Received: 23 November 2017 | Revised: 21 February 2018 | Accepted: 28 February 2018
DOI: 10.1002/zamm.201700329

ORIGINAL PAPER

ZAMM

Resolvent estimates and numerical implementation for the homogenisation of one-dimensional periodic mixed type problems

Sebastian Franz[1] | Marcus Waurick[2]

[1]Institute of Scientific Computing, Technische Universität Dresden, 01062 Dresden, Germany
[2]Dept. of Mathematics and Statistics, University of Strathclyde, Glasgow, UK

Correspondence
Sebastian Franz, Institute of Scientific Computing, Technische Universität Dresden, 01062 Dresden, Germany.
Email: sebastian.franz@tu-dresden.de

We study a homogenisation problem for problems of mixed type in the framework of evolutionary equations. The change of type is highly oscillatory. The numerical treatment is done by a discontinuous Galerkin method in time and a continuous Galerkin method in space.

KEYWORDS
evolutionary equations, fluid-structure model, homogenisation, numerical approximation

MSC(2010)
35M10, 35B35, 35B27, 65M12, 65M60

1 | INTRODUCTION

A standard problem in engineering is the approximation of highly oscillatory coefficients by averaged ones. In fact, given a partial differential equation with variable coefficients, numerical procedures might be too involved for nowadays computing devices so that an effective model is often derived. The process of seeking effective coefficients as replacements for highly oscillatory ones is summarised under the umbrella term of homogenisation. The mathematical theory of homogenisation goes back to the late 1960s. We refer to the standard references [3,5] for a more detailed account.

Standard applications of homogenisation are elliptic, parabolic or hyperbolic divergence form equations. Only quite recently see [16], it has been noticed that for certain problems of mixed type, that is, differential equations changing their type from hyperbolic to parabolic to elliptic on different spatial domains in a highly oscillatory way, one can derive an effective model, which does not change its type anymore and consists of constant coefficients.

In [16], only a qualitative convergence statement was derived. The techniques developed in [4,7], however, suggest that the rate of convergence can be quantified. It is one main result of the present exposition – based on the rationale outlined in [4,7] – that a quantified convergence rate for problems of the type discussed in [16] can be derived. We refer to Section 3 for the precise equations.

Given the low dimensionality of the problem to be discussed in this paper, we will furthermore numerically study the partial differential equation with highly oscillatory coefficients and provide a quantitative convergence statement that for highly oscillatory coefficients the corresponding numerical solution approximates the true solution of the homogenised model. In fact, the results in [8] show that for mixed type equations one can derive a numerical scheme. It consists of a discontinuous Galerkin method in time, see e.g. [6,14,15], combined with a continuous Galerkin method in space. The framework developed in [8] for a slightly different setting can be extended to our present problem easily and approximation properties proved therein can be transferred.

In Section 3, we introduce the model under consideration and provide the desired convergence statement. In Section 4 we recall the numerical scheme derived in [8] and provide the estimate that the numerical solution of the equation with highly oscillatory coefficients approximates the solution of the effective equation in a certain controlled way. We conclude this paper with a short numerical example in Section 4.3.

Before, however, we turn to the main body of this article, we shall comment on the applications to mechanics of the studied problem in the next section.

2 | MIXED TYPE PROBLEMS AS A SPECIAL CASE OF FLUID-SOLID INTERACTION

In [2], the authors studied the stability of an 1+1-dimensional partial differential equation of mixed type, where a parabolic and a hyperbolic equation are combined. The equations can be formulated as

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, & x \in (0,1/2), t > 0, \\ \partial_t w(t,x) - \partial_x^2 w(t,x) = 0, & x \in (1/2,1), t > 0, \\ u(t,1/2) = w(t,1/2), \partial_x u(t,1/2) = \partial_x w(t,1/2), & t > 0 \end{cases}$$

subject to boundary conditions for u and w at $x = 0$ and $x = 1$, respectively, and initial conditions for u, $\partial_t u$ and w. The motivation for this model stems from a thoroughly simplified model of a problem coupling non-linear elasticity to the Navier–Stokes equations, which have applications in the study of fluids interacting with solids, see in particular [1]. In this line of simplification, the interval $(0, 1/2)$ is the solid domain, where the evolution is governed by elastic waves, where on the interval $(1/2, 1)$ the fluid is described, with the evolution governed by a (parabolic, linearised, one-dimensional) variant of Navier–Stokes equations. While being fully aware of this drastically simplified model, we shall focus on the coupling of the equations with two different types.

As it was mentioned above, in this simplified setting, we shall discuss now multiple changes of type in the domain. Initial conditions are implemented as particular right-hand sides. The problem to study now corresponds to interlacing solids and fluids with smaller and smaller proportion in the occupied space (0, 1). This kind of model has been discussed in [16, Section 4]. More precisely, for $N \in \mathbb{N}$

$$\begin{cases} \partial_t^2 u_N(t,x) - \partial_x^2 u_N(t,x) = \partial_t f(t,x), & x \in \bigcup_{j\in\{1,\dots,N\}} \left(\frac{j-1}{N}, \frac{2j-1}{2N}\right) \\ \partial_t u_N(t,x) - \partial_x^2 u_N(t,x) = f(t,x), & x \in \bigcup_{j\in\{1,\dots,N\}} \left(\frac{2j-1}{2N}, \frac{j}{N}\right) \\ \partial_x u_N(t,0) = \partial_x u_N(t,1) = 0, & \end{cases} \quad (t \in \mathbb{R}), \tag{2.1}$$

subject to homogeneous initial conditions and conditions for continuity at the junction points $\{\frac{j}{2N}\,;\, j \in \{1,\dots,2N-1\}\}$ for u_N, where f is a given source term. This problem can straightforwardly be rewritten as a first order system in the following way (using $q_N := -(1 - 1_N)\partial_x u_N$)

$$\left(\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 1_N \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 - 1_N \end{pmatrix} + \begin{pmatrix} 0 & \partial_{x,0} \\ \partial_x & 0 \end{pmatrix}\right) \begin{pmatrix} u_N \\ q_N \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{2.2}$$

as an equation on $\mathbb{R} \times (0,1)$, where 1_N denotes mutiplication by

$$a_N : x \mapsto a(Nx), \quad \text{where } a := \sum_{k\in\mathbb{Z}} \chi_{[k,k+1/2]} \in L^\infty(\mathbb{R}).$$

