Magnetic susceptibilities of diluted magnetic semiconductors and anomalous Hall-voltage noise

C. Timm,[1,*] F. von Oppen,[1] and F. Höfling[1,2]
[1]*Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany*
[2]*Hahn-Meitner-Institut, Glienicker Straße 100, D-14109 Berlin, Germany*

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The carrier-spin and impurity-spin densities in diluted magnetic semiconductors are considered using a semiclassical approach. Equations of motions for the spin densities and the carrier-spin current density in the paramagnetic phase are derived, exhibiting their coupled diffusive dynamics. The dynamical spin susceptibilities are obtained from these equations. The theory holds for *p*-type and *n*-type semiconductors doped with magnetic ions of arbitrary spin quantum number. Spin-orbit coupling in the valence band is shown to lead to anisotropic spin diffusion and to a suppression of the Curie temperature in *p*-type materials. As an application we derive the Hall-voltage noise in the paramagnetic phase. This quantity is critically enhanced close to the Curie temperature due to the contribution from the anomalous Hall effect.

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I. INTRODUCTION

In recent years, a lot of progress has been made in the physics of diluted magnetic semiconductors (DMS), in particular in III-V materials doped with manganese. In the best studied material (Ga,Mn)As, ferromagnetic transition temperatures around 160 K have been achieved.[1–3] On the theoretical side, a Zener model based on valence-band holes exchange coupled to local impurity spins is very successful in describing this material, at least in the metallic regime.[4–8] In (Ga,Mn)As manganese acts as an acceptor and introduces localized spins $S=5/2$ due to its half-filled *d* shell. The material is *p* type but partly compensated, probably due to arsenic antisites[9,10] and manganese interstitials.[11] In group-IV semiconductors[12] manganese plays a similar role. On the other hand, in II-VI materials manganese introduces a spin but is isovalent with the host cations.

It has also been realized that disorder is crucial for the understanding of the properties of DMS, even in the metallic regime.[13–17] There are two main scattering mechanisms: disorder scattering due to the Coulomb potential of charged donors and acceptors and spin-exchange scattering off randomly distributed impurity spins. The Coulomb interaction is the dominant contribution to disorder. This is due to compensation, which leads to a lower hole concentration and thus on the one hand to the presence of charged defects of either sign and on the other to less effective electronic screening. Due to the large Coulomb interactions, the defects are probably incorporated during growth in partially correlated positions—oppositely charged donors and acceptors prefer to sit on nearby sites—and these correlations may increase with annealing.[15,16] In Ref. 15 it was shown that equilibration of defects during growth or annealing leads to an enormous reduction of the typical width $(\langle V^2\rangle-\langle V\rangle^2)^{1/2}$ of the disorder potential *V* and to a very short correlation length of *V*, of the order of the lattice constant. *Ionic* screening is thus very effective, whereas electronic screening is not. However, the width of the disorder potential is still roughly of the same order as the Fermi energy so that it cannot be neglected.

Since the correlation length is so short, a description in terms of a *δ-function correlated* disorder potential is reasonable. In this approximation, a scattered carrier tends to lose all its momentum information. This allows for a relatively simple description of the scattering in the semiclassical Boltzmann approach.[18] The spin-exchange scattering, though typically weaker than the Coulomb scattering, is expected to become important close to the Curie temperature T_c, where spin fluctuations are enhanced. A systematic study of the effect of both types of scattering on the linear response of DMS and in particular on transport would be desirable. For example, the resistivity ρ of (Ga,Mn)As shows a maximum or at least a shoulder at T_c,[9,19–22] whereas the standard Fisher-Langer theory[23] for fluctuation corrections to the resistivity in ferromagnetic metals predicts an infinite *derivative* of ρ at T_c. The origin of this weak critical behavior is that the resistivity is dominated by scattering events with large momentum transfers $q\sim 2k_F$, where k_F is the Fermi momentum. By contrast, the magnetic susceptibility $\chi(\mathbf{q})$ of ferromagnetic metals, of Ornstein-Zernicke form,[24,25] diverges only at $\mathbf{q}=0$.

