

Homogenisation of parabolic/hyperbolic media

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Abstract We consider an evolutionary problem with rapidly oscillating coefficients. This causes the problem to change frequently between a parabolic and an hyperbolic state. We prove convergence of the homogenisation process in the unit square and present a numerical method to deal with approximations of the resulting equations. A numerical study finalises the contribution.

Key words: evolutionary equations, fluid-structure model, homogenisation, numerical approximation

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1 Introduction

In the present article, we discuss an academic example of a partial differential equation with highly oscillatory change of type. In real-world applications this change of type can be observed, when discussing a solid-fluid interaction model. In these kind of models, the solid is modelled by a (hyperbolic) elasticity equation and the fluid is of parabolic type.

An example of the equations to be studied is the following system of equations in the unit square $\Omega = (0, 1)^2$

$$\begin{aligned}\partial_t u - \Delta u &= f_w \text{ on } \Omega_w, \\ \partial_t^2 u - \Delta u &= f_b \text{ on } \Omega_b.\end{aligned}$$

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Ω should be thought of being a chessboard like structure with Ω_w being the white areas and Ω_b being the black areas. u satisfies natural transmission conditions on the interfaces. Our aim is to study the limit of the white and black squares' diameters tending to zero.

We shall present a convergence estimate for this homogenisation problem as well as a numerical study. Equations with change of type (ranging from elliptic to parabolic to hyperbolic) can be treated with the notion of so-called evolutionary equations, which are due to Picard [7], see also [8]. The notion of evolutionary equations is an abstract class of equations formulated in a Hilbert space setting and comprises partial differential-algebraic problems and may further be described as implicit evolution equation, hence the name 'evolutionary equations'.

More precisely, given a Hilbert space H and bounded linear operator $M_0, M_1 \in L(H)$ as well as a skew-self-adjoint operator A in H , we consider the problem of finding $U: \mathbb{R} \rightarrow H$ for some given right-hand side $F: \mathbb{R} \rightarrow H$ such that

$$(\partial_t M_0 + M_1 + A)U = F, \quad (1)$$

where ∂_t denotes the time derivative. The solution theory for this equation is set up in an exponentially weighted Hilbert space describing space-time. We shall specify the ingredients in the next section.

For evolutionary equations, a numerical framework has been developed in [4]. In particular, this numerical treatment allows for equations with change of type.

Qualitatively, problems with highly oscillatory change of type (varying in between elliptic/parabolic/hyperbolic) have been considered in [10] in a one-dimensional setting. For a higher dimensional setting of highly-oscillatory type in the context of Maxwell's equations, we refer to [11]. For a solid-fluid interaction homogenisation problem with oscillations between hyperbolic and parabolic parts we refer to [3].

A quantitative result for equations with change of type has been obtained in [1, 5]. In the latter reference, we have employed results and techniques stemming from [2] to transfer operator-norm estimates on (static) problems posed on \mathbb{R}^n to corresponding estimates for periodic time-dependent problems on the one-dimensional unit cell.

The present contribution is very much in line with the approach presented in [5]. The major difference, however, is the transference to a higher-dimensional setting.

For the sake of the argument, we restrict ourselves to two spatial dimensions. The higher-dimensional case is then adopted without further difficulties.

We shortly comment on the organisation of this paper. We start by presenting the analytical background in the next section. In this section, we shall also derive the necessary convergence estimates for the homogenisation problem.

Our numerical approach will be provided in Section 3. We conclude the article with a small case study.

2 Analytical background

In this section, we rephrase and summarise some results from [2]. The key ingredients are [2, Theorem 3.9] as well as [2, Proposition 3.16].

First of all, we properly define the operators involved. Let $\Omega = (0, 1)^2$. Then we define

$$\tilde{\text{grad}}: C_{\#}^1(\Omega) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)^2, \phi \mapsto (\partial_j \phi)_{j \in \{1,2\}},$$

where $C_{\#}^1(\Omega) := \{\phi|_{\Omega}; \phi \in C^1(\mathbb{R}^2), \phi(\cdot) = \phi(\cdot + k) \quad (k \in \mathbb{Z}^2)\}$. Note that $\tilde{\text{grad}}$ is densely defined as $C_c^1(\Omega) \subseteq C_{\#}^1(\Omega)$. We define $\text{div}_{\#} := -\tilde{\text{grad}}^*$. It is easy to see, that $C_{\#}^1(\Omega)^2 \subseteq \text{dom}(\text{div}_{\#})$ and so $\text{grad}_{\#} := -\text{div}_{\#}^*$ is a well-defined operator extending $\tilde{\text{grad}}$. Note that it can be shown that

$$\text{dom}(\text{grad}_{\#}) = H_{\#}^1(\Omega) := \{\phi|_{\Omega}; \phi \in H_{\text{loc}}^1(\mathbb{R}^2), \phi(\cdot) = \phi(\cdot + k) \quad (k \in \mathbb{Z}^2)\}.$$