Here $\partial_{x,0}$ denotes the spatial derivative with a domain consisting of (weakly) differentiable functions vanishing at $x = 0$ and $x = 1$; ∂_x acts a (weak) derivative on weakly differentiable functions. By the Sobolev embedding theorem, weakly differentiable functions are actually continuous. Thus, u_N (or q_N) being in the domain of ∂_x (or $\partial_{x,0}$) particularly implies continuity at the junction points. This also serves as the implementation of the interface conditions imposed on u and w (then $N = 2$) mentioned above for the model from [2].

In the following we shall discuss (among other things) the numerical implementation of the above problem for fixed N. Moreover, we shall further analyse (and quantify) the convergence for the model of when $N \to \infty$. Furthermore, we numerically implement the limit model and compare the numerical solution for the problem with fixed N and the numerical solution of the limit model. For technical reasons (in the derivation of the quantified estimates), we use periodic boundary conditions instead of the discussed Neumann boundary conditions.

3 | RESOLVENT ESTIMATES FOR THE CONTINUOUS IN-TIME HOMOGENISATION PROBLEM

We aim to establish something, which is in spirit similar to the approach developed in [4,7]. The main ingredients for this one-dimensional situation can readily be found in [4]. The main difference between the cases treated in [4] or [7] is the underlying spatial domain. In fact, the cited work focused on $\mathbb{R}$ and $\mathbb{R}^d$ as underlying spatial domain.

In the present case, we treat a linear pde on the unit interval, instead. The homogenisation problem, we like to address is thus formulated on $\mathbb{R} \times [0,1]$ as underlying domain. Here $\mathbb{R}$ describes time and $(0,1)$ describes the spatial scale. The coefficients, we are dealing with, are highly oscillatory in the sense that we treat multiplication operators and model high oscillations using $N \in \mathbb{R}$ as a parameter describing these oscillations. We like to address the limit of $N \to \infty$ eventually.

More precisely, using the formulation in [16], one can write the problem in question as the following 2×2-block operator matrix system: For a given $F : \mathbb{R} \times (0,1) \to \mathbb{C}^2$ and $N \in \mathbb{N}$ find $U_N : \mathbb{R} \times (0,1) \to \mathbb{C}^2$ such that

$$\left(\partial_t M_0(Nx) + M_1(Nx) + \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix}\right) U_N(t,x) = F(t,x), \quad (t \in \mathbb{R}, x \in (0,1)) \tag{3.1}$$

where $\partial_\#$ is the weak derivative on $(0,1)$ with periodic boundary conditions, M_0, M_1 are 1-periodic, measurable bounded $\mathbb{C}^{2\times 2}$-valued functions with the additional property that $M_0(x) = M_0(x)^* \geq 0$ (in the sense of positive definiteness of matrices) and that there exists $\rho > 0$ and $c > 0$ such that

$$\rho\langle M_0(x)\xi, \xi\rangle_{\mathbb{C}^2} + \mathrm{Re}\langle M_1(x)\xi, \xi\rangle_{\mathbb{C}^2} \geq c\langle \xi, \xi\rangle_{\mathbb{C}^2} \quad (\xi \in \mathbb{C}^2).$$

The special case mentioned in Section 2 is realised with

$$M_0(Nx) = \begin{pmatrix} 1 & 0 \\ 0 & 1_N(x) \end{pmatrix} \text{ and } M_1(Nx) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - 1_N(x) \end{pmatrix} \quad (x \in (0,1)).$$

As equation (3.1) is formulated on $(0,1)$, the continuous Gelfand transformation used in [4] to divide the problem on the whole space has to be replaced by its discrete analogue. In the next two subsections, we will derive an estimate for the static case, which will eventually be applied to the dynamic case by going into the frequency domain.

3.1 | The static case

We start out with the discrete analogue of the Gelfand transformation as introduced in [7].

Definition 3.1. Let $N \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{C}$. Then we define

$$\mathcal{V}_N f(\theta, y) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f(y+k) e^{-i\theta k} \quad (y \in [0,1), \theta \in \{2\pi k/N; k \in \{0,\ldots,N-1\}).$$

Proposition 3.2. *The operator* $\mathcal{V}_N : L^2_\#(0,N) \to L^2(0,1)^N$ *given by*

$$f \mapsto \left(\mathcal{V}_N f(2\pi(k-1)/N, \cdot)\right)_{k\in\{1,\ldots,N\}}$$

is unitary, where $L^2_\#(0,N) := \{f \in L^2_{\mathrm{loc}}(\mathbb{R}); f(\cdot + Nk) = f \quad (k \in \mathbb{Z})\}$ *endowed with the norm of* $L^2(0,N)$.

