RESEARCH ARTICLE

WILEY

A mixed *hp* FEM for the approximation of fourth-order singularly perturbed problems on smooth domains

P. Constantinou[1] | **S. Franz[2]** | **L. Ludwig[2]** | **C. Xenophontos[1]** ⊙

[1]Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus

[2]Technische Universität Dresden, Institut für Wissenschaftliches Rechnen, Dresden 01062, Germany

Correspondence

Christos Xenophontos, Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus.
Email: xenophontos@ucy.ac.cy

We consider fourth-order singularly perturbed problems posed on smooth domains and the approximation of their solution by a *mixed hp* Finite Element Method on the so-called *Spectral Boundary Layer Mesh*. We show that the method converges uniformly, with respect to the singular perturbation parameter, at an exponential rate when the error is measured in the energy norm. Numerical examples illustrate our theoretical findings.

KEYWORDS

boundary layers, fourth-order singularly perturbed problem, mixed *hp* finite element method, spectral boundary layer mesh, uniform exponential convergence

1 | INTRODUCTION

The numerical solution of fourth-order singularly perturbed problems has only recently received attention within the research community, even though second order problems have been studied extensively (see, e.g., the books [1–3] and the references therein). As is well known, a main difficulty in these problems is the presence of *boundary layers* in the solution (for second-order problems) or in the first derivative of the solution (for fourth-order problems) [4]. The accurate approximation of the layers, independently of the singular perturbation parameter, is of great importance for the overall quality of the approximate solution to be considered reliable. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the *h* version on nonuniform, layer-adapted meshes (such as the Shishkin [5] or Bakhvalov [6] mesh), or the use of the high order *p* and *hp* versions on the so-called *Spectral Boundary Layer Mesh* [7, 8]. Regarding fourth-order singularly perturbed problems, the number of available references is scarce; some notable exceptions are [4, 9–14]. One reason for the lack of results is the fact that for fourth-order problems a C^1 approximation must be constructed, something that, until recently, was not preferred (see [15] for a C^1 construction

---

of hierarchic *hp* bases). This may be avoided if one uses, for example, the Discontinuous Galerkin FEM or a mixed FEM formulation. The latter has been used in [10] in conjunction with the *h* version of the FEM on a Shishkin mesh for the approximation of a fourth-order problem. The purpose of this article is to extend the results of [10] to the *hp* version of the FEM on the so-called *Spectral Boundary Layer mesh* (see Definition 3 ahead), applied to two-dimensional fourth-order singularly perturbed problems posed on smooth domains. We assume that the solution has a certain structure (see Proposition 2 ahead), something that has been studied in [16]. In particular, the solution is decomposed into a smooth part, a boundary layer part which has support only near the boundary of the domain and an exponentially small remainder. Using an approximation operator from [17, 18], we are able to prove uniform exponential convergence for our scheme. We also comment on the case of domains with corners.

The rest of the paper is organized as follows: in Section 2 we present the model problem, the regularity of its solution and its mixed formulation. The discretization is presented in Section 3, and in Section 4 we present our main result of robust exponential convergence in the energy norm (see Equations 5 and 8 ahead). Finally, Section [5] shows the results of numerical computations that illustrate our theoretical findings.

With $\Omega \subset \mathbb{R}^2$ a bounded open set with boundary $\partial\Omega$ and measure $|\Omega|$, we will denote by $C^k(\Omega)$ the space of continuous functions on Ω with continuous derivatives up to order k. We will use the usual Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ of functions on Ω with $0, 1, 2, \ldots, k$ generalized derivatives in $L^2(\Omega)$, equipped with the norm and seminorm $\| \cdot \|_{k,\Omega}$ and $| \cdot |_{k,\Omega}$, respectively. We will also use the spaces

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : u|_{\partial\Omega} = 0 \right\}, \ H_0^2(\Omega) = \left\{ u \in H^2(\Omega) : u|_{\partial\Omega} = 0, \ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0 \right\},$$

where $\partial u / \partial n$ denotes the normal derivative. The norm of the space $L^\infty(\Omega)$ of essentially bounded functions is denoted by $\| \cdot \|_{\infty,\Omega}$. Finally, the notation "$a \lesssim b$" means "$a \leq Cb$" with C being a generic positive constant, independent of any discretization or singular perturbation parameters.

2 | THE MODEL PROBLEM AND ITS MIXED FORMULATION

We consider the following model problem: Find u such that

$$\varepsilon^2 \Delta^2 u - b \Delta u + cu = f \text{ in } \Omega \subset \mathbb{R}^2, \tag{1}$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{2}$$

where $0 < \varepsilon \leq 1$ is a given parameter, Δ denotes the Laplacian (and Δ^2 the biharmonic) operator, $b, c > 0$ are given constants, Ω is an open bounded domain with $\partial\Omega$ an analytic curve and $f(x, y)$ is a given function, which satisfies

$$\|\nabla^n f\|_{\infty,\Omega} \lesssim n! \gamma_f^n \ \forall n = 0, 1, 2, \ldots, \tag{3}$$

for some positive constant γ_f independent of ε. Here we have used the shorthand notation

$$|\nabla^n f|^2 := \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} |D^\alpha f|^2 = \sum_{\beta_1,\ldots,\beta_n=1}^{2} |D^{\beta_1 \cdots \beta_n} f|^2,$$

with D^m denoting differentiation of order $|m|$. The variational formulation of (1)–(2) reads: Find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 \langle \Delta u, \Delta v \rangle + b \langle \nabla u, \nabla v \rangle + c \langle u, v \rangle = \langle f, v \rangle \quad \forall v \in H_0^2(\Omega), \tag{4}$$

where $\langle \cdot, \cdot \rangle$ is the usual $L^2(\Omega)$ inner product. Associated with the above problem is the *energy norm*

$$\|u\|_E^2 := \varepsilon^2 \langle \Delta u, \Delta u \rangle + b \langle \nabla u, \nabla u \rangle + c \langle u, u \rangle. \tag{5}$$