As a step towards a comprehensive theory of disorder effects on linear response and transport in DMS, we present a semiclassical theory for the paramagnetic phase of DMS in the metallic regime. Starting from the Zener model[4–7] and semiclassical Boltzmann equations, hydrodynamic equations of motion for the carrier- and impurity-spin magnetizations are derived in Sec. II, including Coulomb scattering and spin-exchange scattering off magnetic impurities. Because of the semiclassical approach, these equations hold for small momenta $\mathbf{q}$ and frequencies ω. The theory is rather general in that it applies to both the conduction and the valence band, III-V, II-VI, and group-IV host semiconductors, and impurities with general spin *S*. From the equations of motion, the dynamical spin susceptibilities of carriers and impurities are derived for small $\mathbf{q}$ and ω. The resulting semiclassical susceptibility is not of Ornstein-Zernicke form. However, this form is presumably restored by quantum effects for *q* of the order of k_F. The semiclassical results exhibit the detailed dependence on the various sources of scattering. We find significant differences between the conduction-band (n-type) and valence-band (p-type) cases due to the pronounced spin-orbit coupling in the latter. For example, spin diffusion in the valence band is anisotropic. On the other hand, we show that semiclassically Berry-phase effects[26,27] are absent from the linear susceptibility even in the valence-band case.

It would be interesting to study the effect of spin fluctuations on the electrical conductivity close to T_c in DMS.[23] This requires the inclusion of quantum effects at the scale of k_F and thus goes beyond the Boltzmann approach. The present theory should be a good starting point for this generalization.

We briefly comment on related work. Sinova *et al.*[28] consider the damping of spin waves in the *ferromagnetic* phase in the limit $\mathbf{q}=0$ within a Green-function approach. Disorder scattering is incorporated by assuming a constant nonzero quasiparticle lifetime. Galitski *et al.*[29] derive the local dynamical spin susceptibility close to T_c for the strongly localized regime, opposite to the case of weak disorder scattering considered here. In the strongly localized case the system can be mapped onto a disordered ferromagnetic Heisenberg model and Griffiths-McCoy singularities are important above T_c.[29] Qi and Zhang[30] consider spin diffusion in nonmagnetic materials within the Boltzmann approach. The present work goes beyond Ref. 30 in that we derive the coupled dynamics of carrier and impurity spins in DMS, consider both conduction and valence bands explicitly, and derive the dynamical susceptibility.

As an application we derive the fluctuations of the anomalous Hall voltage in the *paramagnetic* phase in Sec. III. In the absence of an external magnetic field the average anomalous Hall voltage is zero since the average magnetization vanishes. However, fluctuations of the magnetization lead to nonzero Hall-voltage *noise*. Three mechanisms of the anomalous Hall effect (AHE) are discussed in the literature: *skew scattering*[31] and *side-jump scattering*[32] rely on the imbalance of scattering to the right and to the left due to spin-orbit coupling. On the other hand, *Berry-phase effects*[27] lead to an AHE in the presence of spin-orbit coupling even without scattering. Since Jungwirth *et al.*[26] show that the latter contribution can explain the experimental results for DMS in the ferromagnetic phase, we also assume this mechanism.

II. SEMICLASSICAL THEORY

In this section we present the semiclassical theory for the linear response of the carrier- and impurity-spin magnetizations in DMS in the paramagnetic phase. We first derive hydrodynamic equations of motion for these magnetizations and for the carrier magnetization current. (Some details are given in Appendixes A and B. In Appendix C we show that Berry-phase corrections are absent from the equations of motion.) We solve these equations to obtain the spin susceptibility. The derivation is carried through for both the conduction and the valence band, and for arbitrary impurity spin S. We use $\hbar=k_B=1$.

A. Hydrodynamic equations, conduction band

We start with the simpler case of conduction-band electrons exchange coupled to impurity spins. Spin-orbit effects can be neglected here since the conduction band has mainly s-orbital character. This description is appropriate for n-type DMS. Ferromagnetism in n-type DMS is hard to achieve due to the small exchange interaction between electron and impurity spins and is restricted to very low temperatures.[33] We assume a spherically symmetric band $\epsilon_{\mathbf{p}}$ to avoid inessential complications.