Proof. Let $f : \mathbb{R} \to \mathbb{C}$ be bounded, continuous with $f(\cdot + Nk) = f$ for all $k \in \mathbb{Z}$. Then, we compute with $\theta_\ell = 2\pi\ell/N$

$$\begin{aligned} N\|\mathcal{V}_N f\|^2_{L^2(0,1)^N} &= \sum_{\ell=0}^{N-1} \|\mathcal{V}_N f(2\pi\ell/N, \cdot)\|^2_{L^2(0,1)} \\ &= \sum_{\ell=0}^{N-1}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} e^{-i\theta_\ell(k_1-k_2)} \int_{(0,1)} f(y+k_1)\overline{f(y+k_2)} dy. \end{aligned}$$

We shall argue next that for all $k_1, k_2 \in \{0, \dots, N-1\}$ with $k_1 \neq k_2$, we have

$$\sum_{\ell=0}^{N-1} e^{-i\theta_\ell(k_1-k_2)} = 0. \tag{3.2}$$

For this, denote $n := k_1 - k_2 \neq 0$ and consider the homomorphism

$$\varphi : \mathbb{Z}_N \to G := \{e^{-i\frac{2\pi n}{N}\ell}; \ell \in \{0, \dots, N-1\}\}$$
$$\ell \mapsto e^{-i\frac{2\pi n}{N}\ell}.$$

By the fundamental theorem on homomorphisms, $G = \operatorname{ran}(\varphi) \cong \mathbb{Z}_N / \ker(\varphi)$. In particular, $|G|$ divides N. Furthermore, since $\mathbb{Z}_N$ is cyclic, we obtain that $\mathbb{Z}_N / \ker(\varphi)$ is cyclic and thus G is cyclic. Let $z_* \in G$ generate G. Thus, $G = \{z_*^0, \dots, z_*^{k-1}\}$ are the k unique, distinct kth unit roots. In particular, we obtain for all $z \in \mathbb{C}$

$$z^k - 1 = (z - z_*^0) \cdot \ldots \cdot (z - z_*^{k-1}).$$

Expanding the right-hand side and comparing the coefficient of z^{k-1} of both sides, we deduce that

$$\sum_{z \in G} z = 0.$$

Hence,

$$\sum_{\ell=0}^{N-1} e^{-i\theta_\ell n} = \sum_{\ell=0}^{N-1} \varphi(\ell) = \frac{N}{|G|} \sum_{z \in G} z = 0,$$

which settles (3.2). Therefore, we obtain

$$N\|\mathcal{V}_N f\|^2_{L^2(0,1)^N} = \sum_{\ell=0}^{N-1} \sum_{k_1=0}^{N-1} \int_{(0,1)} f(y+k_1)\overline{f(y+k_1)}dy = N\|f\|^2_{L^2(0,N)}.$$

Moreover, note that for $\varphi = (\varphi_{k+1})_{k \in \{0,\dots,N-1\}} \in C_c(0,1)^N$, we have that the N-periodic extension of f given by $f(x) = e^{i2\pi k/N}\varphi_{k+1}(x)$ for $x \in [k, k+1)$ with $k \in \{0, \dots, N-1\}$ leads to $N\mathcal{V}_N f = \varphi$. Hence, $\mathcal{V}_N$ has dense range. Thus, $\mathcal{V}_N$ is unitary. □

We shall furthermore introduce the following unitary scaling transformation that scales a problem on $(0,1)$ onto $(0,N)$:

Definition 3.3. Let $N \in \mathbb{N}$. Then define for $f \in L^2(0,1)$

$$\mathcal{T}_N f := \tfrac{1}{\sqrt{N}} f\left(\tfrac{\cdot}{N}\right)$$

and $\mathcal{G}_N := \mathcal{V}_N \mathcal{T}_N$

Note that the scaling of the transformations $\mathcal{T}_N$ and $\mathcal{V}_N$ is chosen in a way that $\mathcal{G}_N$ becomes unitary (as a composition of two unitaries) and such that $\mathcal{G}_N$ captures precisely the 'magnitude' of oscillations in the problem (3.1).

Furthermore, we define

$$\partial_\theta : H^1_\theta(0,1) \subseteq L^2_\#(0,1) \to L^2_\#(0,1), f \mapsto f'$$

and $H^1_\theta(0,1) = \{f \in H^1(0,1); f(1) = e^{i\theta} f(0)\}$. We use $\partial_\#$ and $H^1_\#(0,1)$, if $\theta = 0$.

Proposition 3.4. *Let $N \in \mathbb{N}$. Then*

(a) *$\mathcal{T}_N \partial_\# = N\partial_{\#,N}\mathcal{T}_N$, where $\partial_{\#,N}$ is the weak derivative with periodic boundary conditions,*

(b) *$\mathcal{G}_N \partial_\# = N \operatorname{diag}\left((\partial_{\theta_k})_{k\in\{0,\dots,N-1\}}\right)\mathcal{G}_N$, where $\theta_k = 2\pi k/N$.*

(c) *For all $a \in L^\infty_\#(0,1)$ we obtain $\mathcal{G}_N a(N\cdot) = \operatorname{diag}\left((a(\cdot))_{k\in\{0,\dots,N-1\}}\right)\mathcal{G}_N$.*

Proof. The proof follows along elementary calculations. Note that for (a) and (b) it suffices to prove the assertions for smooth functions, only. □

Next, we introduce a static version of the problem in question:

Definition 3.5. Let $c > 0$ and

$$\mathcal{M}_c := \{M \in L^\infty(0,1)_\#^{2\times 2}; \mathrm{Re} M \geq c 1_{2\times 2}\},$$

where $L^\infty(0,1)_\# := \{a \in L^\infty(\mathbb{R}); a(\cdot + k) = a \quad (k \in \mathbb{Z})\}$ and $1_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the inequality in the definition of $\mathcal{M}_c$ holds in the sense of positive definiteness almost everywhere.