Next, let $s_0, s_1: \mathbb{R}^2 \rightarrow \mathbb{C}$ be measurable, bounded, $(0, 1)^2$ -periodic functions satisfying $s_0(x) = s_0(x)^* \geq 0$ for all $x \in \mathbb{R}^2$ and

$$\rho_0 s_0(x) + \Re s_1(x) \geq c$$

for some $\rho_0 \geq 0$ and $c > 0$ and all $x \in \mathbb{R}^2$.

We define $M_0 \in L(L^2(\Omega)^3)$ by

$$M_0(\phi)_{j \in \{1,2,3\}} := \begin{pmatrix} s_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ (\phi_j)_{j \in \{2,3\}} \end{pmatrix} := \begin{pmatrix} (\Omega \ni x \mapsto s_0(x)\phi_1(x)) \\ (\phi_j)_{j \in \{2,3\}} \end{pmatrix},$$

and M_1 similarly replacing s_0 by s_1 . Note that we have

$$\rho_0 M_0 + \Re M_1 \geq c$$

in the sense of positive definiteness; furthermore M_0 is selfadjoint. A straight forward application of [7, Solution Theory] leads to the following result. We recall that ∂_t is the distributional derivative with respect to the first variable in the space

$$L_{\rho}^2(H) := \{f \in L_{\text{loc}}^2(\mathbb{R}; H); \int_{\mathbb{R}} \|f(t)\|_H^2 \exp(-2\rho t) dt < \infty\}$$

with maximal domain $H_{\rho}^1(H)$, that is,

$$\partial_t: H_{\rho}^1(H) \subseteq L_{\rho}^2(H) \rightarrow L_{\rho}^2(H), \phi \mapsto \phi'.$$

It will be obvious from the context, which ρ and which Hilbert space H is chosen. In the next theorem, we have $H = L^2(\Omega)^3$.

Theorem 1 ([7]). *Let $\rho \geq \rho_0$. Then*

$$S := \overline{\partial_t M_0 + M_1 + \begin{pmatrix} 0 & \text{div}_{\#} \\ \text{grad}_{\#} & 0 \end{pmatrix}}^{-1} \in L(L_{\rho}^2(L^2(\Omega)^3))$$

and $\|\mathcal{S}\| \leq 1/c$.

Remark 2. Note that it can be shown ([10, Remark 2.3]) that if $F \in \text{dom}(\partial_t)$, then $\mathcal{S}F \in \text{dom}(\partial_t) \cap \text{dom}\left(\begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix}\right)$. Moreover, let $F = (f, 0) \in L^2_\rho(L^2(\Omega) \oplus L^2(\Omega)^2)$. Then $U = (u, v) = \mathcal{S}F$ satisfies the following two equations

$$\begin{aligned} \partial_t s_0 u + s_1 u + \text{div}_\# v &= f \\ \partial_t v &= -\text{grad}_\# u. \end{aligned}$$

Substituting the second equation into the first one, we obtain

$$\partial_t^2 s_0 u + \partial_t s_1 u + \text{div}_\# \text{grad}_\# u = \partial_t f,$$

which is a damped wave equation, if $s_0 > 0$ everywhere. The conditions imposed on s_0 and s_1 , however, also allow for regions, where $s_0 = 0$ (or $s_0 = 0$ entirely). On these regions, the equation is a heat-type equation. If there are regions where either s_0 or s_1 vanish (but not both on the same region), the resulting equation is of mixed type. We emphasise, that transmission conditions are not necessary for the formulation of the equations but are rather a consequence of U being a solution to the equation; see also [10, Remark 3.2].