In [10], the following *mixed* formulation was presented: find $(u, w) \in H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} \varepsilon \langle \nabla u, \nabla \phi \rangle + \langle w, \phi \rangle = 0 \, \forall \, \phi \in H^1(\Omega), \\ b \langle \nabla u, \nabla \psi \rangle + c \langle u, \psi \rangle - \varepsilon \langle \nabla w, \nabla \psi \rangle = \langle f, \psi \rangle \, \forall \, \psi \in H_0^1(\Omega), \end{cases} \tag{6}$$

where (cf. [19, 20])

$$w = \varepsilon \Delta u \in H^2(\Omega).$$

The fact that $w \in H^2(\Omega)$ is a consequence of the smoothness of f and $\partial \Omega$. Define

$$B((u, w), (\psi, \phi)) := \varepsilon \langle \nabla u, \nabla \phi \rangle + \langle w, \phi \rangle + b \langle \nabla u, \nabla \psi \rangle + c \langle u, \psi \rangle - \varepsilon \langle \nabla w, \nabla \psi \rangle \tag{7}$$

and

$$\||(u, w)\||^2 := \|w\|_{0,\Omega}^2 + b \|\nabla u\|_{0,\Omega}^2 + c \|u\|_{0,\Omega}^2. \tag{8}$$

Then there holds [10],

$$\||(u, w)\||^2 = \||(u, \varepsilon \Delta u)\||^2 = \varepsilon^2 \|\Delta u\|_{0,\Omega}^2 + b \|\nabla u\|_{0,\Omega}^2 + c \|u\|_{0,\Omega}^2 = \|u\|_E^2,$$

that is, the norm given by (8) is equivalent to the energy norm (5). Moreover, in [10] it was shown that $B((\cdot, \cdot), (\cdot, \cdot))$ given by (7), is *coercive* in the norm (8), that is,

$$B((u, w), (u, w)) \geq \||(u, w)\||^2. \tag{9}$$

It is well known (see, e.g., [21]) that the solution to *second* order singularly perturbed problems may be decomposed into a smooth part and a boundary layer part, with the latter having support only in a neighborhood of the boundary $\partial \Omega$. Recently in [16], the fourth-order problem (1)–(2) was studied and a similar decomposition was derived. To describe this decomposition, following [22] we define *boundary fitted co-ordinates* (ρ, θ) in a neighborhood of the boundary as follows: Let $(X(\theta), Y(\theta)), \theta \in [0, L]$ be a parametrization of $\partial \Omega$ by arclength and let Ω_0 be a tubular neighborhood of $\partial \Omega$ in Ω. For each point $z = (x, y) \in \Omega_0$ there is a unique nearest point $z_0 \in \partial \Omega$, so with θ the arclength parameter (with counterclockwise orientation), we set $\rho = |z - z_0|$ which measures the distance from the point z to $\partial \Omega$. Explicitly,

$$\Omega_0 = \{z - \rho \vec{n}_z : z \in \partial \Omega, 0 < \rho < \rho_0 < \text{min. radius of curvature of } \partial \Omega\}, \tag{10}$$

where $\vec{n}_z$ is the outward unit normal at $z \in \partial\Omega$, and

$$x = X(\theta) - \rho Y'(\theta), y = Y(\theta) + \rho X'(\theta),$$

with $\rho \in (0, \rho_0), \theta \in (0, L)$. The determinant of the Jacobian matrix of the transformation is given by $J = 1 - \kappa(\theta)\rho$, where $\kappa(\theta)$ is the curvature of $\partial\Omega$ (see [22] for more details).

Remark 1 Since we assume that $\partial\Omega$ is a smooth (analytic) curve, we have $X^{(k)}(\theta), Y^{(k)}(\theta) \lesssim 1 \forall k = 0, 1, 2, \ldots$, as well as $J, J^{-1} \lesssim 1$. Thus for a function $v(x, y)$ defined in Ω_0, the above change of variables produces $v(x, y) = v(X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta))$ as well as

$$\frac{\partial v}{\partial x} = \frac{1}{1 - \kappa(\theta)\rho} \left\{ \frac{\partial v}{\partial \theta} X'(\theta) - \frac{\partial v}{\partial \rho} \left(Y'(\theta) + \rho X''(\theta) \right) \right\},$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 - \kappa(\theta)\rho} \left\{ \frac{\partial v}{\partial \rho} \left(X'(\theta) - \rho Y''(\theta) \right) + \frac{\partial v}{\partial \theta} Y'(\theta) \right\}.$$

This shows that the first derivatives with respect to the (physical) x, y variables are bounded by the first derivatives with respect to the ρ, θ variables.

We describe the decomposition from [16] in the following

Proposition 2 *The BVP (1)–(2) has a solution u which can be decomposed as a smooth part u^S, a boundary layer part u^{BL} and a remainder r, viz.*

$$u = u^S + \chi u^{BL} + r, \tag{11}$$

where χ is a smooth cut-off function, satisfying

$$\chi = \begin{cases} 1 & for \ 0 < \rho < \rho_0/3 \\ 0 & for \ \rho > 2\rho_0/3 \end{cases}.$$

Moreover, there exist constants $K_1, K_2, \omega, \delta > 0$, independent of ε but depending on the data, such that

$$\|D^n u^S\|_{0,\Omega} \lesssim |n|! K_1^{|n|} \ \forall n \in \mathbb{N}_0^2, \tag{12}$$

$$\left| \frac{\partial^{m+n} u^{BL}(\rho, \theta)}{\partial \rho^m \partial \theta^n} \right| \lesssim n! K_2^{m+n} \varepsilon^{1-m} e^{-\omega\rho/\varepsilon} \ \forall m, n \in \mathbb{N}, (\rho, \theta) \in \overline{\Omega}_0, \tag{13}$$

$$\|r\|_E \lesssim e^{-\delta/\varepsilon}. \tag{14}$$

Finally, there exist constants $C, K > 0$, depending only on the data, such that

$$\|D^n u\|_{0,\Omega} \leq C K^{|n|} \max \left\{ |n|^{|n|}, \varepsilon^{1-|n|} \right\} \ \forall n \in \mathbb{N}_0^2. \tag{15}$$

Equation 15 gives classical differentiability regularity, while (12)–(14) corresponds to regularity obtained through asymptotic expansions (see [16] for more details).

