturbed. Two-parameter singularly perturbed problems arise in chemical flow reactor theory [10] and in the case of boundary layers controlled by suction, or blowing of some fluid [13]. It is well known that standard numerical methods are unsuitable for singularly perturbed problems and fail to give accurate solutions unless the mesh size is at least as small as the perturbation parameter(s), which is unrealistic in practice. There is a vast literature dealing with numerical methods for various singularly perturbed problems (see for example the book by Roos et al. [11] for an overview).

One of the commonly used methods for solving singularly perturbed problems is the finite element method (FEM) combined with a layer-adapted mesh. The standard Galerkin finite element method has good approximation properties, but for the singularly perturbed problems it is less stable then for non-singularly perturbed problems. This produces oscillations in the computed solution which do not exist in the true solution. For that reason the Galerkin FEM should be appropriately stabilized. The most frequently studied and most popular stabilized FEM is the streamline-diffusion finite element method (SDFEM) which adds weighted residuals to the Galerkin FEM. It was proposed by Hughes and Brooks [4] and applied to several classes of problems. Mathematical analysis of this method has been derived by Johnson and Navert [5], Johnson et al. [6], Niijima [9] and Zhou [19]. Compared with the standard Galerkin FEM, the SDFEM provides additional control over the convective derivative in the streamline direction because of the definition of the induced streamline-diffusion norm. This additional bound prevents the discrete solution from oscillating over a large part of the domain. When studying the SDFEM, the main question is the choice of stabilization parameters on problem subdomains (more on different choices of the SD parameter can be found in [2,3,7,11,14,15]).

Two-parameter problems are numerically treated with the SDFEM on an appropriately designed Shishkin mesh only in the one-dimensional case in Roos and Uzelac [12]. The choice of stabilization parameters in [12] is based on the analysis of the structure of the coefficient matrix. The aim was to obtain an M-matrix. For a two-dimensional problem, this approach can not be applied. We also emphasize that the stabilization in two-parameter problems is rather different from the stabilization in the case when $\varepsilon_2 = 1$. That is because in the two-parameter case, stabilization is very sensitive to the relation between perturbation parameters ε_1 and ε_2, and the number of mesh intervals N used for the discretization. This fine tuning of stabilization parameter is achieved with very careful error analysis. To the best of our knowledge, the SDFEM has not been applied on any two-parameter problem in two dimensions so far.

In this paper we consider the following singularly perturbed elliptic two-parameter problem

$$
\begin{aligned}
Lu := -\varepsilon_1 \Delta u + \varepsilon_2 b(x,y) u_x + c(x,y) u = f(x,y) \quad &\text{in} \quad \Omega = (0,1)\times(0,1), \\
u = 0 \quad &\text{on} \quad \partial\Omega,
\end{aligned}
\tag{1.1}
$$

with

$$b(x, y) \geq b_0 > 0, \quad c(x, y) \geq c_0 > 0, \quad (x, y) \in \bar{\Omega}, \tag{1.2}$$

where b, c and f are sufficiently smooth functions, b_0, c_0 are constants, ε_1, ε_2 are small perturbation parameters and f satisfies the compatibility conditions

$$f(0, 0) = f(0, 1) = f(1, 0) = f(1, 1) = 0. \tag{1.3}$$

Under these assumptions there exists a classical solution u of the problem (1.1) such that $u \in C^{3,\alpha}(\bar{\Omega})$ for $\alpha \in (0, 1)$, [8]. We also assume that ε_2 is small enough to have

$$c(x, y) - \frac{\varepsilon_2}{2} b_x(x, y) \geq \gamma > 0, \quad (x, y) \in \Omega, \tag{1.4}$$

for some constant γ. The problem (1.1) is characterized by exponential layers at $x = 0$ and $x = 1$, characteristic layers at $y = 0$ and $y = 1$ and corner layers at four corners of Ω. The width of the exponential layers depends on the relation between ε_1 and ε_2. For $\varepsilon_2 = 0$ the problem (1.1) is a reaction–diffusion problem as opposed to $\varepsilon_2 = 1$ when it becomes a convection–diffusion problem. In the literature so far, convection–diffusion problems and reaction-diffusion problems are mostly handled separately. Here, we consider problems where $0 < \varepsilon_1, \varepsilon_2 \ll 1$ for arbitrary relation between ε_1 and ε_2.

The outline of the paper is as follows. In Sect. 2 we present information about a solution decomposition together with necessary assumptions. A description of the Shishkin mesh is given in Sect. 3. Section 4 contains a short review of the FEM and related error estimates, as well as definition of the SDFEM. In Sect. 5 we present our main theoretical result on superconvergence and suggest the choice of the stabilization parameter. The theoretical findings are experimentally verified in Sect. 6.

Notation 1 *For a set D, a standard notation for Banach spaces $L^p(D)$, Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$, norms $\|\cdot\|_{L^p(D)}$ and seminorms $|\cdot|_{H^k(D)}$ is used. Specially, if $p = 2$ we use notation $\|\cdot\|_{0,D}$. We write $(\cdot, \cdot)_D$ for the standard $L^2(D)$ inner product. If $D = \Omega$ we drop the Ω from the notation. Throughout the paper, C will denote a generic positive constant independent of the perturbation parameters ε_1, ε_2 and of the mesh.*

2 Solution properties

For the characterization of exponential layers in the solution of (1.1), we use an extension of the results from [16] where the problem (1.1) with $b = b(x)$ and $c = c(x)$ is studied. The widths of the layers depend on the values of

$$\mu_0 = \frac{-\varepsilon_2 B + \sqrt{\varepsilon_2^2 B^2 + 4\varepsilon_1 c_0}}{2\varepsilon_1} \quad \text{and} \quad \mu_1 = \frac{\varepsilon_2 b_0 + \sqrt{\varepsilon_2^2 b_0^2 + 4\varepsilon_1 c_0}}{2\varepsilon_1},$$

where $B = \max_{(x,y)\in\bar{\Omega}} b(x,y)$. In the paper we will frequently use the following properties of μ_0 and μ_1 taken from [16]:

$$\begin{aligned}
&\max\{\mu_0^{-1}, \varepsilon_1\mu_1\} \le C\left(\varepsilon_2 + \varepsilon_1^{1/2}\right), \\
&\varepsilon_2\mu_0 \le b_0^{-1}\|c\|_{L^\infty}, \qquad \sqrt{\varepsilon_1}\mu_0 \le 1/\sqrt{c_0}, \qquad (2.1)\\
&\varepsilon_2(\varepsilon_1\mu_1)^{-1/2} \le C\varepsilon_2^{1/2}, \\
&c_0\left(\|b\|_{L^\infty} + \sqrt{\|c\|_{L^\infty}}\right)^{-1} \le \mu_0 \le \mu_1. \qquad (2.2)
\end{aligned}$$

In the error analysis we assume that the solution decomposition and appropriate estimates of the solution components and their derivatives given in the following theorem from [16] are valid also for the problem (1.1).