Next, we aim to study the limit behaviour of \mathcal{S}_N , which is given as \mathcal{S} but with $s_0(N\cdot)$ and $s_1(N\cdot)$ respectively replacing s_0 and s_1 . In particular, our aim is to establish the following theorem. For this, we define

$$H_\rho^k(H) := \text{dom}(\partial_t^k)$$

endowed with the graph norm of ∂_t^k acting as an operator from $L^2_\rho(H)$ into itself. It can be shown that given $\rho > \rho_0$ that ∂_t is continuously invertible in $L^2_\rho(H)$; so that $u \mapsto \|\partial_t^k u\|$ is equivalent to the graph norm on $H_\rho^k(H)$.

Theorem 3. *Let $\rho > \rho_0$. There exists $\kappa \geq 0$ such that for all $N \in \mathbb{N}$ and $f \in H_\rho^2(L^2(\Omega))$ we have*

$$\begin{aligned} & \left\| \left(\partial_t \begin{pmatrix} s_0(N\cdot) & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s_1(N\cdot) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \right. \\ & \quad \left. - \left(\partial_t \begin{pmatrix} \langle s_0 \rangle & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \langle s_1 \rangle & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \right\|_{L^2_\rho(L^2(\Omega)^3)} \\ & \leq \frac{\kappa}{N} \|\partial_t^2 f\|_{L^2_\rho(L^2(\Omega))}, \end{aligned}$$

where

$$\langle s_j \rangle := \int_\Omega s_j(x) dx \quad (j \in \{0, 1\}).$$

In order to prove this theorem, we need to introduce the Fourier–Laplace transformation: Let H be a Hilbert space. For $\phi \in C_c(\mathbb{R}; H)$ we define

$$\mathcal{L}_\rho \phi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(t) \exp(-it\xi - \rho t) dt.$$

A variant of Plancherel’s theorem yields that \mathcal{L}_ρ extends to a unitary operator from $L^2_\rho(H)$ into $L^2(H)$. A remarkable property of \mathcal{L}_ρ is that

$$\partial_t = \mathcal{L}_\rho^*(im + \rho)\mathcal{L}_\rho,$$

where m is the multiplication by argument operator in $L^2(\mathbb{R}; H)$ with maximal domain; see [6, Corollary 2.5]. Thus, applying the Fourier–Laplace transformation to the norms on either side of the inequality in Theorem 3, we deduce that it suffices to show that there exists $\kappa \geq 0$ such that for all $N \in \mathbb{N}$, $z \in \mathbb{C}_{\Re \geq \rho}$ and $f \in L^2(\Omega)$ we have

$$\begin{aligned} & \left\| \left(z \begin{pmatrix} s_0(N \cdot) & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s_1(N \cdot) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \right. \\ & \quad \left. - \left(z \begin{pmatrix} \langle s_0 \rangle & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \langle s_1 \rangle & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \right\|_{L^2(\Omega)^3} \\ & \leq \frac{\kappa}{N} \|z^2 f\|_{L^2(\Omega)}. \end{aligned} \quad (2)$$

This inequality will be shown using the results of [2]. For this we need some auxiliary statements.

Lemma 4. *The space $\text{ran}(\text{grad}_\#)$ is closed on $L^2(\Omega)^2$.*

Proof. Since Ω has continuous boundary, we get that $H^1(\Omega)$ embeds compactly into $L^2(\Omega)$. Since $\text{grad}_\# \subseteq \text{grad}$, where $\text{grad}: H^1(\Omega) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)^2$ is the distributional gradient and $\text{grad}_\#$ is closed, we obtain that $H^1_\#(\Omega)$ is compactly embedded into $L^2(\Omega)$, as well. It is now standard to show that $\text{ran}(\text{grad}_\#) \subseteq L^2(\Omega)^2$ is closed, see e.g. [9, Lemma 4.1(b)].

Using Lemma 4, we define

$$\iota: \text{ran}(\text{grad}_\#) \hookrightarrow L^2(\Omega)^2, \phi \mapsto \phi$$

and obtain that

$$\iota^*: L^2(\Omega)^2 \rightarrow \text{ran}(\text{grad}_\#)$$

is the (surjective) orthogonal projection according to the decomposition $L^2(\Omega)^2 = \ker(\text{div}_\#) \oplus \text{ran}(\text{grad}_\#)$.