For all $N \in \mathbb{N}$, find $\begin{pmatrix} u_N \\ v_N \end{pmatrix} \in L^2(0,1)^2$ such that

$$\left(M(N\cdot) + \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix} \right) \begin{pmatrix} u_N \\ v_N \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{3.3}$$

for some $f, g \in L^2(0,1)^2$. Note that (3.3) is well-posed by [7, Lemma 2.5]. With the help of Proposition 3.4, we obtain an equivalent formulation of (3.3)

Corollary 3.6. *Let $N \in \mathbb{N}$. Then*

$$\begin{pmatrix} \mathcal{G}_N & 0 \\ 0 & \mathcal{G}_N \end{pmatrix} \left(M(N\cdot) + \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix} \right) \begin{pmatrix} \mathcal{G}_N & 0 \\ 0 & \mathcal{G}_N \end{pmatrix}^* = \left(\mathrm{diag}(M(\cdot))_{k\in\{0,\dots,N-1\}} + N \,\mathrm{diag}\left(\begin{pmatrix} 0 & \partial_{\theta_k} \\ \partial_{\theta_k} & 0 \end{pmatrix} \right)_{k\in\{0,\dots,N-1\}} \right).$$

As it has been demonstrated in [7, Section 3], we obtain that [7, Theorem 2.4 and Theorem 2.2] applies to the setting in [7, Equation (10)]. Here we recall the results found there for the particular case of $n = d = 1$. Note that by [7, Remark 4.6] the one-dimensional homogenised coefficient is given by the integral mean.

Theorem 3.7. *For all $N \in \mathbb{N}$ and $k \in \{0, \dots, N-1\}$, we have*

$$\left\| \left(M(\cdot) + N \begin{pmatrix} 0 & \partial_{\theta_k} \\ \partial_{\theta_k} & 0 \end{pmatrix} \right)^{-1} - \left(\int_{(0,1)} M(y)dy + N \begin{pmatrix} 0 & \partial_{\theta_k} \\ \partial_{\theta_k} & 0 \end{pmatrix} \right)^{-1} \right\|_{L(L^2(0,1)^2)} \leq \frac{1}{\pi} \left(2 \left(1 + \frac{\|M\|_\infty}{c} \right)^2 + 1 \right) \frac{1}{N}.$$

3.2 | The dynamic case

With the estimate in the latter theorem, we also obtain results for the full time-dependent problem. The strategy has been outlined in the concluding sections of [4] already. We will, however, provide the necessary notions and a corresponding estimate in this exposition, as well. For $\rho > 0$ and a Hilbert space H, we define

$$L^2_\rho(\mathbb{R}; H) := \{f : \mathbb{R} \to H; f \text{ measurable}, \int_{\mathbb{R}} \|f(t)\|_H^2 \exp(-2\rho t)dt < \infty\},$$

endowed with the obvious scalar product. Employing the usual identification of functions being equal almost everywhere, we obtain that $L^2_\rho(\mathbb{R}; H)$ is a Hilbert space. We denote by $H^1_\rho(\mathbb{R}; H)$ the first Sobolev space of weakly differentiable functions with weak derivative being representable as an element of $L^2_\rho(\mathbb{R}; H)$. Then we put

$$\partial_t : H^1_\rho(\mathbb{R}; H) \subseteq L^2_\rho(\mathbb{R}; H) \to L^2_\rho(\mathbb{R}; H), f \mapsto f'.$$

A spectral representation of ∂_t as multiplication operator is given by the *Fourier–Laplace transformation*, that is, the unitary extension of the operator $\mathcal{L}_\rho : L^2_\rho(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$ given by

$$\mathcal{L}_\rho \varphi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t) \exp(-it\xi - \rho t)dt \quad (\varphi \in C_c(\mathbb{R}; H)),$$

where $C_c(\mathbb{R}; H)$ is the space of continuous functions with compact support. The spectral representation reads as follows:

Theorem 3.8 [9, Corollary 2.5]**.** *Let $\rho > 0$. Then*

$$\partial_t = \mathcal{L}_\rho^*(im+\rho)\mathcal{L}_\rho,$$

where

$$m : \{f \in L_2(\mathbb{R};H); (t \mapsto tf(t)) \in L_2(\mathbb{R};H)\} \subseteq L_2(\mathbb{R};H) \to L_2(\mathbb{R};H)$$
$$f \mapsto (t \mapsto tf(t))$$

is the multiplication by the argument operator with maximal domain.

Next, we recall an elementary version of the well-posedness theorem for evolutionary equations, which is particularly relevant to the case studied here. For this, note that we will use the same notation for an operator acting in H and its corresponding lift as an abstract multiplication operator on $L^2_\rho(\mathbb{R};H)$.

Theorem 3.9 [12, Solution Theory], [13, Theorem 6.2.5]**.** *Let A be a skew-selfadjoint operator in H, $0 \le M_0 = M_0^*, M_1 \in L(H)$. Assume there exists $c, \rho > 0$ with*

$$\rho\langle M_0\varphi,\varphi\rangle + \mathrm{Re}\langle M_1\varphi,\varphi\rangle \ge c\langle\varphi,\varphi\rangle \quad (\varphi \in H). \tag{3.4}$$

Then the operator $\mathcal{B} := \partial_t M_0 + M_1 + A$ with $D(\mathcal{B}) = D(\partial_t) \cap D(\mathcal{A})$ is closable in $L^2_\rho(\mathbb{R};H)$. Moreover, $\mathcal{S}_\rho := \overline{\mathcal{B}}^{-1}$ is well-defined, continuous and bounded with $\|\mathcal{S}_\rho\|_{L(L^2_\rho)} \le 1/c$.

We can now state and prove the full time-dependent version of Theorem 3.7. We shall also refer to [4, Theorem 7.1] for a corresponding result with $\mathbb{R}$ instead of $(0,1)$ as underlying state space.