Theorem 1 [16] *Let the elliptic problem* (1.1) *on the unit square* $\bar{\Omega}$ *be given where the functions* b, c *and* f *are sufficiently smooth on* $\bar{\Omega}$ *satisfying conditions* (1.2)–(1.4)*, and let* $p \in (0,1)$ *and* $k \in (0, 1/2)$ *be arbitrary. Assume that* $2\varepsilon_2\|b'\|_{L^\infty} \le k(1-p)c_0$*. Furthermore, let* δ *be a positive constant satisfying* $\delta^2 \le (1-p)c_0/2$*. Then the solution* u *of the boundary value problem* (1.1) *can be decomposed as*

$$u = S + E_{10} + E_{11} + E_{20} + E_{21} + E_{31} + E_{32} + E_{33} + E_{34},$$

where for all $(x,y) \in \bar{\Omega}$ *and* $0 \le i + j \le 2$*, the regular part* S *satisfies*

$$\left|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}\right| \le C, \qquad (2.3)$$

the exponential and parabolic layer components satisfy

$$\left|\frac{\partial^{i+j} E_{10}}{\partial x^i \partial y^j}\right| \le C\mu_0^i e^{-p\mu_0 x}, \qquad \left|\frac{\partial^{i+j} E_{11}}{\partial x^i \partial y^j}\right| \le C\mu_1^i e^{-p\mu_1(1-x)}, \qquad (2.4)$$

$$\left|\frac{\partial^{i+j} E_{20}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-\frac{j}{2}} e^{-\frac{\delta y}{\sqrt{\varepsilon_1}}}, \qquad \left|\frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-\frac{j}{2}} e^{-\frac{\delta(1-y)}{\sqrt{\varepsilon_1}}}, \qquad (2.5)$$

while the corner layer components satisfy the following estimates

$$\left|\frac{\partial^{i+j} E_{31}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-\frac{j}{2}}\mu_0^i e^{-p\mu_0 x} e^{-\frac{\delta y}{\sqrt{\varepsilon_1}}}, \left|\frac{\partial^{i+j} E_{34}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-\frac{j}{2}}\mu_0^i e^{-p\mu_0 x} e^{-\frac{\delta(1-y)}{\sqrt{\varepsilon_1}}}, \qquad (2.6)$$

$$\left|\frac{\partial^{i+j} E_{32}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-\frac{j}{2}}\mu_1^i e^{-p\mu_1(1-x)} e^{-\frac{\delta y}{\sqrt{\varepsilon_1}}}, \left|\frac{\partial^{i+j} E_{33}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-\frac{j}{2}}\mu_1^i e^{-p\mu_1(1-x)} e^{-\frac{\delta(1-y)}{\sqrt{\varepsilon_1}}}. \qquad (2.7)$$

For the superconvergence analysis, in the sequel we also assume the following L^2-estimates for the components of the exact solution.

Assumption 2.1 For the solution decomposition in Theorem 1, let

$$\left\|\frac{\partial^{i+j}S}{\partial x^i\partial y^j}\right\|_0 \le C,\ \left\|\frac{\partial^{i+j}E_{1s}}{\partial x^i\partial y^j}\right\|_0 \le C\mu_s^{i-1/2},\ s=0,1, \tag{2.8}$$

$$\left\|\frac{\partial^{i+j}E_2}{\partial x^i\partial y^j}\right\|_0 \le C\varepsilon_1^{-j/2+1/4},\ \left\|\frac{\partial^{i+j}E_3^s}{\partial x^i\partial y^j}\right\|_0 \le C\varepsilon_1^{-j/2+1/4}\mu_s^{i-1/2},\ s=0,1, \tag{2.9}$$

for $0 \le i+j \le 3$, where $E_2 = E_{20} + E_{21}$, $E_3^0 = E_{31} + E_{34}$ and $E_3^1 = E_{32} + E_{33}$.

The estimates given in Assumption 2.1 for $0 \le i+j \le 2$ follow directly from Theorem 1. For our error analysis it is sufficient to have L^2-estimates for the above third-order derivatives. This assumption is weaker then requiring all third-order derivatives of u to be pointwise bounded. The similar assumption is also set in [2,15,18] and references therein.

3 Layer-adapted mesh

Let the number of subintervals for the discretization in both x and y direction be N, and let N be a positive integer divisible by 4. Based on the decomposition we take $p = 1/2$ and introduce a corresponding tensor product Shishkin mesh.

Along the x-coordinate axis a piecewise uniform mesh $\Omega^N_{\lambda_0,\lambda_1}$ is constructed with three distinct uniform meshes separated by transition points located at λ_0 and $1-\lambda_1$. Along the y-coordinate axis we construct in the same way a piecewise uniform mesh $\Omega^N_{\lambda_y}$ with transition points located at λ_y and $1-\lambda_y$. Each of the intervals $[0,\lambda_0]$, $[1-\lambda_1,1]$, $[0,\lambda_y]$, $[1-\lambda_y,1]$ is uniformly divided into $N/4$ subintervals, and each of the intervals $[\lambda_0, 1-\lambda_1]$, $[\lambda_y, 1-\lambda_y]$ is uniformly divided into $N/2$ subintervals. Then the mesh on $\bar{\Omega}$ is given by tensor product

$$\Omega^N = \Omega^N_{\lambda_0,\lambda_1} \times \Omega^N_{\lambda_y},$$

where $\lambda_0, \lambda_1, \lambda_y$ are defined by

$$\lambda_s = \min\left\{\frac{1}{4}, \frac{2\sigma}{\mu_s}\ln N\right\},\ s=0,1, \qquad \lambda_y = \min\left\{\frac{1}{4}, \frac{\sigma}{\delta}\sqrt{\varepsilon_1}\ln N\right\}.$$

The parameter σ is typically chosen equal to the formal order of the method but for our superconvergence analysis, we use $\sigma \ge 5/2$ like in [2,15,18]. We assume that λ_0, λ_1 and λ_y are less then $1/4$, since otherwise the method can be analyzed using standard classical techniques.

According to the transition points, the domain $\bar{\Omega}$ is divided into subdomains Ω_C, Ω_{00}, Ω_{0y}, Ω_{11}, Ω_{1y}, Ω_y:

$$\Omega_C = [\lambda_0, 1-\lambda_1] \times [\lambda_y, 1-\lambda_y],$$
$$\Omega_{00} = [0, \lambda_0] \times [\lambda_y, 1-\lambda_y],$$

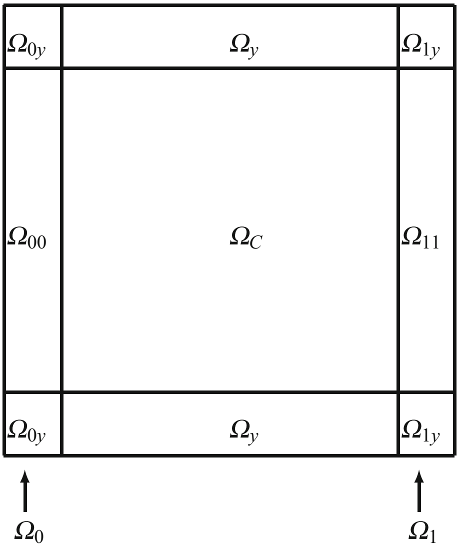

Fig. 1 Partitioning of the domain Ω

$$\begin{aligned}
\Omega_{0y} &= [0, \lambda_0] \times [0, \lambda_y] \cup [0, \lambda_0] \times [1 - \lambda_y, 1], \\
\Omega_{11} &= [1 - \lambda_1, 1] \times [\lambda_y, 1 - \lambda_y], \\
\Omega_{1y} &= [1 - \lambda_1, 1] \times [0, \lambda_y] \cup [1 - \lambda_1, 1] \times [1 - \lambda_y, 1], \\
\Omega_y &= [\lambda_0, 1 - \lambda_1] \times [0, \lambda_y] \cup [\lambda_0, 1 - \lambda_1] \times [1 - \lambda_y, 1],
\end{aligned}$$

see Fig. 1.

For an element $\tau_{i,j} = (x_{i-1}, x_i) \times (y_{j-1}, y_j) \in \Omega^N$ and element sizes in x and y directions, we also use notation $h_{x,\tau} = x_i - x_{i-1}$, $h_{y,\tau} = y_j - y_{j-1}$. The small and the large step sizes of the mesh are given by

$$\begin{aligned}
&h_{x,\Omega_0} = 8\sigma\mu_0^{-1} N^{-1} \ln N, \quad H_x = 2N^{-1}(1 - \lambda_0 - \lambda_1), \quad h_{x,\Omega_1} = 8\sigma\mu_1^{-1} N^{-1} \ln N, \\
&h_y = 4\sqrt{\varepsilon_1} N^{-1} \ln N, \qquad H_y = 2N^{-1}(1 - 2\lambda_y).
\end{aligned}$$

From now on, let $\tau \in \Omega^N$ represent any element of the Shishkin mesh.