Proposition 5 ([2, Proposition 3.8]). *Let $f \in L^2(\Omega)$. Then the following conditions are equivalent:*

1. $u \in \text{dom}(\text{div}_\# \text{grad}_\#)$ satisfies

$$-\text{div}_\# \text{grad}_\# u + z^2 s_0 + z s_1 u = f$$

2. $u \in \text{dom}(\text{grad}_\#)$ and $q \in \text{dom}(\text{div}_\#)$ satisfy

$$\left(\begin{pmatrix} z s_0 + s_1 & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} z^{-1} f \\ 0 \end{pmatrix}.$$

3. $u \in \text{dom}(\text{grad}_\#)$ and $q \in \text{dom}(\text{div}_\#) \cap \text{ran}(\text{grad}_\#)$ satisfy

$$\left(\begin{pmatrix} z s_0 + s_1 & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \iota \\ \iota^* \text{grad}_\# & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} z^{-1} f \\ 0 \end{pmatrix}.$$

Proof. The equivalence of 1 and 2 follows from [2, Proposition 3.8] by multiplying 1 by z^{-1} and by putting $\varepsilon = 1$, $\theta = 0$, $n = 1$, $s = z s_0 + s_1$ and $a = z^{-1}$ in [2, Proposition 3.8]. The implication from 3 to 1 follows upon realising that $\text{div}_\# \text{grad}_\# = \text{div}_\# \iota^* \text{grad}_\#$. Thus, it remains to establish that 2 is sufficient for 3. For this implication, however, note that the second equation in 2, implies that $zq \in \text{ran}(\text{grad}_\#)$ and, hence, $q \in \text{ran}(\text{grad}_\#)$. Therefore, $zq = \iota \iota^* zq = z \iota^* q$.

Next, we introduce the Floquet–Bloch or Gelfand transformation:

Definition 6. Let $N \in \mathbb{N}$, $f: \mathbb{R}^2 \rightarrow \mathbb{C}$. Then define

$$\mathcal{V}_N f(\theta, y) := \frac{1}{N} \sum_{k \in \{0, \dots, N-1\}^2} f(y+k) e^{-i\theta \cdot k} \\ (y \in [0, 1)^2, \theta \in \{2\pi k/N; k \in \{0, \dots, N-1\}^2\})$$

and for $f \in L^2(0, 1)$

$$T_N f := \frac{1}{N} f \left(\frac{\cdot}{N} \right).$$

As in [5] one can show the following result:

Theorem 7. (a) The mapping $V_N: L_\#^2((0, N)^2) \rightarrow L^2(0, 1)^{N^2}$ given by

$$f \mapsto (\mathcal{V}_N f(2\pi k/N, \cdot))_{k \in \{0, \dots, N-1\}^2}$$

is unitary, where $L_\#^2((0, N)^2)$ denotes the set of $(0, N)^2$ -periodic $L_{\text{loc}}^2(\mathbb{R}^2)$ functions endowed with the scalar product from $L^2((0, N)^2)$.

(b) The mapping $G_N := V_N T_N$ is unitary.

The mapping G_N in the previous theorem is also called the Floquet–Bloch or Gelfand transformation. With this transformation at hand, we are in the position to transform the inequality in (2) into an equivalent form such that [2, Section 2] is applicable. The reason is the following representation:

Proposition 8 ([2] and [5]). *Let $N \in \mathbb{N}$, $k \in \{0, \dots, N-1\}^2$ and $\theta := 2\pi k/N$, $f \in L^2(\Omega)$. Then we have*

$$\begin{aligned} & \left(G_N \left(\begin{pmatrix} z s_0(N \cdot) + s_1(N \cdot) & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\# \iota \\ \iota^* \operatorname{grad}_\# & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} G_N^* \right)_k \\ &= \left(\begin{pmatrix} z s_0(\cdot) + s_1(\cdot) & 0 \\ 0 & z \end{pmatrix} + \frac{1}{N} \begin{pmatrix} 0 & \operatorname{div}_\theta \iota_\theta \\ \iota_\theta^* \operatorname{grad}_\theta & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} (G_N f G_N^*)_k \\ 0 \end{pmatrix}, \end{aligned}$$

where $\operatorname{div}_\theta$ and $\operatorname{grad}_\theta$ as well as ι_θ are given as in [2, Section 3].