Theorem 3.10. *Let $\rho > 0$, $H = L^2(0,1)^2$, $M_0, M_1 \in L^\infty(0,1)^{2\times 2}_\#(\subseteq L(H))$, $M_0 = M_0^* \ge 0$. Assume there exists $c > 0$ such that*

$$\rho\langle M_0\varphi,\varphi\rangle + \mathrm{Re}\langle M_1\varphi,\varphi\rangle \ge c\langle\varphi,\varphi\rangle \quad (\varphi \in H),$$

set $A := \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix}$. Then, there exists $\kappa \ge 0$ such that for all $N \in \mathbb{N}$, we have

$$\left\| \left((\partial_t M_0(N\cdot) + M_1(N\cdot) + A)^{-1} - (\partial_t M_0^{\mathrm{av}} + M_1^{\mathrm{av}} + A)^{-1} \right) \partial_t^{-2} \right\|_{L(L^2_\rho(\mathbb{R};H))} \le \frac{\kappa}{N},$$

where $M_j^{\mathrm{av}} := \int_{(0,1)} M_j(y)dy$ for all $j \in \{0,1\}$.

Proof. Using the unitarity of the Fourier–Laplace transformation, we deduce that the claim is equivalent to showing that there exists $\kappa \ge 0$ such that for all $N \in \mathbb{N}$ and $\xi \in \mathbb{R}$:

$$\left\| \left(((i\xi+\rho)M_0(N\cdot) + M_1(N\cdot) + A)^{-1} - ((i\xi+\rho)M_0^{\mathrm{av}} + M_1^{\mathrm{av}} + A)^{-1} \right) (i\xi+\rho)^{-2} \right\|_{L(H)} \le \frac{\kappa}{N}. \tag{3.5}$$

For this, we deduce from the positive definiteness estimate imposed on M_0 and M_1 that

$$(i\xi+\rho)M_0(\cdot) + M_1(\cdot) \in \mathcal{M}_c$$

for all $\xi \in \mathbb{R}$. Hence, using Theorem 3.7 and Corollary 3.6, we obtain the existence of $\kappa \ge 0$ such that for all $N \in \mathbb{N}$ and $\xi \in \mathbb{R}$

$$\left\| \left(((i\xi+\rho)M_0(N\cdot) + M_1(N\cdot) + A)^{-1} - ((i\xi+\rho)M_0^{\mathrm{av}} + M_1^{\mathrm{av}} + A)^{-1} \right) \right\|_{L(H)} \le \frac{\kappa}{N}(1+|\xi|^2)(1+\|M_0\|_\infty + \|M_1\|_\infty)^2.$$

Thus, we conclude

$$\begin{aligned} &\left\| \left(((i\xi+\rho)M_0(N\cdot) + M_1(N\cdot) + A)^{-1} - ((i\xi+\rho)M_0^{\mathrm{av}} + M_1^{\mathrm{av}} + A)^{-1} \right) (i\xi+\rho)^{-2} \right\|_{L(H)} \\ &\le \frac{\kappa}{N}(1+|\xi|^2)(1+\|M_0\|_\infty + \|M_1\|_\infty)^2 \frac{1}{|i\xi+\rho|^2} = \frac{\kappa}{N}\frac{1+\xi^2}{\rho^2+\xi^2}(1+\|M_0\|_\infty + \|M_1\|_\infty)^2, \end{aligned}$$

which implies (3.5) and, thus, the assertion. □

4 | NUMERICAL IMPLEMENTATION

We use as numerical method a discontinuous Galerkin method in time and a continuous Galerkin method in space. For a similar problem this approach is already considered and analysed in [8]. Therefore, we will only describe the method here shortly and point to the differences in the numerical analysis.

4.1 | Numerical method

We will start by describing the method and providing a convergence result for an arbitrary problem of type (3.1), that is, we shall focus on problems of the type

$$\left(\partial_t M_0 + M_1 + A\right) U = F, \tag{4.1}$$

where $A = \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix}$ and $M_0 = M_0^* \geq 0$, M_1 are in $L^\infty(\Omega)^{2\times 2}$, which are readily extended to operators acting on $L^2_\rho(\mathbb{R}; L^2(\Omega)^2)$. Throughout, we shall assume the condition (3.4).

Let the time-interval $[0, T]$ be partitioned into subintervals $I_m = (t_{m-1}, t_m]$ of length τ_m for $m \in \{1, 2, \dots, M\}$ with $t_0 = 0$ and $t_M = T$. Let the space-interval $\Omega := (0, 1)$ also be partitioned into subintervals $J_k = [x_{k-1}, x_k]$ of length h_k for $k \in \{1, 2, \dots, K\}$ with $x_0 = 0$ and $x_K = 1$. Furthermore, let a temporal-polynomial degree $q \in \mathbb{N}$ and a spatial-polynomial degree $p \in \mathbb{N}$ be given.

Then we define the discrete space

$$\mathcal{U}^{h,\tau} := \left\{ (u_h, v_h) \in H_\rho(\mathbb{R}; H); u_h|_{I_m}, v_h|_{I_m} \in \mathcal{P}_q(I_m; V(\Omega)), m \in \{1, \dots, M\} \right\},$$

where the spatial space is

$$V(\Omega) := \left\{ v \in H^1_\#(\Omega);\ v|_{J_k} \in \mathcal{P}_p(J_k), k \in \{1, \dots, K\} \right\}.$$

Furthermore, $\mathcal{P}_q(I_m)$ is the space of polynomials of degree up to q on the interval I_m and similarly $\mathcal{P}_p(J_k)$. Thus our discrete space consists of functions that are piece-wise polynomials of degree p and continuous w.r.t. the space variable, and piece-wise polynomial of degree q and discontinuous at the time-points t_k w.r.t. time.