4 The finite element method

For the problem (1.1), the standard weak formulation is:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a_G(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega),$$

with the bilinear form

$$a_G(w, v) := \varepsilon_1(\nabla w, \nabla v) + (\varepsilon_2 b w_x + c w, v), \quad w, v \in H_0^1(\Omega).$$

Let $V^N \subset H_0^1(\Omega)$ be the finite element space of piecewise bilinear functions defined on the Shishkin mesh. The Galerkin method is characterized by:

$$\text{Find } u^N \in V^N \text{ such that } a_G(u^N, v^N) = (f, v^N), \quad \forall v^N \in V^N. \tag{4.1}$$

The bilinear form satisfies the Galerkin orthogonality property. Due to assumption (1.4), the bilinear form $a_G(\cdot, \cdot)$ is coercive with respect to the energy norm $|||v|||_E^2 := \varepsilon_1 \|\nabla v\|_0^2 + \gamma \|v\|_0^2$. Hence, the standard weak formulation and the Galerkin method have unique solutions.

If $\sigma \geq 2$, from [17] we know that for problem (1.1) the following estimates are valid

$$\|u - u^I\|_{0,\Omega_C} \leq C N^{-2}, \tag{4.2}$$
$$\|u - u^I\|_{0,\Omega_0} \leq C \mu_0^{-1/2} N^{-2} \ln^2 N, \tag{4.3}$$
$$\|u - u^I\|_{0,\Omega_1} \leq C \mu_1^{-1/2} N^{-2} \ln^2 N,$$
$$\|u - u^I\|_{0,\Omega_y} \leq C \varepsilon_1^{1/4} N^{-2} \ln^2 N, \tag{4.4}$$

where u^I denotes the piecewise bilinear function that interpolates u at the mesh nodes of the Shishkin mesh. Also, a superconvergence result

$$|||u^I - u^N|||_E \leq C \left(N^{-2} + \left(\varepsilon_2 + \varepsilon_1^{1/2} \right)^{1/2} N^{-2} \ln^2 N \right)$$

is proved in [18, Theorem 4.1] for $\sigma \geq 5/2$.

Now we introduce the streamline-diffusion FEM in order to stabilize the discretization given by the standard Galerkin FEM

$$a_G(w, v) + \sum_{\tau \in \Omega^N} \delta_\tau (Lw - f, \varepsilon_2 b(x, y) v_x)_\tau = (f, v), \tag{4.5}$$

where $\delta_\tau \geq 0$ is a user chosen parameter. This modification is consistent with (1.1), i.e., the solution u of (1.1) is also a solution of (4.5). Its discretization reads:

Find $u^N \in V^N$ such that

$$a_{SD}(u^N, v^N) := a_G(u^N, v^N) + a_{stab}(u^N, v^N) = f_{SD}(v^N) \text{ for all } v^N \in V^N,$$

where

$$a_{stab}(w, v) := \sum_{\tau \in \Omega^N} \delta_\tau (-\varepsilon_1 \Delta w + \varepsilon_2 b w_x + c w, \varepsilon_2 b v_x)_\tau,$$
$$f_{SD}(v) := (f, v) + \sum_{\tau \in \Omega^N} \delta_\tau (f, \varepsilon_2 b v_x)_\tau.$$

This bilinear form also satisfies the orthogonality condition. Now we define a streamline diffusion norm

$$|||v|||_{SD}^2 := |||v|||_E^2 + \varepsilon_2^2 \sum_{\tau \in \Omega^N} \delta_\tau \|bv_x\|_{0,\tau}^2.$$

Lemma 4.1 *Assume that*

$$0 \le \delta_\tau \le \gamma / \|c\|_{L^\infty(\tau)}^2 \text{ for all } \tau \in \Omega^N. \tag{4.6}$$

Then the discrete bilinear form of the SDFEM is coercive with

$$a_{SD}\left(v^N, v^N\right) \ge \frac{1}{2}|||v^N|||_{SD}^2, \quad \forall v^N \in V^N.$$

Proof From the definitions of the norms, for any $v^N \in V^N$ we have

$$\begin{aligned}
a_{SD}(v^N, v^N) &= \varepsilon_1 \left(\nabla v^N, \nabla v^N\right) + \left(\varepsilon_2 b v_x^N, v^N\right) + \left(c v^N, v^N\right) \\
&\quad + \sum_{\tau \in \Omega^N} \delta_\tau \left(-\varepsilon_1 \Delta v^N + \varepsilon_2 b v_x^N + c v^N, \varepsilon_2 b v_x^N\right)_\tau \\
&\ge \varepsilon_1 \left|v^N\right|_1^2 + \gamma \left\|v^N\right\|_0^2 + \varepsilon_2^2 \sum_{\tau \in \Omega^N} \delta_\tau \left\|b v_x^N\right\|_{0,\tau}^2 \\
&\quad + \varepsilon_2 \sum_{\tau \in \Omega^N} \delta_\tau \left(c v^N, b v_x^N\right)_\tau .
\end{aligned}$$

The Cauchy–Schwarz inequality implies

$$\begin{aligned}
\left|\sum_{\tau \in \Omega^N} \delta_\tau \left(c v^N, \varepsilon_2 b v_x^N\right)_\tau\right| &\le \sum_{\tau \in \Omega^N} \delta_\tau \|c v^N\|_{0,\tau} \|\varepsilon_2 b v_x^N\|_{0,\tau} \\
&= \sum_{\tau \in \Omega^N} \delta_\tau \sqrt{\int_\tau \left(c v^N\right)^2 \int_\tau \left(\varepsilon_2 b v_x^N\right)^2} \le \frac{1}{2} \sum_{\tau \in \Omega^N} \delta_\tau \int_\tau \left(c v^N\right)^2 \\
&\quad + \frac{\varepsilon_2^2}{2} \sum_{\tau \in \Omega^N} \delta_\tau \int_\tau \left(b v_x^N\right)^2 \\
&\le \frac{1}{2} \sum_{\tau \in \Omega^N} \delta_\tau \|c\|_{L^\infty(\tau)}^2 \|v^N\|_{0,\tau}^2 + \frac{\varepsilon_2^2}{2} \sum_{\tau \in \Omega^N} \delta_\tau \|b v_x^N\|_{0,\tau}^2 \\
&\le \frac{\gamma}{2} \|v^N\|_0^2 + \frac{\varepsilon_2^2}{2} \sum_{\tau \in \Omega^N} \delta_\tau \|b v_x^N\|_{0,\tau}^2
\end{aligned}$$

for $\delta_\tau \le \gamma / \|c\|^2_{L^\infty(\tau)}$. Since

$$\sum_{\tau\in\Omega^N} \delta_\tau \left(cv^N, \varepsilon_2 b v_x^N\right)_\tau \ge -\frac{\gamma}{2}\|v^N\|_0^2 - \frac{\varepsilon_2^2}{2}\sum_{\tau\in\Omega^N} \delta_\tau \|bv_x^N\|^2_{0,\tau},$$

we conclude

$$\begin{aligned} &a_{SD}(v^N, v^N) \\ &\ge \varepsilon_1 |v^N|_1^2 + \gamma\|v^N\|_0^2 + \varepsilon_2^2 \sum_{\tau\in\Omega^N} \delta_\tau \|bv_x^N\|^2_{0,\tau} - \frac{\gamma}{2}\|v^N\|_0^2 - \frac{\varepsilon_2^2}{2}\sum_{\tau\in\Omega^N} \delta_\tau \|bv_x^N\|^2_{0,\tau} \\ &= \frac{\varepsilon_1}{2}|v^N|_1^2 + \frac{\varepsilon_1}{2}|v^N|_1^2 + \frac{\gamma}{2}\|v^N\|_0^2 + \frac{\varepsilon_2^2}{2}\sum_{\tau\in\Omega^N} \delta_\tau \|bv_x^N\|^2_{0,\tau} \ge \frac{1}{2}|||v^N|||^2_{SD}. \end{aligned}$$