We first briefly motivate the Boltzmann equations for the electron density $n_{\mathbf{p}\sigma}(\mathbf{r})$, where $\sigma=\pm 1/2$ is the spin orientation, and for the occupation fraction f_m of impurity spins with quantum number m of S^z. The Hamiltonian reads

$$H=H_{\mathrm{kin}}+J\int d^3r\,\mathbf{m}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})+g_{\mathrm{e}}\mu_{\mathrm{B}}\int d^3r\,\mathbf{m}(\mathbf{r})\cdot\mathbf{B}_{\mathrm{e}}^{\mathrm{ext}} +g_{\mathrm{i}}\mu_{\mathrm{B}}\int d^3r\,\mathbf{M}(\mathbf{r})\cdot\mathbf{B}_{\mathrm{i}}^{\mathrm{ext}}, \tag{1}$$

where $\mathbf{m}$ and $\mathbf{M}$ are the electron- and impurity-spin densities (oriented oppositely to the magnetizations), respectively, and their coupling is described by the exchange integral $J=50\pm 5\ \mathrm{meV\,nm}^3$.[19] $J>0$ ($J<0$) corresponds to antiferromagnetic (ferromagnetic) coupling. We have introduced two distinct external magnetic fields $\mathbf{B}_{\mathrm{e}}^{\mathrm{ext}}$ and $\mathbf{B}_{\mathrm{i}}^{\mathrm{ext}}$ acting on electron and impurity spins, respectively, in order to obtain the linear response of each species separately, which will prove useful in Sec. III.

The exchange term is decoupled at the mean-field level. We can restrict ourselves to collinear spin configurations since the paramagnetic susceptibility is proportional to the unit matrix in our spherical model. We choose the magnetization direction as the z axis. The mean-field Hamiltonian of the electrons and the impurities is then

$$H_{\mathrm{e}}=H_{\mathrm{kin}}+g_{\mathrm{e}}\mu_{\mathrm{B}}\int d^3r\,m(\mathbf{r})\,B_{\mathrm{e}}, \tag{2}$$

$$H_{\mathrm{i}}=g_{\mathrm{i}}\mu_{\mathrm{B}}\int d^3r\,M(\mathbf{r})\,B_{\mathrm{i}}, \tag{3}$$

respectively. In terms of the spin magnetizations $\mu_{\mathrm{e}}=-g_{\mathrm{e}}\mu_{\mathrm{B}}\langle m\rangle$, $\mu_{\mathrm{i}}=-g_{\mathrm{i}}\mu_{\mathrm{B}}\langle M\rangle$, the effective fields read

$$B_{\mathrm{e}}=B_{\mathrm{e}}^{\mathrm{ext}}-\frac{J}{g_{\mathrm{e}}g_{\mathrm{i}}\mu_{\mathrm{B}}^2}\,\mu_{\mathrm{i}}, \tag{4}$$

$$B_{\mathrm{i}}=B_{\mathrm{i}}^{\mathrm{ext}}-\frac{J}{g_{\mathrm{e}}g_{\mathrm{i}}\mu_{\mathrm{B}}^2}\,\mu_{\mathrm{e}}. \tag{5}$$

The single-particle energy of an electron with momentum $\mathbf{p}$ and spin $\sigma=\pm 1/2$ is $E^{\mathrm{e}}_{\mathbf{p}\sigma}=\epsilon_{\mathbf{p}}+g_{\mathrm{e}}\mu_{\mathrm{B}}\sigma B_{\mathrm{e}}$. The energy of an impurity spin with magnetic quantum number m is $E^{\mathrm{i}}_m=g_{\mathrm{i}}\mu_{\mathrm{B}}mB_{\mathrm{i}}$. In the absence of scattering, the semiclassical equation of motion for the electron density $n_{\mathbf{p}\sigma}(\mathbf{r})$ is given by the Poisson bracket