The method reads: For given $F \in \mathcal{U}^{h,\tau}$ and $x_0 \in H$, find $\mathcal{U} \in \mathcal{U}^{h,\tau}$, such that for all $\Phi \in \mathcal{U}^{h,\tau}$ and $m \in \{1, 2, \dots, M\}$ it holds

$$Q_m\left[(\partial_t M_0 + M_1 + A)\mathcal{U}, \Phi\right]_\rho + \langle M_0 [\![\mathcal{U}]\!]^{x_0}_{m-1}, \Phi^+_{m-1}\rangle = Q_m\left[F, \Phi\right]_\rho. \tag{4.2}$$

Here, we denote by

$$[\![\mathcal{U}]\!]^{x_0}_{m-1} := \begin{cases} \mathcal{U}(t_{m-1}+) - \mathcal{U}(t_{m-1}-), & m \in \{2, \dots, M\} \\ \mathcal{U}(t_0+) - x_0, & m = 1, \end{cases}$$

the jump at t_{m-1}, by $\Phi^+_{m-1} := \Phi(t_{m-1}+)$ and by

$$Q_m\left[a, b\right]_\rho := \frac{\tau_m}{2} \sum_{i=0}^{q} \omega_i^m \langle a(t_{m,i}), b(t_{m,i})\rangle$$

a right-sided weighted Gauß–Radau quadrature formula on I_m approximating

$$\langle a, b\rangle_{\rho,m} := \int_{t_{m-1}}^{t_m} \langle a(t), b(t)\rangle \exp(-2\rho(t - t_{m-1}))\mathrm{d}t,$$

see [8] for further details. We denote by $U_N^{h,\tau}$ the numerical solution obtained by above method (4.2) for the problem with periodic, rough coefficients and by $U^{h,\tau}$ for the homogenised data.

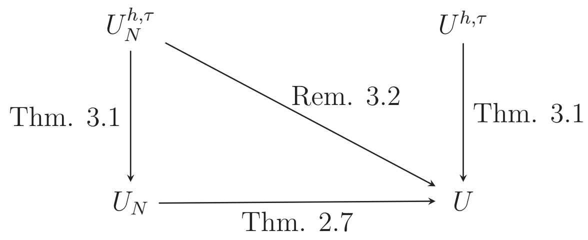

FIGURE 1 Diagram showing the connections between the different problems

4.2 | Numerical analysis

We are ready to provide the convergence result for the above method assuming enough regularity of the solution of Example (4.1) measuring the error in an L^∞-L^2 sense with

$$E^2_{\text{sup}}(a) := \sup_{t\in[0,T]} \langle M_0 a(t), a(t)\rangle_{L^2(\Omega)^2}$$

and with the discrete version of the $L^2_\rho(\mathbb{R}; H)$-norm, given by

$$E^2_Q(a) := \mathrm{e}^{2\rho T} \sum_{m=1}^{M} Q_m\,[a,a]_\rho\, \mathrm{e}^{-2\rho t_{m-1}}.$$

Theorem 4.1. *We assume for the solution $U = (u, v)$ of Example* (4.1) *the regularity*

$$U \in H^1_\rho\big(\mathbb{R}; H^p_\#(\Omega) \times H^p_\#(\Omega)\big) \cap H^{q+3}_\rho(\mathbb{R}; L^2(\Omega) \times L^2(\Omega))$$

as well as

$$AU \in H_\rho\big(\mathbb{R}; H^p_\#(\Omega) \times H^p_\#(\Omega)\big).$$

Then we have for the error of the numerical solution by (4.2) *with a generic constant C*

$$E^2_{\text{sup}}(U - U^{h,\tau}) + E^2_Q(U - U^{h,\tau}) \le C\mathrm{e}^{2\rho T}(\tau^{2(q+1)} + Th^{2p}).$$

Proof. The proof is basically identical to the one given in [8]. The only difference being the periodic boundary condition instead of the homogeneous Dirichlet condition. But all estimates are the same, as only local estimates in space are used, independent of boundary conditions. □

Considering now the problem coming from the homogenisation process, we essentially have two different problems we can approximate numerically, see Figure 1, where in addition U_N denotes the solution to the problem with rough coefficients.

Remark 4.2. Following the diagram in Figure 1, we have by the Theorems 3.10 and 4.1 for a suitable choice of polynomial degrees $p = q + 1 \ge 1$ and meshwidths $\tau = c_1 h = \frac{c_2}{N}$, $c_1, c_2 > 0$ the convergence result

$$E_Q\big(U_N^{h,\tau} - U\big) \le E_Q\big(U_N^{h,\tau} - U_N\big) + E_Q(U_N - U) \le E_Q\big(U_N^{h,\tau} - U_N\big) + C\|U_N - U\|_{H^1_\rho(\mathbb{R},H)} \le CN^{-1},$$

where the second inequality comes from Sobolev's embedding theorem (see e.g. [9, Lemma 5.2]) and the final one from applying Theorems 3.10 and 4.1. Note that for this estimate to hold we have to impose suitable regularity in time for the right-hand side in (3.1) (or (4.1)).

4.3 | Numerical example

We shall revisit the example mentioned in Section 2 in the following. Another interpretation of the model is a case of one-dimensional Maxwell's equations with eddy current type approximation on the parabolic parts, see [11] for a 3-dimensional