□

5 Superconvergence and choice of the stabilization parameter

In this section we consider the discretization error. The streamline-diffusion parameter δ_τ is chosen to be constant on each mesh subdomain of Ω, i.e., we set $\delta_\tau = \delta_k$ if $\tau \subset \Omega_k$ for $k \in \{y, C, 00, 0y, 11, 1y\}$. Throughout the following analysis we will often use the following inverse estimates. Let v be a polynomial on the mesh rectangle τ. Then

$$\|v_x\|_{L^p(\tau)} \le Ch^{-1}_{x,\tau}\|v\|_{L^p(\tau)}, \quad \|v_y\|_{L^p(\tau)} \le Ch^{-1}_{y,\tau}\|v\|_{L^p(\tau)} \tag{5.1}$$

and

$$\|v\|_{L^q(\tau)} \le C(\text{meas}(\tau))^{1/q-1/p}\|v\|_{L^p(\tau)} \quad \text{for} \quad p, q \in [1,\infty]. \tag{5.2}$$

Furthermore, for $b \in W^1_\infty(\Omega)$, $k \in \{y, C, 00, 0y, 11, 1y\}$, $\tau \in \Omega_k$ and any bilinear function χ we shall use estimate from [2, Proposition 8]:

$$\begin{aligned} &\left|(b(v-v^I)_x, b\chi_x)_{\Omega_k}\right| \\ &\quad \le C\Big[(h_{x,\tau}+h_{y,\tau})(h_{x,\tau}\|v_{xx}\|_{0,\Omega_k} + h_{y,\tau}\|v_{xy}\|_{0,\Omega_k}) + h^2_{y,\tau}\|v_{xyy}\|_{0,\Omega_k}\Big]\|\chi_x\|_{0,\Omega_k}. \end{aligned} \tag{5.3}$$

The main result of the paper is given in the following theorem.

Theorem 2 *Let the estimates of Theorem 1 and Assumption 2.1 hold true for the solution u to problem (1.1). Let u^N be the streamline-diffusion approximation to u and u^I be the piecewise bilinear function that interpolates u at the mesh nodes of the*

Shishkin mesh with $\sigma \geq 5/2$. Suppose that stabilization parameter δ_τ satisfies (4.6) and

$$\begin{aligned}
\delta_{00} \leq & C^* N^{-1} \min\left\{\varepsilon_1^{-1/4} N^{-1}, 1\right\}, \\
\delta_{0y} \leq & C^* N^{-1} \min\left\{N^{-1} \min\left\{\varepsilon_1^{-1/4}, \varepsilon_2^{-1} N^{-1/2}\right\}, 1\right\}, \\
\delta_{y} \leq & C^* N^{-1} \min\left\{\varepsilon_2^{-1} N^{-3/2}, \varepsilon_1^{-1/2}\right\}, \\
\delta_{11} \leq & C^* N^{-1} \min\left\{\varepsilon_1 \varepsilon_2^{-1} N^{-1}, \varepsilon_1\right\}, \\
\delta_{1y} \leq & C^* N^{-1} \min\left\{\varepsilon_2^{-1} N^{-1} \min\left\{\varepsilon_1^{3/2}, N^{-1/2}\right\}, \varepsilon_1\right\}, \\
\delta_{C} \leq & C^* N^{-1} \min\left\{\varepsilon_1^{-1/2} \varepsilon_2^{-1} N^{-1} \min\left\{\varepsilon_1^{-1/2}, (\varepsilon_2 + \varepsilon_1^{1/2})^{-1/2}\right\}, 1\right\},
\end{aligned} \tag{5.4}$$

with some positive constant C^* independent of $\varepsilon_1, \varepsilon_2$ and N. Then we have a superconvergence result

$$|||u^I - u^N|||_{SD} \leq C\left(N^{-2} + \left(\varepsilon_2 + \varepsilon_1^{1/2}\right)^{1/2} N^{-2} \ln^2 N\right). \tag{5.5}$$

Proof Our error analysis starts from the coercivity and Galerkin orthogonality:

$$\begin{aligned}
\frac{1}{2}|||u^I - u^N|||_{SD}^2 &\leq a_G(u^I - u^N, u^I - u^N) + a_{stab}(u^I - u^N, u^I - u^N) \\
&= a_G(u^I - u, u^I - u^N) + a_{stab}(u^I - u, u^I - u^N).
\end{aligned} \tag{5.6}$$

Let $u^I - u^N = \chi$ and $u^I - u = \eta$. For the first term on the right-hand side from [18, p.751] we have

$$|a_G(\eta, \chi)| \leq C\left(N^{-2} + \left(\varepsilon_2 + \varepsilon_1^{1/2}\right)^{1/2} N^{-2} \ln^2 N\right) |||\chi|||_E \quad \text{for all} \quad \chi \in V^N. \tag{5.7}$$

Then by (5.6) and (5.7), we have

$$\frac{1}{2}|||\chi|||_{SD}^2 \leq a_{stab}(\eta, \chi) + C\left(N^{-2} + \left(\varepsilon_2 + \varepsilon_1^{1/2}\right)^{1/2} N^{-2} \ln^2 N\right) |||\chi|||_{SD}.$$

Therefore, we have to estimate a_{stab}:

$$\begin{aligned}
a_{stab}(\eta, \chi) &= \sum_{\tau \in \Omega^N} \delta_\tau \left[-\varepsilon_1 \varepsilon_2 (\Delta\eta, b\chi_x)_\tau + \varepsilon_2^2 (b\eta_x, b\chi_x)_\tau + \varepsilon_2 (c\eta, b\chi_x)_\tau \right] \\
&\leq \sum_{\tau \in \Omega^N} \delta_\tau \left[\varepsilon_1 \varepsilon_2 |(\Delta u, b\chi_x)_\tau| + \varepsilon_2^2 |(b(u - u^I)_x, b\chi_x)_\tau| \right.
\end{aligned}$$

$$+\varepsilon_2|(c(u-u^I), b\chi_x)_\tau|\Big]. \tag{5.8}$$

Let us first consider Ω_y which is a part of the domain where parabolic boundary layers dominate. For the third term of (5.8) we use (4.4) and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}\varepsilon_2\delta_y\Big|(c(u-u^I), b\chi_x)_{\Omega_y}\Big| &\le C\varepsilon_2\delta_y\varepsilon_1^{1/4}N^{-2}\ln^2 N\|b\chi_x\|_{0,\Omega_y}\\ &\le C\varepsilon_1^{1/4}\delta_y^{1/2}N^{-2}\ln^2 N|||\chi|||_{SD}.\end{aligned}$$