$$\partial_t n_{\mathbf{p}\sigma} = -\{n_{\mathbf{p}\sigma}, E^{\rm e}_{\mathbf{p}\sigma}\} = -\mathbf{F}_\sigma \cdot \boldsymbol{\nabla}_{\mathbf{p}} n_{\mathbf{p}\sigma} - \mathbf{v}_{\mathbf{p}} \boldsymbol{\nabla}_{\mathbf{r}} n_{\mathbf{p}\sigma} \tag{6}$$

with the spin-dependent force $\mathbf{F}_\sigma = -g_{\rm e}\mu_{\rm B}\sigma\boldsymbol{\nabla}_{\mathbf{r}} B_{\rm e}$ and the velocity $\mathbf{v}_{\mathbf{p}} \cong \mathbf{p}/m_{\rm cb}$, where $m_{\rm cb}$ is the effective mass. Including scattering, we obtain the Boltzmann equation

$$(\partial_t + \mathbf{v}_{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{r}} + \mathbf{F}_\sigma \cdot \boldsymbol{\nabla}_{\mathbf{p}}) n_{\mathbf{p}\sigma}(\mathbf{r}) = \mathcal{S}^{\rm dis}_{\mathbf{p}\sigma} + \mathcal{S}^{\rm flip}_{\mathbf{p}\sigma} + \sum_m \mathcal{S}^{\rm spin}_{\mathbf{p}\sigma m} \tag{7}$$

with collision integrals $\mathcal{S}$ discussed below.

For the impurity spins we define the occupation fraction of spins with magnetic quantum number m as f_m, where $\Sigma_m f_m = 1$. The corresponding density is $n_{\rm i} f_m$ with the density $n_{\rm i}$ of magnetically active impurities. We neglect the contribution of interstitial magnetic impurities.[3,11,34]

We now discuss the collision integrals. The simplest one describes disorder scattering of the electrons,[18]

$$\mathcal{S}^{\rm dis}_{\mathbf{p}\sigma} = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{N(0)\tau} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'})(n_{\mathbf{p}'\sigma} - n_{\mathbf{p}\sigma}). \tag{8}$$

Here, $N(0)$ is the density of states per spin component and $1/\tau$ is the transport scattering rate.

The next contribution is spin-exchange scattering between electron and impurity spins. For this we need the transition probabilities between spin states. We write the spin operator of the electron (impurity) as $\mathbf{s}$ ($\mathbf{S}$). The joint spin state is denoted by $|\sigma m\rangle$. The matrix elements of the exchange coupling are

$$\begin{aligned}\langle \sigma m | \mathbf{s}\cdot\mathbf{S} | \sigma' m' \rangle &= \frac{1}{2}\delta_{\sigma,1/2}\delta_{\sigma',-1/2}\delta_{m+1,m'}\sqrt{S(S+1)-m(m+1)} \\ &\quad + \frac{1}{2}\delta_{\sigma,-1/2}\delta_{\sigma',1/2}\delta_{m-1,m'}\sqrt{S(S+1)-m(m-1)} \\ &\quad + \delta_{\sigma\sigma'}\delta_{mm'}\sigma m.\end{aligned} \tag{9}$$

Note that only the $\mathbf{p}' = \mathbf{p}$ contributions to the $s^z S^z$ term are taken care of by the mean-field decoupling. For $\mathbf{p}' \neq \mathbf{p}$ this term expresses that carriers can also scatter off impurities due to the exchange interaction without flipping the spins. The transition probabilities between the states are given by $P_{\sigma m,\sigma' m'} = |\langle \sigma m | \mathbf{s}\cdot\mathbf{S} | \sigma' m' \rangle|^2$. The collision integral for electron-impurity spin scattering can then be written as