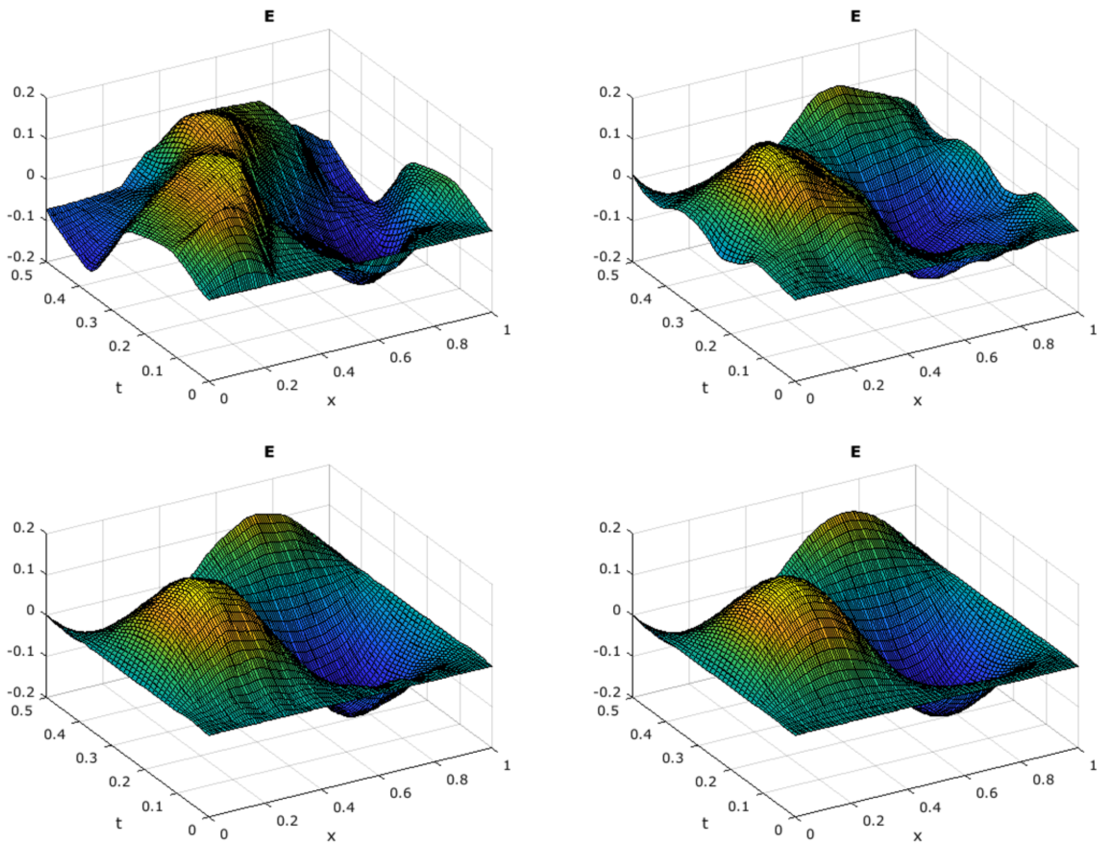

FIGURE 2 Computed solutions E_N for $N \in \{4, 8, 16\}$ and E from top left to bottom right

setting. To also stress this way of interpreting the equations studied, we rename the coefficients and functions in the following. Let $N \in \mathbb{N}$ be even and with

$$\varepsilon_N(x) := \begin{cases} 1, & \exists i \in \mathbb{N}_0 : x \in \left[\frac{2i}{N}, \frac{2i+1}{N}\right) \\ 0, & otherwise \end{cases}, \quad \sigma_N(x) := 1 - \varepsilon_N(x)$$

we consider the rough-coefficient problem for $U_N = (E_N, H_N)$

$$\left(\partial_t \begin{pmatrix} \varepsilon_N & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \sigma_N & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix}\right) \begin{pmatrix} E_N \\ H_N \end{pmatrix} = \begin{pmatrix} J \\ K \end{pmatrix} \tag{4.3}$$

and the homogenised problem for $U = (E, H)$

$$\left(\partial_t \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_\# \\ \partial_\# & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} J \\ K \end{pmatrix}, \tag{4.4}$$

where $J(t, x) = \sin(2\pi x) \cdot \min\{1, 10t\}$ and $K(t, x) = 0$ for all $t > 0, x \in [0, 1]$. For our numerical experiment we use the Matlab/Octave software SOFE.[10] The exact solutions are unknown. Therefore, we use reference solutions computed on a very fine grid and higher polynomial degree in the computation of the errors. Figures 2 and 3 present computed pictures of the solutions for rough coefficients ($N \in \{4, 8, 16\}$) and the homogenised solutions. It is clearly visible that the homogenised data will be the limiting case.

In Table 1 we present the simulation results of $U_N^{h,\tau} = (E_N^{h,\tau}, H_N^{h,\tau})$ for $h = 1/K$, $\tau = 1/M$ and $M = 2K = 8N$ and polynomial degrees $p = q + 1 = 2$. In the second and third column we see almost second order convergence of $U_N^{h,\tau}$ towards $U_N = (E_N, H_N)$ in accordance with Theorem 4.1, while in the last two columns we observe first order convergence of $U_N^{h,\tau}$ towards $U = (E, H)$ in accordance with Remark 4.2.