For the second term of (5.8) we proceed as follows. Let $w = S + E_2$ and $\tilde{w} = E_{10} + E_{11} + E_3^0 + E_3^1$. Using (5.3), (2.3), (2.5), (2.8) and (2.9) we get

$$\begin{aligned}&\varepsilon_2^2\delta_y\Big|(b(w-w^I)_x, b\chi_x)_{\Omega_y}\Big|\\ &\le C\varepsilon_2^2\delta_y\Big[(N^{-1}+N^{-1}\sqrt{\varepsilon_1}\ln N)(N^{-1}\|w_{xx}\|_{0,\Omega_y}+N^{-1}\sqrt{\varepsilon_1}\ln N\|w_{xy}\|_{0,\Omega_y})\\ &\quad + N^{-2}\varepsilon_1\ln^2 N\|w_{xyy}\|_{0,\Omega_y}\Big]\|b\chi_x\|_{0,\Omega_y}\\ &\le C\varepsilon_1^{1/4}\varepsilon_2\delta_y^{1/2}N^{-2}\ln^2 N|||\chi|||_{SD}.\end{aligned}$$

Using (5.1), (5.2) and (2.4), (2.6), (2.7) we obtain

$$\begin{aligned}&\varepsilon_2^2\delta_y\Big|(b(\tilde{w}-\tilde{w}^I)_x, b\chi_x)_{\Omega_y}\Big|\\ &\le C\varepsilon_2^2\delta_y\Big[\|\tilde{w}_x\|_{L^1(\Omega_y)}\|\chi_x\|_{L^\infty(\Omega_y)}+\|(\tilde{w}^I)_x\|_{0,\Omega_y}\|b\chi_x\|_{0,\Omega_y}\Big]\\ &\le C\varepsilon_1^{1/4}\varepsilon_2\delta_y^{1/2}N^{1-\sigma}\ln^{1/2}N|||\chi|||_{SD}.\end{aligned}$$

For the first term of (5.8) we also use the splitting $u = w+\tilde{w}$. Using (5.2) and estimates from Theorem 1, we have

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_y\Big|(\Delta\tilde{w}, b\chi_x)_{\Omega_y}\Big| &\le \varepsilon_1\varepsilon_2\delta_y\|\Delta\tilde{w}\|_{L^1(\Omega_y)}\|b\chi_x\|_{L^\infty(\Omega_y)}\\ &\le C\varepsilon_1^{1/4}(\varepsilon_2+\varepsilon_1^{1/2})\delta_y^{1/2}N^{1-\sigma}\ln^{1/2}N|||\chi|||_{SD}.\end{aligned}$$

Further, if we use the Cauchy–Schwarz inequality and estimate (2.3) and (2.5), we get

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_y|(\Delta w, b\chi_x)_{\Omega_y}| &\le \varepsilon_1\varepsilon_2\delta_y\|\Delta w\|_{0,\Omega_y}\|\chi_x\|_{0,\Omega_y} \le C\varepsilon_1^{1/4}\varepsilon_2\delta_y\|\chi_x\|_{0,\Omega_y}\\ &\le C\varepsilon_1^{-1/4}\varepsilon_2\delta_y|||\chi|||_E.\end{aligned} \tag{5.9}$$

Alternatively, if we use an inverse inequality we get

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_y|(\Delta w, b\chi_x)_{\Omega_y}| &\le \varepsilon_1\varepsilon_2\delta_y\|\Delta w\|_{0,\Omega_y}\|\chi_x\|_{0,\Omega_y} \le C\varepsilon_1^{1/4}\varepsilon_2\delta_y\|\chi_x\|_{0,\Omega_y}\\ &\le C\varepsilon_1^{1/4}\varepsilon_2\delta_y N\|\chi\|_{0,\Omega_y} \le C\varepsilon_1^{1/4}\varepsilon_2\delta_y N|||\chi|||_E.\end{aligned} \tag{5.10}$$

Combining (5.9) and (5.10) we have

$$\varepsilon_1\varepsilon_2\delta_y|(\Delta w, b\chi_x)_{\Omega_y}| \le C\left(\varepsilon_1^{-1/4}\varepsilon_2\delta_y|||\chi|||_E\right)^{1/2}\left(\varepsilon_1^{1/4}\varepsilon_2\delta_y N|||\chi|||_E\right)^{1/2}$$
$$\le C\varepsilon_2\delta_y N^{1/2}|||\chi|||_E. \tag{5.11}$$

Collecting the above results, we get

$$|a_{stab}(\eta,\chi)_{\Omega_y}| \le C\left(\varepsilon_2\delta_y N^{1/2} + \varepsilon_1^{1/4}\delta_y^{1/2}\left(N^{-2}+N^{1-\sigma}\right)\ln^2 N\right)|||\chi|||_{SD}. \tag{5.12}$$

On Ω_{00} for third term of (5.8) we use (4.3) and the Cauchy–Schwarz inequality to obtain

$$\varepsilon_2\delta_{00}\left|\left(c(u-u^I), b\chi_x\right)_{\Omega_{00}}\right| \le C\varepsilon_2\delta_{00}\mu_0^{-1/2}N^{-2}\ln^2 N\|b\chi_x\|_{0,\Omega_{00}}$$
$$\le C\delta_{00}^{1/2}\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}N^{-2}\ln^2 N|||\chi|||_{SD}.$$

For the second term of (5.8) let $w = S + E_{10}$ and $\tilde{w} = E_2 + E_{11} + E_3^0 + E_3^1$. Using (5.3) and estimates from Theorem 1 we get

$$\varepsilon_2^2\delta_{00}\left|\left(b(w-w^I)_x, b\chi_x\right)_{\Omega_{00}}\right|$$
$$\le C\varepsilon_2^2\delta_{00}\Big[\left(\mu_0^{-1}N^{-1}\ln N + N^{-1}\right)\left(\mu_0^{-1}N^{-1}\ln N\|w_{xx}\|_{0,\Omega_{00}} + N^{-1}\|w_{xy}\|_{0,\Omega_{00}}\right)$$
$$+\, N^{-2}\|w_{xyy}\|_{0,\Omega_{00}}\Big]\|b\chi_x\|_{0,\Omega_{00}}$$
$$\le C\varepsilon_2^{1/2}\delta_{00}^{1/2}N^{-2}\ln N|||\chi|||_{SD}.$$

Using (5.1), (5.2) and estimates from Theorem 1 we obtain

$$\varepsilon_2^2\delta_{00}\left|\left(b(\tilde{w}-\tilde{w}^I)_x, b\chi_x\right)_{\Omega_{00}}\right|$$
$$\le C\varepsilon_2^2\delta_{00}\Big[\|\tilde{w}_x\|_{L^1(\Omega_{00})}\|b\chi_x\|_{L^\infty(\Omega_{00})} + \|(\tilde{w}^I)_x\|_{0,\Omega_{00}}\|b\chi_x\|_{0,\Omega_{00}}\Big]$$
$$\le C\varepsilon_2^{1/2}\delta_{00}^{1/2}N^{1-\sigma}\ln^{1/2}N|||\chi|||_{SD}.$$

If we use the Cauchy–Schwarz inequality, estimates from Theorem 1 and (2.1) for the first term of (5.8) we have

$$\varepsilon_1\varepsilon_2\delta_{00}\left|(\Delta w, b\chi_x)_{\Omega_{00}}\right| \le C\varepsilon_1\varepsilon_2\delta_{00}\|\Delta w\|_{0,\Omega_{00}}\|b\chi_x\|_{0,\Omega_{00}} \le C\varepsilon_1^{1/4}\delta_{00}|||\chi|||_E.$$

Further,

$$\varepsilon_1\varepsilon_2\delta_{00}\left|(\Delta\tilde{w}, b\chi_x)_{\Omega_{00}}\right| \le \varepsilon_1\varepsilon_2\|\Delta\tilde{w}\|_{L^1(\Omega_{00})}\|b\chi_x\|_{L^\infty(\Omega_{00})}$$

$$\leq C\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\delta_{00}^{1/2}N^{1-\sigma}\ln^{1/2}N|||\chi|||_{SD}.$$