$$\begin{aligned}\mathcal{S}^{\rm spin}_{\mathbf{p}\sigma m} = \int \frac{d^3p'}{(2\pi)^3} \sum_{\sigma' m'} \frac{1}{N(0)\tau_{\rm spin}} \delta(&\epsilon_{\mathbf{p}} + g_{\rm e}\mu_{\rm B}\sigma B_{\rm e} \\ &+ g_{\rm i}\mu_{\rm B} m B_{\rm i} - \epsilon_{\mathbf{p}'} - g_{\rm e}\mu_{\rm B}\sigma' B_{\rm e} \\ &- g_{\rm i}\mu_{\rm B} m' B_{\rm i}) P_{\sigma m,\sigma' m'} [n_{\mathbf{p}'\sigma'}(1-n_{\mathbf{p}\sigma}) f_{m'} \\ &- n_{\mathbf{p}\sigma}(1-n_{\mathbf{p}'\sigma'}) f_m]\end{aligned} \tag{10}$$

with the spin-exchange scattering rate $1/\tau_{\rm spin}$.[35] Due to conservation of the total spin by the process expressed by Eq. (10), the same collision integral also appears in the Boltzmann equation for f_m,

$$\partial_t n_{\rm i} f_m = \int \frac{d^3p}{(2\pi)^3} \sum_\sigma \mathcal{S}^{\rm spin}_{\mathbf{p}\sigma m}. \tag{11}$$

The left-hand side only contains the explicit time derivative since the impurities are assumed to be immobile and purely local. This is the only scattering term we consider for the impurities.

The scattering processes expressed by $\mathcal{S}^{\rm dis}$ and $\mathcal{S}^{\rm spin}$ are not sufficient for a reasonable thermodynamic description, however. The reason is that both processes conserve the total spin. Thus the homogeneous spin susceptibility would be zero. To avoid this problem we include relaxation of the total spin by an additional "spin-flip" scattering term for the electrons. This can be due to the hyperfine interaction with nuclear spins[36] or electron-electron interaction in conjunction with spin-orbit coupling in other bands.[28] This process is expressed by

$$\begin{aligned}\mathcal{S}^{\rm flip}_{\mathbf{p}\sigma} = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{N(0)\tau_{\rm flip}} \delta(&\epsilon_{\mathbf{p}} + g_{\rm e}\mu_{\rm B}\sigma B_{\rm e} - \epsilon_{\mathbf{p}'} \\ &- g_{\rm e}\mu_{\rm B}\bar{\sigma} B_{\rm e})\,(n_{\mathbf{p}'\bar{\sigma}} - n_{\mathbf{p}\sigma}),\end{aligned} \tag{12}$$

where $\bar{\sigma} = -\sigma$.

From Eq. (7) one easily derives the continuity equation for the electron density. Our main goal is to derive corresponding equations for the magnetizations

$$\mu_{\rm e} = -g_{\rm e}\mu_{\rm B} \int \frac{d^3p}{(2\pi)^3} \sum_\sigma \sigma\, n_{\mathbf{p}\sigma}, \tag{13}$$

$$\mu_{\rm i} = -g_{\rm i}\mu_{\rm B} n_{\rm i} \sum_m m\, f_m \tag{14}$$

and the electron magnetization current

$$\mathbf{j}_\mu = -g_{\rm e}\mu_{\rm B} \int \frac{d^3p}{(2\pi)^3} \sum_\sigma \sigma \mathbf{v}_{\mathbf{p}}\, n_{\mathbf{p}\sigma}. \tag{15}$$

We start with the impurity spins. Multiplying Eq. (11) by m and summing over m we obtain

$$\begin{aligned}-\frac{\partial_t \mu_{\rm i}}{g_{\rm i}\mu_{\rm B}} &= \int \frac{d^3p}{(2\pi)^3} \sum_{\sigma m} m\, \mathcal{S}^{\rm spin}_{\mathbf{p}\sigma m} \\ &= -\frac{S(S+1)}{3\tau_{\rm spin}} \frac{\mu_{\rm e}}{g_{\rm e}\mu_{\rm B}} + \frac{N(0)T}{2\tau_{\rm spin}} \frac{\mu_{\rm i}}{g_{\rm e}\mu_{\rm B} n_{\rm i}} \frac{N(0)\,S(S+1)}{6\tau_{\rm spin}} \\ &\quad \times (g_{\rm e}\mu_{\rm B} B_{\rm e} - g_{\rm i}\mu_{\rm B} B_{\rm i}),\end{aligned} \tag{16}$$