3 | DISCRETIZATION BY A MIXED *HP*-FEM

The discrete version of (6) reads: find $(u_N, w_N) \in V_1^N \times V_2^N \subset H_0^1(\Omega) \times H^1(\Omega)$ such that $\forall (\psi, \phi) \in V_1^N \times V_2^N$ there holds

$$\begin{cases} \varepsilon \langle \nabla u_N, \nabla \phi \rangle + \langle w_N, \phi \rangle = 0 \\ b \langle \nabla u_N, \nabla \psi \rangle + c \langle u_N, \psi \rangle - \varepsilon \langle \nabla w_N, \nabla \psi \rangle = \langle f, \psi \rangle \end{cases} \tag{16}$$

with the finite dimensional subspaces V_1^N, V_2^N defined below. Subtracting (16) from (6), we get $\forall (\psi, \phi) \in V_1^N \times V_2^N$

$$\varepsilon \langle \nabla u - \nabla u_N, \nabla \phi \rangle + \langle w - w_N, \phi \rangle + b \langle \nabla u - \nabla u_N, \nabla \psi \rangle$$
$$+ c \langle u - u_N, \psi \rangle - \varepsilon \langle \nabla w - \nabla w_N, \nabla \psi \rangle = 0,$$

or

$$B((u - u_N, w - w_N), (\psi, \phi)) = 0 \, \forall (\psi, \phi) \in V_1^N \times V_2^N. \tag{17}$$

In order to define the spaces V_1^N, V_2^N, we let $\Delta = \{\Omega_i\}_{i=1}^N$ be a mesh consisting of curvilinear quadrilaterals, subject to the usual conditions (see, e.g., [17]) and associate with each Ω_i a bijective mapping $M_i : S_{ST} \to \overline{\Omega}_i$, where $S_{ST} = [0,1]^2$ denotes the usual reference square. With $Q_p(S_{ST})$ the space of polynomials of degree p (in each variable) on S_{ST}, we define

$$\mathcal{S}^p(\Delta) = \left\{ u \in H^1(\Omega) : u|_{\Omega_i}^{\circ} M_i \in Q_p(S_{ST}), \quad i = 1, \ldots, N \right\},$$
$$\mathcal{S}_0^p(\Delta) = \mathcal{S}^p(\Delta) \cap H_0^1(\Omega).$$

We then take $V_1^N = \mathcal{S}_0^p(\Delta), V_2^N = \mathcal{S}^p(\Delta)$, with the mesh Δ chosen following the construction in [17, 18]. We denote by Δ_A a *fixed* (asymptotic) mesh consisting of curvilinear quadrilateral elements $\Omega_i, i = 1, \ldots, N_1$. These elements Ω_i are the images of the reference square S_{ST} under the element mappings $M_{A,i}, i = 1, \ldots, N_1 \in \mathbb{N}$ (the subscript A emphasizes that they correspond to the asymptotic mesh). They are assumed to satisfy conditions (M1)–(M3) of [17] in order for the space $\mathcal{S}^p(\Delta)$ to have the necessary approximation properties. Moreover, the element mappings $M_{A,i}$ are assumed to be analytic (with analytic inverse), or equivalently [17]

$$\|(M_{A,i}')^{-1}\|_{\infty,S_{ST}} \lesssim 1, \qquad \|D^\alpha M_{A,i}\|_{\infty,S_{ST}} \lesssim \alpha! \gamma^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2, \quad i = 1, \ldots, N_1,$$

for some constant $\gamma > 0$. We also assume that the elements do not have a single vertex on the boundary $\partial \Omega$ but only complete, single edges. For convenience, we number the elements along the boundary first, that is, $\Omega_i, i = 1, \ldots, N_2 < N_1$ for some $N_2 \in \mathbb{N}$. We now give the definition of the *Spectral Boundary Layer Mesh* $\Delta_{BL} = \Delta_{BL}(\kappa, p)$

> **Definition 3** (Spectral Boundary Layer mesh $\Delta_{BL}(\kappa, p)$). [18] Given parameters $\kappa > 0$, $p \in \mathbb{N}$, $\varepsilon \in (0, 1]$ and the (asymptotic) mesh Δ_A, the Spectral Boundary Layer mesh $\Delta_{BL}(\kappa, p)$ is defined as follows:
>
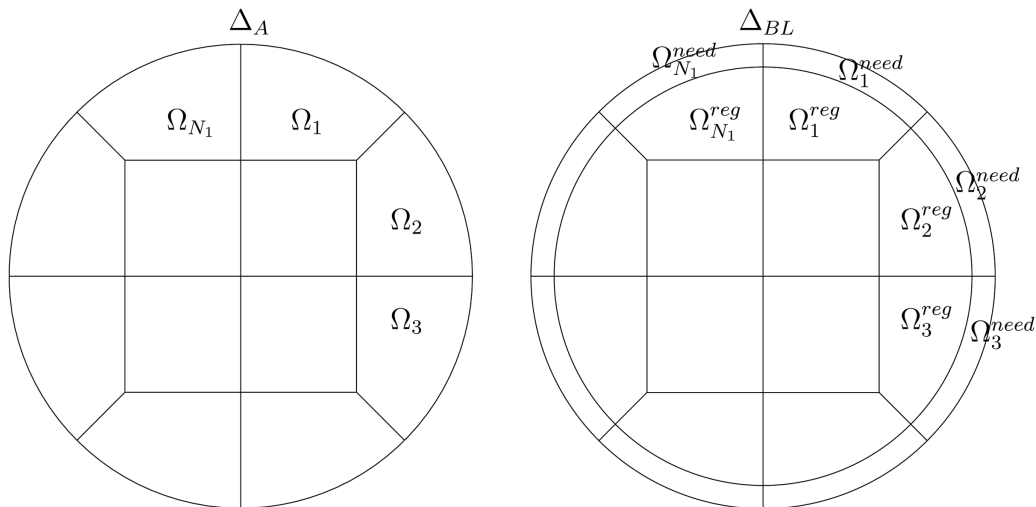> 1. If $\kappa p \varepsilon \geq 1/2$ then we are in the asymptotic range of p and we use the mesh Δ_A.