Proof. Let $(u, q) := \left(\begin{pmatrix} z s_0(N \cdot) + s_1(N \cdot) & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\# \iota \\ \iota^* \operatorname{grad}_\# & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix}$. By Proposition 5, we have that

$$-\operatorname{div}_\# \operatorname{grad}_\# u + z^2 s_0(\cdot) u + z s_1(\cdot) u = z f, \quad z q = -\operatorname{grad}_\# u.$$

Then, by the argument just after [2, Proposition 3.5] (use an adapted version of [2, Proposition 3.5], where the Gelfand transform used there is replaced by the discrete version introduced here), it follows that $u_k := (G_N u G_N^*)_k$ satisfies

$$-\frac{1}{N^2} \operatorname{div}_\theta \operatorname{grad}_\theta u_k + z^2 s_0(\cdot) u_k + z s_1(\cdot) u_k = z (G_N f G_N^*)_k =: z f_k.$$

Applying G_N to $z q = -\operatorname{grad}_\# u$, we obtain

$$z (G_N q)_k = -\operatorname{grad}_\theta (G_N u)_k = -\iota_\theta^* \operatorname{grad}_\theta (G_N u)_k,$$

which yields the assertion.

Now, along the lines of [2, Section 3] it is possible to show the following result, which eventually implies Theorem 3.

Theorem 9 ([2, Proof of Theorem 3.1; Eq (14)]. *There exists $\kappa \geq 0$ such that for all $N \in \mathbb{N}$, $k \in \{0, \dots, N-1\}^2$ with $\theta = 2\pi k/N$ and $f \in L^2(\Omega)$ we have*

$$\begin{aligned} & \left\| \left(\begin{pmatrix} z s_0(\cdot) + s_1(\cdot) & 0 \\ 0 & z \end{pmatrix} + \frac{1}{N} \begin{pmatrix} 0 & \operatorname{div}_\theta \iota_\theta \\ \iota_\theta \operatorname{grad}_\theta & 0 \end{pmatrix} \right)^{-1} \right. \\ & \left. - \left(\begin{pmatrix} z \langle s_0 \rangle + \langle s_1 \rangle & 0 \\ 0 & z \end{pmatrix} + \frac{1}{N} \begin{pmatrix} 0 & \operatorname{div}_\theta \iota_\theta \\ \iota_\theta \operatorname{grad}_\theta & 0 \end{pmatrix} \right)^{-1} \right\| \leq \frac{\kappa}{N} \|z^2 f\|_{L^2(\Omega)}. \end{aligned}$$

With this theorem, the assertion of Theorem 3 follows upon applying the inverse Gelfand transformation first and afterwards the inverse Fourier–Laplace transformation; see also [5, Proof of Theorem 3.10] for the precise argument.

3 Numerical method

In this whole section, we address solving the equation

$$\left(\partial_t M_0 + M_1 + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right) U = F \quad (3)$$

The analytical results (Theorem 1 and Remark 2) state that given $F \in H_\rho^1(H)$, the solution U of (3) yields $U \in L_\rho^2(H_\#^1(\Omega) \times H(\text{div}_\#, \Omega))$ such that $M_0 U \in H_\rho^1(H)$ with $H = L^2(\Omega)^3$. We will use a discontinuous Galerkin method in time and a conforming Galerkin method in space. For that let $0 = t_0 < t_1 < \dots < t_M = T$ be a mesh for the time interval $[0, T]$ using M equidistant intervals $I_m = (t_{m-1}, t_m)$ of length $\tau = t_m - t_{m-1} = \frac{T}{M}$, $m \in \{1, \dots, M\}$. The method could also be defined on a non-uniform mesh in time with the obvious changes. For the discretisation of $\bar{\Omega} = [0, 1]^2$ we use an equidistant tensor-product mesh with mesh-cells $K_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$, where $x_i = \frac{i}{N}$, $i \in \{0, \dots, N\}$ and $y_j = \frac{j}{N}$, $j \in \{0, \dots, N\}$. Again a non-equidistant tensor product mesh with different mesh-sizes in the different dimensions is also possible.