to linear order in the effective fields and magnetizations, cf. Appendix A. In the last expression we can identify the Pauli susceptibility of free electrons with density of states $N(0)$ per spin component and the Curie susceptibility of noninteracting impurity spins with spin quantum number S and density $n_{\rm i}$:[37]

$$\chi_{\mathrm{Pauli}}=\frac{N(0)g_{\mathrm{e}}^2\mu_{\mathrm{B}}^2}{2}, \tag{17}$$

$$\chi_{\mathrm{Curie}}=\frac{S(S+1)\,g_{\mathrm{i}}^2\mu_{\mathrm{B}}^2 n_{\mathrm{i}}}{3T}. \tag{18}$$

Using these susceptibilities we write

$$\partial_t\mu_{\mathrm{i}}=\frac{S(S+1)}{3\tau_{\mathrm{spin}}}\frac{g_{\mathrm{i}}}{g_{\mathrm{e}}}(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}}) -\frac{1}{2\tau_{\mathrm{spin}}}\frac{N(0)T}{n_{\mathrm{i}}}(\mu_{\mathrm{i}}-\chi_{\mathrm{Curie}}B_{\mathrm{i}}). \tag{19}$$

The rate of change of the impurity magnetization μ_{i} thus depends linearly on the deviations of μ_{e} and μ_{i} from their respective equilibrium values, which is quite reasonable. Note that this and the following equations of motion do not contain a precession term since this term would be of second order in the magnetization.[37]

Multiplying the Boltzmann equation (7) by σ and summing over $\mathbf{p}$, σ we obtain

$$-\frac{\partial_t\mu_{\mathrm{e}}}{g_{\mathrm{e}}\mu_{\mathrm{B}}}-\frac{\nabla_{\mathbf{r}}\cdot\mathbf{j}_\mu}{g_{\mathrm{e}}\mu_{\mathrm{B}}} =\int\frac{d^3p}{(2\pi)^3}\sum_\sigma\sigma\left(\mathcal{S}^{\mathrm{dis}}_{\mathbf{p}\sigma}+\mathcal{S}^{\mathrm{flip}}_{\mathbf{p}\sigma}+\sum_m\mathcal{S}^{\mathrm{spin}}_{\mathbf{p}\sigma m}\right), \tag{20}$$

where the force term on the left-hand side vanishes since the integrand is a total $\mathbf{p}$ gradient. The right-hand side can be evaluated similarly to the calculation in Appendix A and expressed using χ_{Pauli} and χ_{Curie},

$$\partial_t\mu_{\mathrm{e}}+\nabla_{\mathbf{r}}\cdot\mathbf{j}_\mu=-\left(\frac{2}{\tau_{\mathrm{flip}}}+\frac{S(S+1)}{3\tau_{\mathrm{spin}}}\right)(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}}) +\frac{1}{2\tau_{\mathrm{spin}}}\frac{N(0)T}{n_{\mathrm{i}}}\frac{g_{\mathrm{e}}}{g_{\mathrm{i}}}(\mu_{\mathrm{i}}-\chi_{\mathrm{Curie}}B_{\mathrm{i}}). \tag{21}$$

To eliminate the magnetization current $\mathbf{j}_\mu$, we derive its equation of motion by multiplying Eq. (7) by $\sigma\mathbf{v}_{\mathbf{p}}$ and summing over $\mathbf{p}$ and σ,