FIGURE 1 Example of an admissible mesh. Left: asymptotic mesh Δ_A. Right: boundary layer mesh Δ_{BL}

2. If $\kappa p \varepsilon < 1/2$, we need to define so-called needle elements. We do so by splitting the elements $\Omega_i, i = 1, \ldots, N_2$ into two elements Ω_i^{need} and Ω_i^{reg}. To this end, split the reference square S_{ST} into two elements

$$S^{\text{need}} = [0, \kappa p \varepsilon] \times [0, 1], \qquad S^{\text{reg}} = [\kappa p \varepsilon, 1] \times [0, 1],$$

and define the elements Ω_i^{need}, Ω_i^{reg} as the images of these two elements under the element map $M_{A,i}$ and the corresponding element maps as the concatenation of the affine maps

$$A^{\text{need}} : S_{ST} \to S^{\text{need}}, \qquad (\xi, \eta) \to (\kappa p \varepsilon \xi, \eta),$$
$$A^{\text{reg}} : S_{ST} \to S^{\text{reg}}, \qquad (\xi, \eta) \to (\kappa p \varepsilon + (1 - \kappa p \varepsilon)\xi, \eta)$$

with the element map $M_{A,i}$, that is, $M_i^{\text{need}} = M_{A,i} \circ A^{\text{need}}$ and $M_i^{\text{reg}} = M_{A,i} \circ A^{\text{reg}}$. Explicitly:

$$\Omega_i^{\text{need}} = M_{A,i}\left(S^{\text{need}}\right), \Omega_i^{\text{reg}} = M_{A,i}\left(S^{\text{reg}}\right),$$
$$M_i^{\text{need}}(\xi, \eta) = M_{A,i}\left(\kappa p \varepsilon \xi, \eta\right), M_i^{\text{reg}}(\xi, \eta) = M_{A,i}\left(\kappa p \varepsilon + (1 - \kappa p \varepsilon)\xi, \eta\right).$$

In Figure 1 we show an example of such a mesh construction on the unit circle.

In total, the mesh $\Delta_{BL}(\kappa, p)$ consists of $N = N_1 + N_2$ elements if $\kappa p \varepsilon < 1/2$. By construction, the resulting mesh

$$\Delta_{BL} = \Delta_{BL}(\kappa, p) = \left\{\Omega_1^{\text{need}}, \ldots, \Omega_{N_1}^{\text{need}}, \Omega_1^{\text{reg}}, \ldots, \Omega_{N_1}^{\text{reg}}, \Omega_{N_1+1}, \ldots, \Omega_N\right\}$$

is a regular admissible mesh in the sense of [17].

4 | ERROR ESTIMATES

Our approximation will be based on the (element-wise) Gauß-Lobatto interpolant from [[17], Prop. 3.11] (see also [21]) and its improvement in [18]. We have the following

Lemma 4 *Let (u, w) be the solution to (6) and assume that (3) holds. Then there exist constants κ_0, κ_1, C, $\beta > 0$ independent of $\varepsilon \in (0, 1]$ and $p \in \mathbb{N}$, such that the following is true: For every p and every $\kappa \in (0, \kappa_0]$ with $\kappa p \geq \kappa_1$, there exist $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p))$, $\pi_p w \in \mathcal{S}^p(\Delta_{BL}(\kappa, p))$ such that*

$$\max \left\{ \|u - \pi_p u\|_{\infty, \Omega}, \|\nabla(u - \pi_p u)\|_{\infty, \Omega}, \|w - \pi_p w\|_{\infty, \Omega}, \varepsilon^{1/2} \|\nabla(w - \pi_p w)\|_{0, \Omega} \right\} \lesssim e^{-\beta p \kappa}$$

Proof The proof is separated into two cases.

Case 1: $\kappa p \varepsilon \geq 1/2$ (asymptotic case).

In this case we use the asymptotic mesh Δ_A and u satisfies (15). Inspecting the proof of Corollary 3.5 of [18], we see that we can find $\pi_p u \in \mathcal{S}_0^p(\Delta_A)$ such that

$$\|u - \pi_p u\|_{\infty, \Omega} + \|\nabla(u - \pi_p u)\|_{\infty, \Omega} \lesssim p^2 (\ln p + 1)^2 e^{-\beta p \kappa} \tag{18}$$

(due to the fact that for u the boundary layers are in the derivative, hence we have an extra power of ε in estimate (15)). For $w = \varepsilon \Delta u$, we have

$$\|D^\alpha w\|_{0, \Omega} \lesssim \varepsilon K^{|\alpha| + 2} \max \left\{ (|\alpha| + 2)^{|\alpha| + 2}, \varepsilon^{1 - (|\alpha| + 2)} \right\} \quad \forall |\alpha| \in \mathbb{N}_0^2.$$

and by Corollary 3.5 of [18], there exists $\pi_p w \in \mathcal{S}^p(\Delta_A)$ such that

$$\|w - \pi_p w\|_{\infty, \Omega} + \varepsilon^{1/2} \|\nabla(w - \pi_p w)\|_{0, \Omega} \lesssim p^2 (\ln p + 1)^2 e^{-\beta p \kappa}. \tag{19}$$

This gives the result in the asymptotic case, once we absorb the powers of p in the exponential term and adjusting the constants.

Case 2: $\kappa p \varepsilon < 1/2$ (preasymptotic case).

In this case we use the *Spectral Boundary Layer* mesh Δ_{BL} and u is decomposed as

$$u = u^S + \chi u^{BL} + r.$$

The approximation of u^S and r is constructed as in Case 1 above (basically taken to be that of [17]) and estimates like (18) may be obtained. For u^{BL} we use the approximation of Lemma 3.4 in [18], taking advantage of the extra power of ε in the regularity estimates. Ultimately, we get $\pi_p u \in \mathcal{S}_0^p(\Delta_A)$ such that (18) holds and for $w = \varepsilon \Delta u$, a similar argument gives (19). ∎

The previous lemma allows us to measure the error between the solution (u, w) and its interpolant $(\pi_p u, \pi_p w)$. The following one allows us to measure the error between the interpolant and the finite element solution (u_N, w_N).