Combining these estimates, we have on Ω_{00}

$$|a_{stab}(\eta,\chi)_{\Omega_{00}}|\leq C\left(\varepsilon_1^{1/4}\delta_{00}+\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\delta_{00}^{1/2}\left(N^{-2}+N^{1-\sigma}\right)\ln^2 N\right)|||\chi|||_{SD}. \tag{5.13}$$

On Ω_{0y} for the third term of (5.8) we use (4.3) and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}\varepsilon_2\delta_{0y}\left|(c(u-u^I),b\chi_x)_{\Omega_{0y}}\right| &\leq C\varepsilon_2\delta_{0y}\mu_0^{-1/2}N^{-2}\ln^2 N\|b\chi_x\|_{0,\Omega_{0y}}\\ &\leq C\delta_{0y}^{1/2}\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}N^{-2}\ln^2 N|||\chi|||_{SD}.\end{aligned}$$

For the second term of (5.8) let $w=S+E_2+E_{10}+E_3^0$ and $\tilde{w}=E_{11}+E_3^1$. Using (5.3) and estimates from Theorem 1 we get

$$\begin{aligned}&\varepsilon_2^2\delta_{0y}\left|\left(b(w-w^I)_x,b\chi_x\right)_{\Omega_{0y}}\right|\\ &\leq C\varepsilon_2^2\delta_{0y}\Big[\left(\mu_0^{-1}N^{-1}\ln N+\sqrt{\varepsilon_1}N^{-1}\ln N\right)\left(\mu_0^{-1}N^{-1}\ln N\|w_{xx}\|_{0,\Omega_{0y}}\right.\\ &\quad\left.+\sqrt{\varepsilon_1}N^{-1}\ln N\|w_{xy}\|_{0,\Omega_{0y}}\right)+\varepsilon_1N^{-2}\ln^2 N\|w_{xyy}\|_{0,\Omega_{0y}}\Big]\|b\chi_x\|_{0,\Omega_{0y}}\\ &\leq C\varepsilon_2\delta_{0y}^{1/2}N^{-2}\ln^2 N|||\chi|||_{SD}.\end{aligned}$$

Using (5.1), (5.2) and estimates from Theorem 1 we obtain

$$\begin{aligned}&\varepsilon_2^2\delta_{0y}\left|\left(b(\tilde{w}-\tilde{w}^I)_x,b\chi_x\right)_{\Omega_{0y}}\right|\\ &\leq C\varepsilon_2^2\delta_{0y}\left[\|\tilde{w}_x\|_{L^1(\Omega_{0y})}\|b\chi_x\|_{L^\infty(\Omega_{0y})}+\|(\tilde{w}^I)_x\|_{0,\Omega_{0y}}\|b\chi_x\|_{0,\Omega_{0y}}\right]\\ &\leq C\varepsilon_2\delta_{0y}^{1/2}N^{1-\sigma}|||\chi|||_{SD}.\end{aligned}$$

For the first part of (5.8) we have

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_{0y}\left|(\Delta\tilde{w},b\chi_x)_{\Omega_{0y}}\right| &\leq \varepsilon_1\varepsilon_2\|\Delta\tilde{w}\|_{L^1(\Omega_{0y})}\|b\chi_x\|_{L^\infty(\Omega_{0y})}\\ &\leq C\left(\varepsilon_2+\varepsilon_1^{1/2}\right)\delta_{0y}^{1/2}N^{1-\sigma}|||\chi|||_{SD}.\end{aligned}$$

Further, if we use (2.1) and estimates from Theorem 1, we obtain

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_{0y}\left|(\Delta(S+E_{10}),b\chi_x)_{\Omega_{0y}}\right| &\leq \varepsilon_1\varepsilon_2\delta_{0y}\|\Delta(S+E_{10})\|_{0,\Omega_{0y}}\|\chi_x\|_{0,\Omega_{0y}}\\ &\leq C\varepsilon_1^{1/2}\delta_{0y}\ln N|||\chi|||_E.\end{aligned}$$

If we do apply the Cauchy–Schwarz inequality for $\varepsilon_1\varepsilon_2\delta_{0y}\left|\left(\Delta(E_2+E_3^0), b\chi_x\right)_{\Omega_{0y}}\right|$, the estimate is not sharp enough. Thus we have to use an alternative way. We know that it holds

$$(\Delta E_2, b\chi_x)_{\Omega_{0y}} = -\left((b\Delta E_2)_x, \chi\right)_{\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}} - (\Delta E_2, b\chi_x)_{\Omega_y} - (\Delta E_2, b\chi_x)_{\Omega_{1y}}$$

Estimating each term on the right side separately, we have

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_{0y}\left|(\Delta E_2, b\chi_x)_{\Omega_{1y}}\right| &\le C\varepsilon_1\varepsilon_2\delta_{0y}\|\Delta E_2\|_{0,\Omega_{1y}}\|b\chi_x\|_{0,\Omega_{1y}}\\ &\le C\varepsilon_1^{1/4}\varepsilon_2^{1/2}\delta_{0y}\ln^{1/2}N|||\chi|||_E,\end{aligned} \tag{5.14}$$

where we use (2.2), and

$$\begin{aligned}&\varepsilon_1\varepsilon_2\delta_{0y}\left|((b\Delta E_2)_x, \chi)_{\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}}\right|\\ &\quad\le C\varepsilon_1\varepsilon_2\delta_{0y}\|(\Delta E_2)_x\|_{0,\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}}\|\chi\|_{0,\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}} \le C\varepsilon_1^{1/4}\varepsilon_2\delta_{0y}|||\chi|||_E.\end{aligned} \tag{5.15}$$

In (5.11), we determined that

$$\varepsilon_1\varepsilon_2\delta_{0y}\left|(\Delta E_2, b\chi_x)_{\Omega_y}\right| \le C\varepsilon_2\delta_{0y}N^{1/2}|||\chi|||_E. \tag{5.16}$$

From (5.14), (5.15) and (5.16) we obtain

$$\varepsilon_1\varepsilon_2\delta_{0y}\left|(\Delta E_2, b\chi_x)_{\Omega_{0y}}\right| \le C\delta_{0y}\left(\varepsilon_2 N^{1/2} + \varepsilon_1^{1/4}\varepsilon_2^{1/2}\ln^{1/2}N + \varepsilon_1^{1/4}\varepsilon_2\right)|||\chi|||_E.$$

Similarly, we get

$$\begin{aligned}&\varepsilon_1\varepsilon_2\delta_{0y}\left|\left(\Delta E_3^0, b\chi_x\right)_{\Omega_{1y}}\right| \le C\varepsilon_1\varepsilon_2\delta_{0y}\|\Delta E_3^0\|_{0,\Omega_{1y}}\|b\chi_x\|_{0,\Omega_{1y}}\\ &\le C\left(\varepsilon_1^{1/2}\delta_{0y}N^{-\sigma}|||\chi|||_E + \varepsilon_1^{1/4}\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\delta_{0y}^{1/2}N^{-\sigma}|||\chi|||_{SD}\right),\\ &\varepsilon_1\varepsilon_2\delta_{0y}\left|\left((b\Delta E_3^0)_x, \chi\right)_{\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}}\right|\\ &\le C\varepsilon_1\varepsilon_2\delta_{0y}\|(\Delta E_3^0)_x\|_{0,\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}}\|\chi\|_{0,\Omega_{0y}\cup\Omega_y\cup\Omega_{1y}} \le C\varepsilon_1^{1/4}\delta_{0y}|||\chi|||_E,\\ &\varepsilon_1\varepsilon_2\delta_{0y}\left|\left(\Delta E_3^0, b\chi_x\right)_{\Omega_y}\right|\\ &\le C\varepsilon_1\varepsilon_2\delta_{0y}\|\Delta E_3^0\|_{0,\Omega_y}\|b\chi_x\|_{0,\Omega_y} \le C\varepsilon_1^{1/4}\delta_{0y}N^{1-\sigma}|||\chi|||_E,\end{aligned}$$

which gives

$$\varepsilon_1\varepsilon_2\delta_{0y}\left|\left(\Delta E_3^0, b\chi_x\right)_{\Omega_{0y}}\right|$$

$$\leq C\left(\varepsilon_1^{1/4}\delta_{0y}|||\chi|||_E + \varepsilon_1^{1/4}\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\delta_{0y}^{1/2}N^{-\sigma}|||\chi|||_{SD}\right).$$