$$\begin{aligned}
-\frac{\partial_t\mathbf{j}_\mu}{g_{\mathrm{e}}\mu_{\mathrm{B}}}&+\int\frac{d^3p}{(2\pi)^3}\sum_\sigma\sigma\mathbf{v}_{\mathbf{p}}(\mathbf{v}_{\mathbf{p}}\cdot\nabla_{\mathbf{r}}n_{\mathbf{p}\sigma})\\
&-\int\frac{d^3p}{(2\pi)^3}\sum_\sigma\sigma\mathbf{v}_{\mathbf{p}}g_{\mathrm{e}}\mu_{\mathrm{B}}\sigma(\nabla_{\mathbf{r}}B_{\mathrm{e}})\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}\sigma}\\
&\cong-\frac{v_F^2}{3}\frac{\nabla_{\mathbf{r}}\mu_{\mathrm{e}}}{g_{\mathrm{e}}\mu_{\mathrm{B}}}+\frac{g_{\mathrm{e}}\mu_{\mathrm{B}}\rho^{(0)}}{4m_{\mathrm{cb}}}\nabla_{\mathbf{r}}B_{\mathrm{e}}\\
&=-\frac{v_F^2}{3}\frac{1}{g_{\mathrm{e}}\mu_{\mathrm{B}}}\nabla_{\mathbf{r}}(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}})\\
&=\int\frac{d^3p}{(2\pi)^3}\sum_\sigma\sigma\mathbf{v}_{\mathbf{p}}\left(\mathcal{S}^{\mathrm{dis}}_{\mathbf{p}\sigma}+\mathcal{S}^{\mathrm{flip}}_{\mathbf{p}\sigma}+\sum_m\mathcal{S}^{\mathrm{spin}}_{\mathbf{p}\sigma m}\right).
\end{aligned} \tag{22}$$

The first term $-\partial_t\mathbf{j}_\mu/g_{\mathrm{e}}\mu_{\mathrm{B}}$ is neglected since it only becomes relevant for frequencies of the order of the largest scattering rate. In the second term we have replaced $v_{\mathbf{p}}^\alpha v_{\mathbf{p}}^\beta$ in the usual way by $\delta_{\alpha\beta}v_F^2/3$, where v_F is the Fermi velocity. This is valid since $n_{\mathbf{p}\sigma}$ has significant $\mathbf{r}$ dependence only close to the Fermi energy. The third term has been expanded to linear order in the perturbation and in the final step the equilibrium electron density has been written as $\rho^{(0)}=2N(0)m_{\mathrm{cb}}v_F^2/3$ for a parabolic band. Evaluating the integrals, we obtain

$$\mathbf{j}_\mu=-D\,\nabla_{\mathbf{r}}(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}}) \tag{23}$$

with the diffusion constant $D=v_F^2\tau_{\mathrm{tot}}/3$ and the total scattering rate

$$\frac{1}{\tau_{\mathrm{tot}}}=\frac{1}{\tau}+\frac{1}{\tau_{\mathrm{flip}}}+\frac{S(S+1)}{4\tau_{\mathrm{spin}}}. \tag{24}$$

Inserting this result into Eq. (21) we find the equation of motion of the electron-spin magnetization,

$$\partial_t\mu_{\mathrm{e}}=-\left(\frac{2}{\tau_{\mathrm{flip}}}+\frac{S(S+1)}{3\tau_{\mathrm{spin}}}-D\nabla_{\mathbf{r}}^2\right)(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}}) +\frac{1}{2\tau_{\mathrm{spin}}}\frac{N(0)T}{n_{\mathrm{i}}}\frac{g_{\mathrm{e}}}{g_{\mathrm{i}}}(\mu_{\mathrm{i}}-\chi_{\mathrm{Curie}}B_{\mathrm{i}}). \tag{25}$$

We observe that also the rate of change of μ_{e} is linear in the deviations of the hole and impurity magnetization from their equilibrium values. The result that $\partial_t\mu_{\mathrm{e}}$ vanishes in equilibrium must hold in general, not just for the parabolic band assumed above, as expressed by the Einstein relation.