Lemma 5 *Assume that Proposition 2 holds and let* $(u_N, w_N) \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p)) \times \mathcal{S}^p(\Delta_{BL}(\kappa, p))$ *be the solution to (16). Then there exist polynomials* $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p))$, $\pi_p w \in \mathcal{S}^p(\Delta_{BL}(\kappa, p))$ *such that*

$$\||\left(\pi_p u - u_N, \pi_p w - w_N\right)\||^2 \lesssim e^{-\tilde{\beta}p},$$

with $\tilde{\beta} > 0$ *a constant independent of* ε *and* p

Proof Recall that the bilinear form $B\left((\cdot, \cdot), (\cdot, \cdot)\right)$, given by (7) is coercive (see (9)), hence we have, with $\psi = \pi_p u - u_N \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p))$ and $\phi = \pi_p w - w_N \in \mathcal{S}^p(\Delta_{BL}(\kappa, p))$

$$
\begin{aligned}
\||(\psi, \phi)\||^2 &\leq B\left((\pi_p u - u, \pi_p w - w), (\psi, \phi)\right) = \varepsilon\langle \nabla\left(\pi_p u - u\right), \nabla\phi\rangle + \langle \pi_p w - w, \phi\rangle \\
&\quad + b\langle \nabla\left(\pi_p u - u\right), \nabla\psi\rangle + c\langle \pi_p u - u, \psi\rangle - \varepsilon\langle \nabla\left(\pi_p w - w\right), \nabla\psi\rangle \\
&=: I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
$$

Each term is treated using Cauchy-Schwarz and Lemma 4, except for I_1 which also requires the use of an inverse inequality:

$$
\begin{aligned}
|I_1| &= |\varepsilon\langle \nabla\left(\pi_p u - u\right), \nabla\phi\rangle| \leq \varepsilon\|\nabla\left(\pi_p u - u\right)\|_{0,\Omega}\|\nabla\phi\|_{0,\Omega} \\
&\lesssim \|\nabla\left(\pi_p u - u\right)\|_{0,\Omega}\varepsilon(\kappa p\varepsilon)^{-1}p^2\|\phi\|_{0,\Omega} \lesssim pe^{-\beta p}\|\phi\|_{0,\Omega}, \\
|I_2| &= |\langle \pi_p w - w, \phi\rangle| \leq \|\pi_p w - w\|_{0,\Omega}\|\phi\|_{0,\Omega} \lesssim e^{-\beta p}\|\phi\|_{0,\Omega}, \\
|I_3| &= |b\langle \nabla\left(\pi_p u - u\right), \nabla\psi\rangle| \lesssim \|\nabla\left(\pi_p u - u\right)\|_{0,\Omega}\|\nabla\psi\|_{0,\Omega} \lesssim e^{-\beta p}\|\nabla\psi\|_{0,\Omega}, \\
|I_4| &= |c\langle \pi_p u - u, \psi\rangle| \lesssim \|\pi_p u - u\|_{0,\Omega}\|\psi\|_{0,\Omega} \lesssim e^{-\beta p}\|\psi\|_{0,\Omega}, \\
|I_5| &= |\varepsilon\langle \nabla\left(\pi_p w - w\right), \nabla\psi\rangle| \leq \varepsilon\|\nabla\left(\pi_p w - w\right)\|_{0,\Omega}\|\nabla\psi\|_{0,\Omega} \lesssim e^{-\beta p}\|\nabla\psi\|_{0,\Omega}.
\end{aligned}
$$

Hence, after absorbing the factor p into the exponential term in the estimate for I_1, we get

$$\||(\psi, \phi)\||^2 \lesssim e^{-\tilde{\beta}p}\left(\|\nabla\psi\|_{0,\Omega} + \|\phi\|_{0,\Omega} + \|\psi\|_{0,\Omega}\right) \lesssim e^{-\tilde{\beta}p}\||(\psi, \phi)\||$$

and the proof is complete. ∎

Combining Lemmas 4 and 5 we establish our main result:

Theorem 6 *Let* $(u, w) \in H_0^1(\Omega) \times H^1(\Omega)$, $(u_N, w_N) \in V_1^N \times V_2^N$ *be the solutions to (6) and (16), respectively, and assume Proposition 2 holds. Then there exists a positive constant* β, *independent of* ε *but depending on* κ, *such that*

$$\||\left(u - u_N, w - w_N\right)\|| \lesssim e^{-\beta p}.$$

Proof The triangle inequality gives

$$\||\left(u - u_N, w - w_N\right)\|| \leq \||\left(u - \pi_p u, w - \pi_p w\right)\|| + \||\left(\pi_p u - u_N, \pi_p w - w_N\right)\||$$

and we then use Lemmas 4 and 5. ∎

Remark 7 The dependence of the constant β above on κ, has been studied for one-dimensional second-order singularly perturbed problems in [8]. In the present work, we have investigated this issue computationally and we have found that the choice κ = 1 yields better results than all other choices we have tried.

Remark 8 We note that the result of the previous theorem gives robust exponential convergence of the *hp* version of the FEM on the Spectral Boundary Layer mesh when the error is measured in the (energy) norm (8). As it was pointed out in [10], this norm is not balanced in the sense that, if u^{BL}, u^{S} are layer and smooth components of the solution, respectively, then

$$|||(u^{BL}, \varepsilon \Delta u^{BL})||| = O(\varepsilon^{1/2}), \quad |||(u^{S}, \varepsilon \Delta u^{S})||| = O(1).$$

This means that as $\varepsilon \to 0$, the energy norm "does not see the layer." The convergence of our method in a stronger, balanced norm is currently under investigation.

5 | NUMERICAL RESULTS

In this section, we present the results of numerical computations for three examples. Computations were made using the finite element library $\mathbb{SOFE}$ (https://github.com/SOFE-Developers/SOFE).

Example 1 We consider a smooth domain Ω with a boundary Γ given by a curve $\gamma(\varphi)$ using polar coordinates. We consider the so called *Cranioid*-curve with

$$\gamma(\varphi) = \left(\frac{1}{4} \sin(\varphi) + \frac{1}{2}\sqrt{1 - 0.9\cos(\varphi)^2} + \frac{1}{2}\sqrt{1 - 0.7\cos(\varphi)^2} \right) \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

and $\varphi \in [0, 2\pi)$. On this domain Ω we choose $b = c = 1$ and as right-hand side $f(x, y) = 10x$. Figure 2 shows the (approximated) solutions to this problem for a rather large value of $\varepsilon = 10^{-2}$. The solutions show the expected behavior with a visible layer structure only for w. Note that the mesh consists of eight coarse and six needle, curved quadrilaterals in the boundary layer region. Here the width of the numerical layer region (and therefore of the quadrilaterals) is set to $\kappa p \varepsilon$ with $\kappa = 1$.