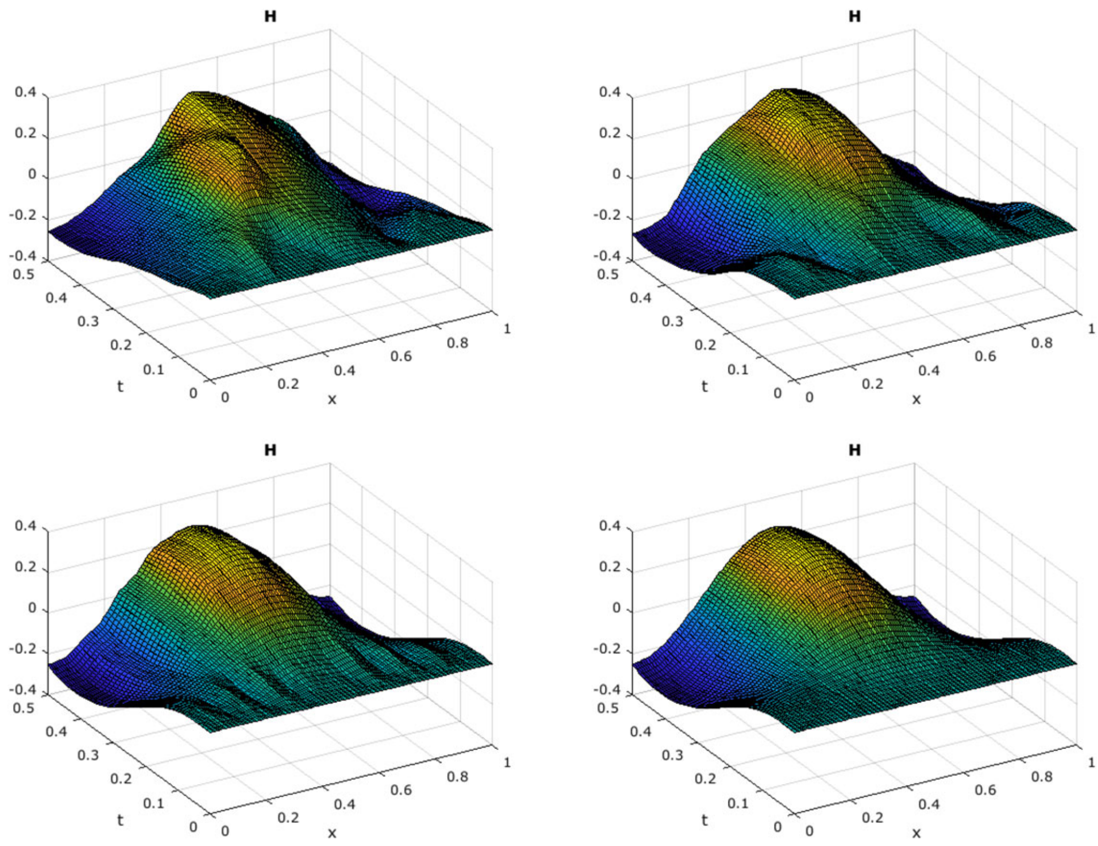

FIGURE 3 Computed solutions H_N for $N \in \{4, 8, 16\}$ and H from top left to bottom right

TABLE 1 Convergence results for $U_N - U_N^h$ and $U - U_N^h$ of problem (4.3)

n	$E_{\sup}(U_N - U_N^{h,\tau})$		$E_Q(U_N - U_N^{h,\tau})$		$E_{\sup}(U - U_N^{h,\tau})$		$E_Q(U - U_N^{h,\tau})$	
4	2.857e-03		1.117e-03		1.381e-01		3.683e-02	
8	9.490e-04	1.59	3.623e-04	1.62	3.418e-02	2.01	1.297e-02	1.51
16	2.802e-04	1.76	1.151e-04	1.65	1.328e-02	1.36	4.463e-03	1.54
32	8.611e-05	1.70	3.713e-05	1.63	5.890e-03	1.17	2.039e-03	1.13
64	2.306e-05	1.90	9.136e-06	2.02	2.802e-03	1.07	9.983e-04	1.03

ACKNOWLEDGEMENTS

The authors wish to thank Shane Cooper for useful discussion on the subject and in particular on the discrete version of the Gelfand transformation presented here.

REFERENCES

[1] G. Avalos, R. Triggiani, Mathematical analysis of PDE systems which govern fluid-structure interactive phenomena, *Bol. Soc. Parana. Mat.* **2007**, *25*, 17.

[2] C. Batty, L. Paunonen, D. Seifert, Optimal energy decay in a one-dimensional coupled wave-heat system, *J. Evol. Equ.* **2016**, *16*, 649.

[3] A. Bensoussan, J. L. Lions, G. Papanicolaou, *Asymptotic Analysis for Periodic Structures, Studies in Mathematics and its Applications*, Vol. *5*, North-Holland Publishing Co., Amsterdam-New York **1978**.

[4] K. Cherednichenko, M. Waurick, Resolvent estimates in homogenisation of periodic problems of fractional elasticity, *J. Differ. Equ.* **2018**, *64*, 3811.

[5] D. Cioranescu, P. Donato, *An Introduction to Homogenization*, Oxford Lecture Series in Mathematics and its Applications, Vol. *17*, The Clarendon Press, Oxford University Press, New York **1999**.

[6] B. Cockburn, G. E. Karniadakis, C. W. Shu, *The Development of Discontinuous Galerkin Methods*, Springer Berlin Heidelberg, Berlin, Heidelberg **2000**, 2000, pp. 3–50.

[7] S. Cooper, M. Waurick, Fibre homogenisation, Tech. rep., U Bath, **2017**, arXiv:1706.00645.

[8] S. Franz, S. Trostorff, M. Waurick, Numerical methods for changing type systems, *IMA J. Numer. Anal.* **2018**, http://doi.org/10.1093/imanum/dry007.

[9] A. Kalauch, R. Picard, S. Siegmund, S. Trostorff, M. Waurick, A Hilbert Space Perspective on Ordinary Differential Equations with Memory Term, *J. Dyn. Differ. Equ.* **2014**, *26*, 369.

[10] L. Ludwig, $\mathbb{SOFE}$, https://github.com/SOFE-Developers/SOFE.

[11] D. Pauly, R. Picard, A note on the justification of the eddy current model in electrodynamics, *Math. Methods Appl. Sci.* **2017**, *40*, 7104.

[12] R. Picard, A structural observation for linear material laws in classical mathematical physics, *Math. Methods Appl. Sci.* **2009**, *32*, 1768.

[13] R. Picard, D. McGhee, *Partial Differential Equations: A unified Hilbert Space Approach*, Expositions in Mathematics, Vol. *55*, DeGruyter, Berlin **2011**.

[14] W. H. Reed, T. R. Hill, *Triangular Mesh Methods for the Neutron Transport Equation*, Submitted to American Nuclear Society Topical Meeting on Mathematical Models and Computational Techniques for Analysis of Nuclear Systems, Los Alamos Laboratory **1973**.

[15] B. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations*, Society for Industrial and Applied Mathematics **2008**.

[16] M. Waurick, Stabilization via Homogenization, *Appl. Math. Lett.* **2016**, *60*, 101.

How to cite this article: Franz S, Waurick M. Resolvent estimates and numerical implementation for the homogenisation of one-dimensional periodic mixed type problems. *Z Angew Math Mech*. 2018;98:1284–1294. https://doi.org/10.1002/zamm.201700329.