Combining the above estimates, on Ω_{0y} we have

$$|a_{stab}(\eta,\chi)_{\Omega_{0y}}| \leq C\left((\varepsilon_1^{1/4}+\varepsilon_2 N^{1/2})\delta_{0y}+\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\delta_{0y}^{1/2}\left(N^{-2}+N^{1-\sigma}\right)\ln^2 N\right)|||\chi|||_{SD}. \tag{5.17}$$

On Ω_{11} we apply the same technique as on Ω_{00} and obtain

$$|a_{stab}(\eta,\chi)_{\Omega_{11}}| \leq C\left(\varepsilon_1^{-1}\varepsilon_2\delta_{11}+\varepsilon_1^{-1/2}\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\delta_{11}^{1/2}\left(N^{-2}+N^{1-\sigma}\right)\ln^2 N\right)|||\chi|||_{SD}. \tag{5.18}$$

Moreover, estimates on Ω_{1y} are obtained in the same way as on Ω_{0y}

$$|a_{stab}(\eta,\chi)_{\Omega_{1y}}| \leq C\left(\left(\varepsilon_1^{-3/2}\varepsilon_2+\varepsilon_2 N^{1/2}\right)\delta_{1y}+\varepsilon_1^{-1/2}\delta_{1y}^{1/2}\left(N^{-2}+N^{1-\sigma}\right)\ln^2 N\right)|||\chi|||_{SD}. \tag{5.19}$$

On Ω_C we set $w = E_{10}+E_{11}+E_2+E_3^0+E_3^1$. For the first term of (5.8), the Hölder inequality and an inverse estimate yield

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_C|(\Delta w, b\chi_x)_{\Omega_C}| &\leq C\varepsilon_1\varepsilon_2\delta_C\|\Delta w\|_{L^1(\Omega_C)}\|b\chi_x\|_{L^\infty(\Omega_C)}\\ &\leq C\varepsilon_2\delta_C\left(\varepsilon_2+\varepsilon_1^{1/2}\right)N^{1-\sigma}\|b\chi_x\|_{0,\Omega_C}\\ &\leq C\delta_C^{1/2}\left(\varepsilon_2+\varepsilon_1^{1/2}\right)N^{1-\sigma}|||\chi|||_{SD}.\end{aligned}$$

The second term is estimated using the inverse inequality

$$\begin{aligned}&\varepsilon_2^2\delta_C|\left(b(w-w^I)_x, b\chi_x\right)_{\Omega_C}|\\ &\quad\leq C\varepsilon_2^2\delta_C\left[\|w_x\|_{L^1(\Omega_C)}\|b\chi_x\|_{L^\infty(\Omega_C)}+\|(w^I)_x\|_{0,\Omega_C}\|b\chi_x\|_{0,\Omega_C}\right]\\ &\quad\leq C\varepsilon_2^2\delta_C N^{1-\sigma}\|b\chi_x\|_{0,\Omega_C}\leq C\varepsilon_2\delta_C^{1/2}N^{1-\sigma}|||\chi|||_{SD}.\end{aligned}$$

With the Cauchy–Schwarz inequality and estimate (4.2) for the third term we get

$$\begin{aligned}\varepsilon_2\delta_C|\left(c(w-w^I), b\chi_x\right)_{\Omega_C}| &\leq C\varepsilon_2\delta_C\|w-w_x\|_{0,\Omega_C}\|b\chi_x\|_{0,\Omega_C}\\ &\leq C\varepsilon_2\delta_C N^{-2}\|b\chi_x\|_{0,\Omega_C}\leq C\delta_C^{1/2}N^{-2}|||\chi|||_{SD}.\end{aligned}$$

The terms containing S must be treated with care, because they are not exponentially small away from the layers. Therefore, we use

$$(\varDelta S, b\chi_x)_{\varOmega_C} + (\varDelta S, b\chi_x)_{\varOmega_{00}} + (\varDelta S, b\chi_x)_{\varOmega_{11}} = -\left((b\varDelta S)_x, \chi\right)_{\varOmega_{00}\cup\varOmega_C\cup\varOmega_{11}}$$

to get for the first term

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta_C|(\varDelta S, b\chi_x)_{\varOmega_C}| &\le C\varepsilon_1\varepsilon_2\delta_C(\|\chi\|_{0,\varOmega_{00}\cup\varOmega_C\cup\varOmega_{11}} + \|\chi_x\|_{L^1(\varOmega_{00})} + \|\chi_x\|_{L^1(\varOmega_{11})})\\ &\le C\varepsilon_1\varepsilon_2\delta_C\left(|||\chi|||_E + \sqrt{\mathrm{meas}(\varOmega_{00})}\|\chi_x\|_{0,\varOmega_{00}}\right.\\ &\quad\left.+\sqrt{\mathrm{meas}(\varOmega_{11})}\|\chi_x\|_{0,\varOmega_{11}}\right)\\ &\le C\left(\varepsilon_1\varepsilon_2 + \varepsilon_1^{1/2}\varepsilon_2\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\ln^{1/2}N\right)\delta_C|||\chi|||_E.\end{aligned}$$

The Lin-identity

$$\left|\left((v-v^I)_x, \chi_x\right)_\tau\right| \le Ch_{y,\tau}^2\|v_{xyy}\|_{0,\tau}\|\chi_x\|_{0,\tau}, \quad \forall v \in C^3(\bar\tau)$$

applied to the second term of (5.8) gives

$$\varepsilon_2^2\delta_C|(b(S-S^I)_x, b\chi_x)_{\varOmega_C}| \le C\varepsilon_2^2\delta_C N^{-2}\|b\chi_x\|_{0,\varOmega_C} \le C\varepsilon_2\delta_C^{1/2}N^{-2}|||\chi|||_{SD}.$$

Using the Cauchy–Schwarz inequality, we get for the third term

$$\varepsilon_2\delta_C|\left(c(S-S^I), b\chi_x\right)_{\varOmega_C}| \le C\delta_C^{1/2}N^{-2}|||\chi|||_{SD}.$$

From the above estimates on $\varOmega_C$ we have

$$\begin{aligned}&|a_{stab}(\eta,\chi)_{\varOmega_C}|\\ &\le C\left(\left(\varepsilon_1\varepsilon_2+\varepsilon_1^{1/2}\varepsilon_2\left(\varepsilon_2+\varepsilon_1^{1/2}\right)^{1/2}\ln^{1/2}N\right)\delta_C + \delta_C^{1/2}\left(N^{-2}+N^{1-\sigma}\right)\right)|||\chi|||_{SD}.\end{aligned} \tag{5.20}$$

Estimates (5.12), (5.13), (5.17)–(5.20) together with the proposed choice of stabilization parameter give the final result (5.5). □

Remark 5.1 Thorough inspection of the choice of parameters from Theorem 2 reveals its accordance with the case $\varepsilon_2 = 1$ (from [2]) in the sense that more stabilization is needed away from the stronger exponential layer.