As the exact solution is unknown we use a numerically computed reference solution in its place. It is computed on a mesh generated by once refining the shown mesh in Figure 2 and with a polynomial degree $p = 18$ that is larger by two that the maximal one used for the simulations.

The results obtained in the energy-norm can be seen in the left picture of Figure 3. We observe a robust exponential convergence, visible as a straight decay in the semilog plot. The error curves for different values of ε lie on top of each other (different to our second example below). Regarding Remark 8 we also investigated the error component $\|w - w_h\|_{0,\Omega}$ separately. The right picture of Figure 3 shows the error curves for this part, normalized again by $|||(u, w)|||$. Obviously, this part of the error decays proportionally to $\varepsilon^{1/2}$ and exponentially in p. Thus, a balancing as indicated in Remark 8 will also give robust error measures.

Example 2 Now we choose $\Omega = (0, 1)^2, b = c = 1$ and $f(x, y)$ chosen so that the exact solution is given by
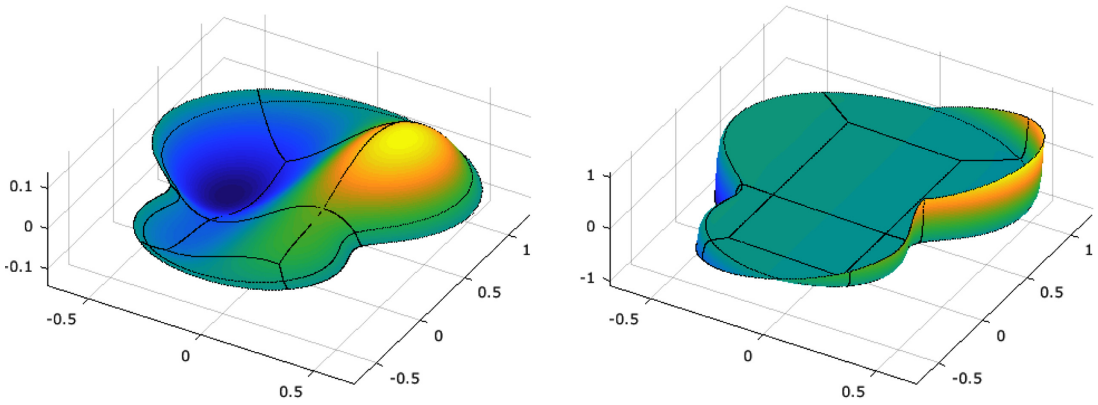
$$u(x, y) = X(x)Y(y),$$

FIGURE 2 Approximated solutions u (left) and w (right), and mesh for $\varepsilon = 10^{-2}$ [Color figure can be viewed at wileyonlinelibrary.com]

where

$$X(x) = \frac{1}{2}\left(\sin(\pi x) + \frac{\pi\varepsilon}{1 - \varepsilon^{-1/\varepsilon}}\left(\varepsilon^{-x/\varepsilon} + \varepsilon^{(x-1)/\varepsilon} - 1 - \varepsilon^{-1/\varepsilon}\right)\right),$$

$$Y(y) = \left(2y(1 - y^2) + \varepsilon\left(\ell d(1 - 2y) - 3\frac{q}{\ell} + \left(\frac{3}{\ell} - d\right)\varepsilon^{-y/\varepsilon} + \left(\frac{3}{\ell} + d\right)\varepsilon^{(y-1)/\varepsilon}\right)\right),$$

with $\ell = 1 - \varepsilon^{-1/\varepsilon}$, $q = 2 - \ell$ and $d = 1/(q - 2\varepsilon\ell)$ (cf. [10, 11]). This function has boundary layers along each side of Ω (and no corner singularities) and the *Spectral Boundary Layer* mesh consists of nine elements, as shown in Figure 4; we used $\kappa = 1$ as other values gave similar results (i.e., exponential convergence with a smaller value of β in Theorem 6). We use polynomials of degree $p = 1, \ldots, 20$ (in each variable) and take $\varepsilon = 10^{-j}, j = 3, \ldots, 9$.
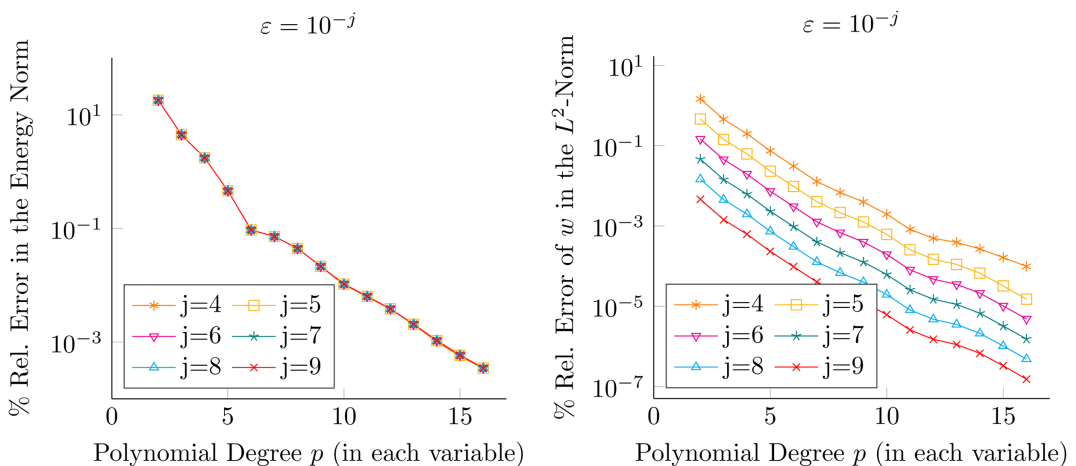