We will approximate $U = (u, v)$ using piecewise polynomials, globally discontinuous in time and piecewise polynomials, globally continuous (H^1 -conforming) in space for u and globally $H(\text{div})$ -conforming for v . Thus our discrete space is given by

$$\mathcal{U}^{h,\tau} := \left\{ U \in H_\rho([0, T]; H) : U|_{I_m} \in \mathcal{P}_q(I_m, \mathcal{V}_u(\Omega) \times \mathcal{V}_v(\Omega)), m \in \{1, \dots, M\} \right\},$$

where the spatial spaces are

$$\begin{aligned} \mathcal{V}_u(\Omega) &:= \left\{ u \in H_\#^1(\Omega) : u|_{K_{ij}} \in \mathcal{Q}_p(K_{ij}), 0 \leq i, j \leq N \right\}, \\ \mathcal{V}_v(\Omega) &:= \left\{ v \in H_\#(\text{div}, \Omega) : v|_{K_{ij}} \in \mathcal{RT}_{p-1}(K_{ij}), 0 \leq i, j \leq N \right\}. \end{aligned}$$

Here, $\mathcal{P}_q(I_m, H)$ is the space of polynomials of degree up to q on the interval I_m with values in H and $\mathcal{Q}_p(K_{ij})$ is the space of polynomials with total degree up to p on the cell $K_{ij} \subseteq \Omega$. Furthermore, $\mathcal{RT}_{p-1}(K_{ij})$ is the Raviart–Thomas space on K_{ij} , defined by

$$\mathcal{RT}_{p-1}(K_{ij}) = (\mathcal{Q}_{p-1}(K_{ij}))^n + x \mathcal{Q}_{p-1}(K_{ij}).$$

Note that

$$\begin{aligned} (\mathcal{Q}_{p-1}(K_{ij}))^n &\subset \mathcal{RT}_{p-1}(K_{ij}) \subset (\mathcal{Q}_p(K_{ij}))^n, \\ \text{div}(\mathcal{RT}_{p-1}(K_{ij})) &\subset \mathcal{Q}_{p-1}(K_{ij}) \quad \text{and} \\ \mathcal{RT}_{p-1}(K_{ij}) \cdot n|_{\partial K_{ij}} &\subset \mathcal{P}_{p-1}(\partial K_{ij}). \end{aligned}$$

Finally, the “#” denotes periodic boundary conditions. This means, that $w \in \mathcal{V}_u(\Omega)$ fulfils

$$w(0, \zeta) = w(1, \zeta), w(\zeta, 0) = w(\zeta, 1), \quad \text{for any } \zeta \in [0, 1]$$

and $w \in \mathcal{V}_v(\Omega)$ fulfils using the outer normal n on $\partial\Omega$

$$(n \cdot w)(0, \zeta) = -(n \cdot w)(1, \zeta), \quad (n \cdot w)(\zeta, 0) = -(n \cdot w)(\zeta, 1), \quad \text{for any } \zeta \in [0, 1].$$

With these notions at hand, we can now properly specify the numerical method. For any given right-hand side $F \in \mathcal{U}^{h,\tau}$ and initial condition $x_0 \in H$, find $\mathcal{U} \in \mathcal{U}^{h,\tau}$, such that for all $\Phi \in \mathcal{U}^{h,\tau}$ and $m \in \{1, 2, \dots, M\}$ it holds

$$\mathcal{Q}_m [(\partial_t M_0 + M_1 + A)\mathcal{U}, \Phi]_\rho + \langle M_0 [\mathcal{U}]_{m-1}^{x_0}, \Phi_{m-1}^+ \rangle = \mathcal{Q}_m [F, \Phi]_\rho. \quad (4)$$

Here, we denote by

$$[\mathcal{U}]_{m-1}^{x_0} := \begin{cases} \mathcal{U}(t_{m-1}+) - \mathcal{U}(t_{m-1}-), & m \in \{2, \dots, M\} \\ \mathcal{U}(t_0+) - x_0, & m = 1, \end{cases}$$

the jump at t_{m-1} , by $\Phi_{m-1}^+ := \Phi(t_{m-1}+)$ the right-sided trace and by

$$\mathcal{Q}_m [a, b]_\rho := \frac{\tau_m}{2} \sum_{i=0}^q \omega_i^m \langle a(t_{m,i}), b(t_{m,i}) \rangle$$

a right-sided weighted Gauß–Radau quadrature formula on I_m approximating

$$\langle a, b \rangle_{\rho, m} := \int_{t_{m-1}}^{t_m} \langle a(t), b(t) \rangle \exp(-2\rho(t - t_{m-1})) dt,$$

see [4] for further details.