Table 1 $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-4}$

N	$\|\|\|u - u^N\|\|\|_E$	Rate	$\|\|\|u^I - u^N\|\|\|_{SD}$	Rate	$\|u - u^N\|_{L^\infty}$	Rate
16	2.176e−03	1.62	7.168e−03	2.87	1.213e−01	1.07
32	1.018e−03	1.04	1.859e−03	2.66	7.349e−02	1.26
64	5.973e−04	1.00	4.778e−04	2.50	3.855e−02	1.50
128	3.485e−04	1.00	1.239e−04	2.39	1.721e−02	1.68
256	1.993e−04	1.00	3.254e−05	2.30	6.710e−03	1.81
512	1.122e−04	1.00	8.667e−06	2.23	2.369e−04	1.89
1024	6.233e−05	1.00	2.339e−06	2.17	7.803e−04	1.94
2048	3.429e−05	–	6.381e−07	–	2.450e−04	–

Table 2 $\varepsilon_1 = 10^{-4}, \varepsilon_2 = 10^{-10}$

N	$\|\|\|u - u^N\|\|\|_E$	Rate	$\|\|\|u^I - u^N\|\|\|_{SD}$	Rate	$\|u - u^N\|_{L^\infty}$	Rate
16	2.493e−02	0.88	1.126e−02	1.65	1.203e−01	1.05
32	1.646e−02	0.90	5.191e−03	1.80	7.336e−02	1.27
64	1.042e−02	0.95	2.074e−03	1.91	3.844e−02	1.49
128	6.252e−03	1.16	7.392e−04	2.34	1.719e−02	2.01
256	3.262e−03	1.20	2.000e−04	2.40	5.587e−03	2.21
512	1.637e−03	1.18	5.027e−05	2.36	1.566e−03	2.26
1024	8.192e−04	1.16	1.259e−05	2.32	4.154e−04	2.27
2048	4.097e−04	–	3.147e−06	–	1.070e−04	–

6 Numerical experiments

Our test problem is

$$\begin{aligned} -\varepsilon_1 \Delta u + \varepsilon_2 (3 - x) u_x + u &= f(x, y) \quad x \in \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where the function $f(x, y)$ is chosen in such a way that

$$\begin{aligned} u(x, y) = \frac{1}{4}\left(1 + \frac{\sin 8x}{2}\right)\left(1 - e^{-\varepsilon_2 k_1 x/(2\varepsilon_1)}\right) \\ \left(1 - e^{-\varepsilon_2 k_2 (1-x)/(2\varepsilon_1)}\right)\left(1 - e^{-y/\sqrt{\varepsilon_1}}\right)\left(1 - e^{-(1-y)/\sqrt{\varepsilon_1}}\right) \end{aligned}$$

is the exact solution and $k_{1,2} = \mp 1 + \sqrt{1 + 16\varepsilon_1/\varepsilon_2^2}$. We chose $\sigma = 5/2$ and $C^* = 1$. The rate of convergence is estimated in the standard way. All calculations were carried out using MATLAB.

Tables 1 and 2 display the errors of the SDFEM for the test problem in various norms for two choices of perturbation parameters. They are illustrations of the first

Table 3 $\varepsilon_2 = 10^{-4}$, $N = 256$

ε_1	$\vert\vert\vert u - u^N \vert\vert\vert_E$	$\vert\vert\vert u^I - u^N \vert\vert\vert_{SD}$	$\Vert u - u^N \Vert_{L^\infty}$
10^{-1}	3.439e−04	1.249e−06	3.252e−06
10^{-2}	8.640e−04	1.386e−05	4.698e−05
10^{-3}	1.686e−03	3.732e−05	6.301e−04
10^{-4}	3.265e−03	2.004e−04	5.587e−03
10^{-5}	2.095e−03	1.403e−04	6.709e−03
10^{-6}	1.190e−03	8.278e−05	6.710e−03
10^{-7}	6.762e−04	5.273e−05	6.710e−03
10^{-8}	3.965e−04	3.908e−05	6.710e−03
10^{-9}	2.588e−04	3.413e−05	6.710e−03
10^{-10}	1.993e−04	3.254e−05	6.710e−03

Table 4 $\varepsilon_1 = 10^{-10}$, $N = 256$

ε_2	$\vert\vert\vert u - u^N \vert\vert\vert_E$	$\vert\vert\vert u^I - u^N \vert\vert\vert_{SD}$	$\Vert u - u^N \Vert_{L^\infty}$
10^{-1}	4.235e−03	3.056e−04	5.388e−03
10^{-2}	1.654e−03	1.202e−04	6.708e−03
10^{-3}	5.349e−04	5.035e−05	6.710e−03
10^{-4}	1.993e−04	3.254e−05	6.710e−03
10^{-5}	1.254e−04	3.018e−05	6.710e−03
10^{-6}	1.195e−04	3.005e−05	6.710e−03
10^{-7}	1.193e−04	3.004e−05	6.710e−03
10^{-8}	1.193e−04	3.004e−05	6.710e−03
10^{-9}	1.193e−04	3.004e−05	6.710e−03
10^{-10}	1.193e−04	3.004e−05	6.710e−03

order convergence results (first column) and the second order superconvergence result (second column). The last column gives the errors in the L^∞ norm for which almost second order convergence is also observed, although we do not have theoretical justification for this behavior. In Table 3 for fixed $\varepsilon_2 = 10^{-4}$, $N = 256$ and in Table 4 for fixed $\varepsilon_1 = 10^{-10}$, $N = 256$, we investigate the dependence of the errors on the perturbation parameters. We observe ε_1-, ε_2-independence of the errors in all given norms.

The numerical experiments here above indicate that the estimates for the SDFEM parameters given in Theorem 2 are sharp. This observation for the δ_C is illustrated in Fig. 2 where the solid line represent $|||u^I - u^N \||||_{SD}$ with all δ chosen to be equal to the upper bound from Theorem 2 (with $C^* = 1$) and the dashed line represent $|||u^I - u^N \||||_{SD}$ when δ_C is chosen to be greater then the upper bound (5.4) and $C^* = 1$ [with minimum replaced by maximum in (5.4)]. That the other conditions on the SDFEM parameters are sharp as well can be justified by the use of similar investigation.

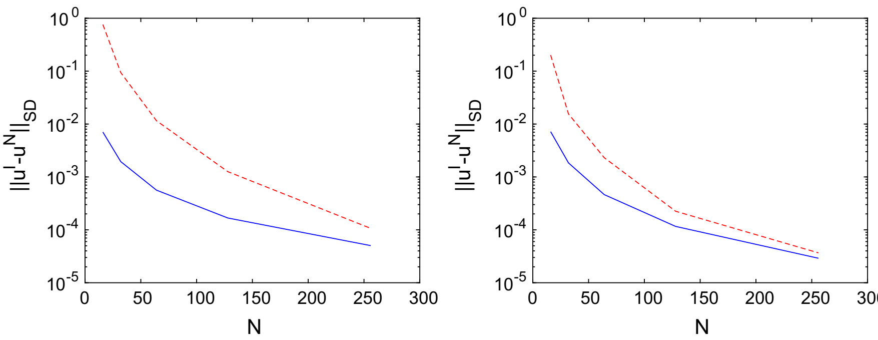

Fig. 2 Convergence behavior in the SD norm for $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-3}$ (left), and $\varepsilon_1 = 10^{-16}$, $\varepsilon_2 = 10^{-10}$ (right)

7 Conclusion

A singularly perturbed elliptic problem with two small independent parameters has been considered. To obtain numerical approximation of the problem, we apply the streamline–diffusion finite element method with bilinear elements on a layer-adapted mesh of Shishkin type. We have proved that such discretization exhibits superconvergence property with the appropriate choice of streamline-diffusion parameters. Numerical tests presented in Fig. 2 indicate that our estimates of the parameters are sharp. Compared with the standard Galerkin method for the problem (1.1) considered in [17,18], the SDFEM is of the equal accuracy but it generates more stable numerical solution with lower computational cost for the solution of the associated discrete system.

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