FIGURE 3 Convergence of the solutions on Cranioid-domain [Color figure can be viewed at wileyonlinelibrary.com]
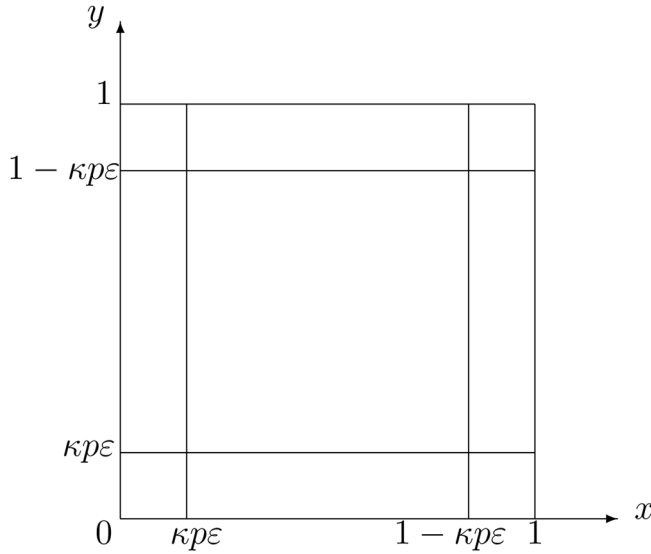
FIGURE 4 The *Spectral Boundary Layer Mesh*, when $\kappa p \varepsilon < 1/2$, used in Example 2

Figure 5 shows the percentage relative error in the energy norm versus the polynomial degree, in a semilog scale. The fact that we observe straight lines (as p is increased) verifies the exponential convergence of the method. As $\varepsilon \to 0$, the errors get smaller, which is a manifestation of the lack of balance in the norm (cf. Remark 8), even though we have robustness.

Example 3 Finally, we choose $b = c = f(x, y) = 1$ and $\Omega = (-1, 1)^2 \setminus (-1, 0)^2$, that is, an L-shaped domain. This example is meant to examine what happens when the domain is a polygon and the data does not satisfy any compatibility conditions (thus the solution contains corner singularities). In second-order singularly perturbed problems, the corner singularities have support only in the layer region [23]. For fourth-order singularly perturbed problems, this is still an open question but we note the following: the limiting problem is (essentially) a Poisson-like problem and it will feature its own (classical) corner singularities. As a result, the Spectral Boundary Layer mesh will need to include geometric refinement toward the re-entrant corner, in addition to the needle elements along the boundary.

Figure 6 shows two meshes: a mesh that includes *only* boundary layer refinement (left) and a mesh with *both* boundary layer and geometric refinement (right). The latter uses three refinements inside the layer region and two outside, with ratio 0.15.

Figure 7 shows the comparison of the two schemes. In particular, we show the percentage relative error in the energy norm versus the polynomial degree p, in a semilog scale. Since there is no exact solution available, we used a reference solution obtained with $p = 21$. Both seem to yield robust exponential convergence once ε is small enough, but the method that uses geometric refinement seems to give better results (at the expense, of course, of using much more degrees of freedom). Based on this experiment, we feel that this issue deserves further study (theoretical and computational) and we intend to do so in the near future.
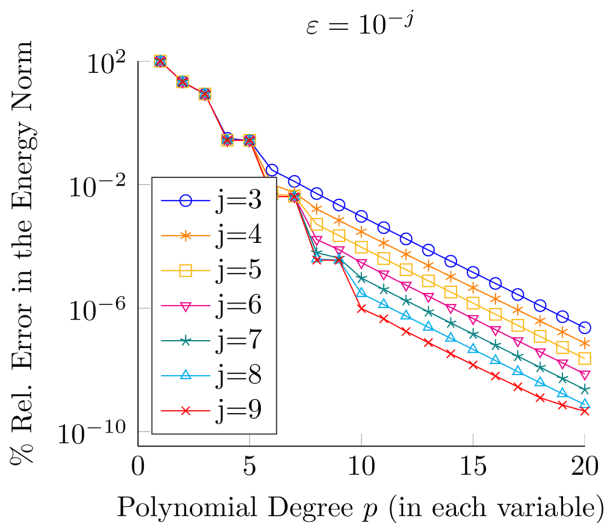
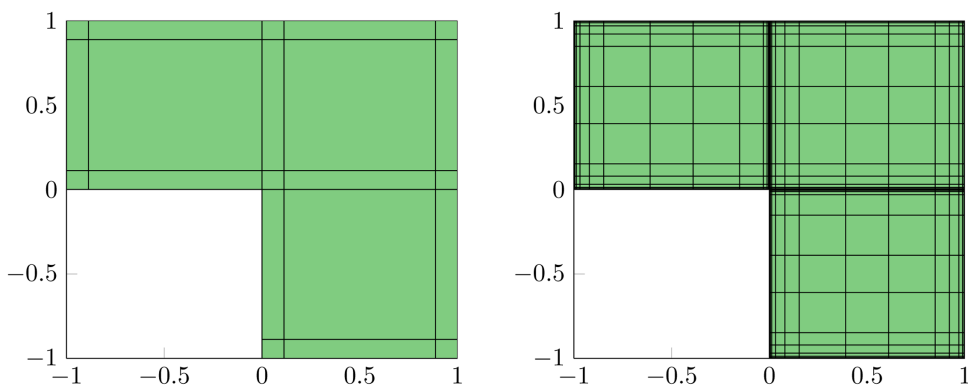**FIGURE 5** Energy norm convergence [Color figure can be viewed at wileyonlinelibrary.com]