We can cite the convergence results from [4] which were for Dirichlet boundary conditions. The proof needs only marginal modifications to hold for the periodic case too. We introduce two measures for the error. The first one measures the error in an L^∞ - L^2 sense with

$$E_{\text{sup}}^2(a) := \sup_{t \in [0, T]} \langle M_0 a(t), a(t) \rangle,$$

while the second is a discrete version of the $L_\rho^2(H)$ -norm, given by

$$E_Q^2(a) := e^{2\rho T} \sum_{m=1}^M \mathcal{Q}_m [a, a]_\rho e^{-2\rho t_{m-1}}.$$

Note that $E_Q(a) = \|a\|$ for $a \in \mathcal{U}^{h,\tau}$.

Theorem 10. *We assume for the solution U of Example (3) the regularity*

$$U \in H_\rho^1(H_\#^p(\Omega) \times H_\#^p(\Omega)^2) \cap H_\rho^{q+3}(L^2(\Omega) \times L^2(\Omega)^2)$$

as well as

$$AU \in L^2_\rho(H^p_\#(\Omega) \times H^p_\#(\Omega)^2).$$

Then we have for the error of the numerical solution $U^{h,\tau}$ of (4) with a generic constant C

$$E_{\text{sup}}^2(U - U^{h,\tau}) + E_Q^2(U - U^{h,\tau}) \leq Ce^{2\rho T}(\tau^{2(q+1)} + Th^{2p}).$$

Note that the spatial regularity is only needed in each cell K_{ij} of the spatial mesh as local interpolation error estimates are used.

4 Numerical study

All computations were done in `SOFE` (<https://github.com/SOFE-Developers/SOFE>), a finite element suite for Matlab/Octave.

For our numerical study let us assume an equidistant rectangular background mesh covering Ω with nodes $(x_i = \frac{i}{N}, y_j = \frac{j}{N})$, $i, j \in \{0, \dots, N\}$ for an even number $N \in \mathbb{N}$. This background mesh will be used in defining the oscillating coefficients.

Our rough coefficient problem is given by

$$\left(\partial_t \begin{pmatrix} \varepsilon_N & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 - \varepsilon_N & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right) U_N = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad (5)$$

where the coefficient function ε_N is defined as

$$\varepsilon_N(x, y) := \begin{cases} 1, & \exists i, j \in \mathbb{N}_0 : (x, y) \in (x_i, x_{i+1}) \times (y_j, y_{j+1}) \text{ and } i + j \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding homogenised problem is then

$$\left(\partial_t \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \text{div}_\# \\ \text{grad}_\# & 0 \end{pmatrix} \right) U_{\text{hom}} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (6)$$

The theoretical results of Sections 2 and 3 provide the following expected convergence behaviour

$$\|U_N - U_{\text{hom}}\|_{H^1_\rho(\mathbb{R}, H)} \leq CN^{-1},$$

$$E_Q(U_{\text{hom}}^{h,\tau} - U_{\text{hom}}) \leq C(h^p + \tau^{q+1}), \quad E_Q(U_N^{h,\tau} - U_N) \leq C(h^p + \tau^{q+1})$$

for smooth solutions U_{hom} and U_N . In general we cannot expect the solutions to be very smooth. Thus, for our experiments we only chose a polynomial order $p = 2$ in space and $q = 1$ in time. Setting furthermore $h = \tau = 1/(2N)$ we combine the above expected estimates and obtain