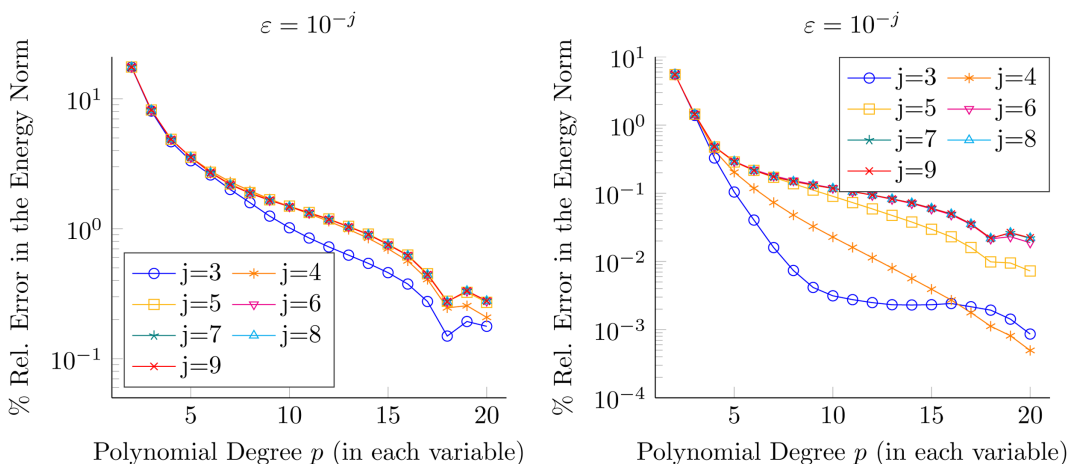**FIGURE 6** The meshes used in Example 3 [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 7 Energy norm convergence for Example 3 [Color figure can be viewed at wileyonlinelibrary.com]

ORCID

C. Xenophontos http://orcid.org/0000-0003-0862-3977

REFERENCES

[1] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Fitted numerical methods singular perturbation problems*, World Scientific, River Edge, NJ, 1996.

[2] K. W. Morton, *Numerical solution of convection-diffusion problems*, Volume 12 of Applied Mathematics and Mathematical Computation, Chapman & Hall, London, 1996.

[3] H.-G. Roos, M. Stynes, L. Tobiska. *Robust numerical methods for singularly perturbed differential equations*, Volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2008. Convection-diffusion-reaction and flow problems.

[4] G. Sun, M. Stynes, *Finite-element methods for singularly perturbed high order elliptic two point boundary value problems I: reaction-diffusion-type problems*, IMA J. Numer. Anal. vol. 15 (1005) pp. 117–139.

[5] G. I. Shishkin, *Grid approximation of singularly perturbed boundary value problems with a regular boundary layer*, Sov. J. Numer. Anal. Math. Model. vol. 4 (1989) pp. 397–417.

[6] N. S. Bakhvalov, *Towards optimization of methods for solving boundary value problems in the presence of boundary layers* (in Russian), Zh. Vychisl. Mat. Mat. Fiz. vol. 9 (1969) pp. 841–859.

[7] J. M. Melenk, C. Xenophontos, L. Oberbroeckling, *Robust exponential convergence of hp-FEM for singularly perturbed systems of reaction-diffusion equations with multiple scales*, IMA J. Numer. Anal. vol. 33 (2013) pp. 609–628.

[8] C. Schwab, M. Suri, *The p and hp versions of the finite element method for problems with boundary layers*, Math. Comput. vol. 65 (1996) pp. 1403–1429.

[9] P. Constantinou, C. Varnava and C. Xenophontos, *An hp finite element method for fourth order singularly perturbed problems*, Numer. Algorithms vol. 73 (2016) pp. 567–590.

[10] S. Franz, H.-G. Roos, *Robust error estimation in energy and balanced norms for singularly perturbed fourth order problems*, Comput. Math. Appl. vol. 72 (2016) pp. 233–247.

[11] S. Franz, H.-G. Roos, A. Wachtel, *A C^0 interior penalty method for a singularly-perturbed fourth-order elliptic problem on a layer-adapted mesh*, Numer. Methods Partial Differ. Equ. vol. 30 (2014) pp. 838–861.

[12] P. Panaseti, A. Zouvani, N. Madden, C. Xenophontos, *A C^1–conforming hp finite element method for fourth order singularly perturbed boundary value problems*, Appl. Numer. Math. vol. 104 (2016) pp. 81–97.

[13] H.-G. Roos, *A uniformly convergent discretization method for a singularly perturbed boundary value problem of the fourth order*, Review of Research, Faculty of Science, Mathematics Series, Univ. Novi Sad vol. 19 (1989) pp. 51–64.

[14] H.-G. Roos, M. Stynes, *A uniformly convergent discretization method for a fourth order singular perturbation problem*, Bonn. Math. Schr. vol. 228 (1991) pp. 30–40.

[15] L. J. F. Ferreira, M. L. Bittencourt, *Hierarachical high-order conforming C^1 bases for quadrangular and triangular finite elements*, Int. J. Numer. Methods Eng. vol. 109 (2017) pp. 936–964.

[16] P. Constantinou, *The hp Finite Element Method for fourth order singularly perturbed problems*, Doctoral Dissertation, Department of Mathematics and Statistics, University of Cyprus, (2018).

[17] J. M. Melenk, C. Schwab, *hp FEM for reaction-diffusion equations I: robust exponential convergence*, SIAM J. Numer. Anal. vol. 35 (1998) pp. 1520–1557.

[18] J. M. Melenk, C. Xenophontos, *Robust exponential convergence of hp-FEM in balanced norms for singularly perturbed reaction-diffusion equations*, Calcolco vol. 53 (2016) pp. 105–132.

[19] C. Bernardi, V. Girault, Y. Maday, *Mixed spectral element approximation of the Navier-Stokes equations in the stream-function and vorticity formulation*, IMA J. Numer. Anal. vol. 12 (1992) pp. 565–608.

[20] P. G. Ciarlet, P.-A. Raviart, *A mixed finite element method for the biharmonic equation*, Mathematical aspects of finite elements in partial differential equations, Proc. Symp., Math. Res. Center, Univ. Wisconsin-Madison, Academic Press, New York, Publication No. 33 (1974) pp. 125–145.

[21] J. M. Melenk, *hp finite element methods for singular perturbations*, Springer-Verlag, Berlin Heidelberg, 2002.

[22] D. N. Arnold, R. S. Falk, *The boundary layer for the Reissner-Mindlin plate model*, SIAM J. Numer. Anal. vol. 21 (1990) pp. 281–312.

[23] H. Han, R. B. Kellogg, *Differentiability properties of solutions to the equation $-\varepsilon^2 \Delta u + ru = f(x, y)$ in a square*, SIAM J. Math. Anal. vol. 21 (1990) pp. 394–408.