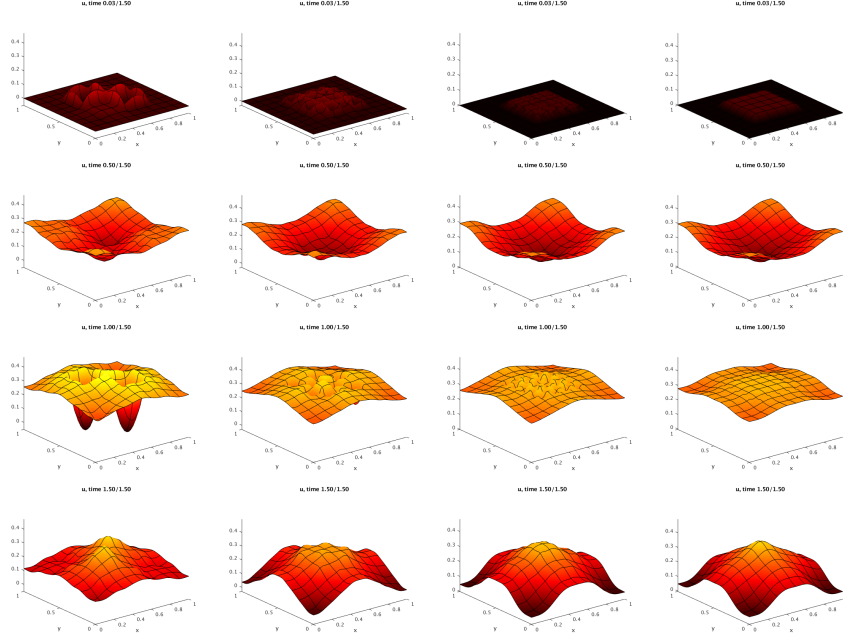


Fig. 1 Solutions U_4 , U_8 , U_{16} and U_{hom} (left to right) at $t = 0.025, 0.5, 1$ and 1.5 (top to bottom)

$$\begin{aligned} E_Q(U_N^{h,\tau} - U_{\text{hom}}) &\leq E_Q(U_N^{h,\tau} - U_N) + E_Q(U_N - U_{\text{hom}}) \\ &\leq E_Q(U_N^{h,\tau} - U_N) + C\|U_N - U\|_{H^1_0(\mathbb{R},H)} \leq CN^{-1}, \end{aligned}$$

where the second inequality comes from Sobolev's embedding theorem (see e.g. [6, Lemma 5.2])

Let us finalise the definition of our problem by setting the right-hand side

$$f(t,x,y) = \begin{cases} 1, & t \in (0,1) \text{ and } \max\{|2x-1|, |2y-1|\} \leq \frac{1}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus f is one in the time-space cube $(0,1) \times [1/4, 3/4]^2$ and otherwise zero. Figure 1 shows (numerical approximations of) the solutions U_4 , U_8 , U_{16} and U_{hom} at different times. In the first row the rough coefficients can be seen quite nicely, while the solution becomes smooth very quickly (lower rows). Furthermore, already for a very coarse background mesh of $N = 16$ the solutions U_N and U_{hom} are very similar. This visualises the homogenisation process.

In Table 1 we see the results for a simulation using polynomial degrees $p = q + 1 = 2$. As no exact solutions to (5) and (6) are known, we use reference solutions \tilde{U}_N and \tilde{U}_{hom} computed with polynomial degree $p = 3$ on a mesh with 256 cells in each space dimension and 384 cells in time dimension. The reference solution mesh is therefore twice as fine as the finest one used in the simulation.

Table 1 Convergence results for $\tilde{U}_N - U_N^{h,\tau}$ and $\tilde{U}_{\text{hom}} - U_N^{h,\tau}$ of problem (5) using $h = \tau = \frac{1}{2N}$

N	$E_{\text{sup}}(\tilde{U}_N - U_N^{h,\tau})$		$E_Q(\tilde{U}_N - U_N^{h,\tau})$		$E_{\text{sup}}(\tilde{U}_{\text{hom}} - U_N^{h,\tau})$		$E_Q(\tilde{U}_{\text{hom}} - U_N^{h,\tau})$	
2	5.046e-02		1.336e-02		7.175e-02		2.778e-02	
4	2.346e-02	1.11	6.692e-03	1.00	4.391e-02	0.71	1.969e-02	0.50
8	1.171e-02	1.00	3.165e-03	1.08	2.256e-02	0.96	8.802e-03	1.16
16	6.063e-03	0.95	1.507e-03	1.07	1.038e-02	1.12	4.186e-03	1.07
32	3.172e-03	0.93	6.633e-04	1.18	5.081e-03	1.03	2.005e-03	1.06
64	1.590e-03	1.00	3.012e-04	1.14	2.383e-03	1.09	9.445e-04	1.09

Note that we also provided the experimental orders of convergence (eoc), calculated for errors E_n and E_{2n} by

$$\text{eoc}_n = \frac{\ln \frac{E_n}{E_{2n}}}{\ln 2}.$$

We observe a first order convergence of the numerical solution $U_N^{h,\tau}$ towards U_N and towards U_{hom} . While the second result confirms the reasoning at the beginning of this section, the first directs to a non-smoothness of the solution as otherwise we would obtain a second order convergence, see Theorem 10. Considering the oscillating coefficients and discontinuous f this reduction is to be expected.

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