The two equations (19) and (25) are coupled explicitly and through the effective fields. They are formally solved by Fourier transformation in space and time,

$$-i\omega\mu_{\mathrm{e}}=-\left(\frac{2}{\tau_{\mathrm{flip}}}+\frac{S(S+1)}{3\tau_{\mathrm{spin}}}+Dq^2\right)(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}}) +\frac{1}{2\tau_{\mathrm{spin}}}\frac{N(0)T}{n_{\mathrm{i}}}\frac{g_{\mathrm{e}}}{g_{\mathrm{i}}}(\mu_{\mathrm{i}}-\chi_{\mathrm{Curie}}B_{\mathrm{i}}), \tag{26}$$

$$-i\omega\mu_{\mathrm{i}}=\frac{S(S+1)}{3\tau_{\mathrm{spin}}}\frac{g_{\mathrm{i}}}{g_{\mathrm{e}}}(\mu_{\mathrm{e}}-\chi_{\mathrm{Pauli}}B_{\mathrm{e}}) -\frac{1}{2\tau_{\mathrm{spin}}}\frac{N(0)T}{n_{\mathrm{i}}}(\mu_{\mathrm{i}}-\chi_{\mathrm{Curie}}B_{\mathrm{i}}). \tag{27}$$

In the absence of external fields, the static, homogeneous magnetizations have nonzero solutions at the mean-field Curie temperature[4,6,38–40]

$$T_c=\frac{S(S+1)}{6}N(0)\,J^2\,n_{\mathrm{i}}. \tag{28}$$

B. Hydrodynamic equations, valence band

We now derive hydrodynamic equations for valence-band holes exchange coupled to impurity spins, relevant for *p*-type DMS. The case of spin quantum number $S=5/2$ corresponds to substitutional Mn in GaAs. The main complication here is the presence of *spin-orbit coupling*. We employ a four-band Kohn-Luttinger Hamiltonian[5,41,42] in the spherical approximation, which is the simplest one incorporating the relevant physics. In the absence of magnetic impurities the Hamiltonian reads[26,43]

$$H=\frac{1}{2m}[(\gamma_1+5\gamma_2/2)\,k^2-2\gamma_2(\mathbf{k}\cdot\mathbf{j})^2] \tag{29}$$

with Kohn-Luttinger parameters γ_1, γ_2 and the angular momentum operator $\mathbf{j}$ of the holes, which in this subspace can be written as a 4×4 matrix and has the Casimir operator $\mathbf{j}\cdot\mathbf{j}=3/2(3/2+1)$. Since the split-off band is neglected, this description only applies to semiconductors with sufficiently strong spin-orbit coupling. The eigenstates of H at $\mathbf{k}$ are characterized by the quantum number $j=\pm1/2,\pm3/2$ of $\hat{\mathbf{k}}\cdot\mathbf{j}$, where $\hat{\mathbf{k}}\equiv\mathbf{k}/k$. We restrict ourselves to the heavy-hole band, which is justified for energies close to the band edge because of the much smaller density of states of the light-hole band.

We introduce the eigenstates $|j\rangle_{\mathbf{k}}$ of $\hat{\mathbf{k}}\cdot\mathbf{j}$ with eigenvalues j. We denote the spin eigenstates with respect to a *fixed* quantization axis $\hat{\mathbf{z}}$ by $|j\rangle$. The former can be expressed in terms of the latter by means of a rotation in spin space,[44,45]

$$|j\rangle_{\mathbf{k}}=e^{-ij^z\phi}e^{-ij^y\theta}|j\rangle, \tag{30}$$

where j^y, j^z are spin operators and θ and ϕ are the polar angles of $\mathbf{k}$.

The states $|j\rangle_{\mathbf{k}}$ can be expressed in terms of eigenstates of orbital angular momentum $\mathbf{l}$ (with $\mathbf{l}\cdot\mathbf{l}=2$) and spin $\mathbf{s}$ with the help of Clebsch-Gordon coefficients. One then easily finds that *all* 4×4 matrix elements of s^z equal the corresponding matrix elements of $j^z/3$. The same holds for the x and y components because of symmetry so that $\mathbf{s}=\mathbf{j}/3$ holds as an operator identity in the four-band subspace.[26] Consequently the heavy-hole states ($j=\pm3/2$) are eigenstates to $\hat{\mathbf{k}}\cdot\mathbf{s}$ with eigenvalues $\pm1/2$. However, the heavy holes do *not* form a spin doublet since the matrix elements of $s^{\pm}=j^{\pm}/3$ all vanish in the heavy-hole subspace—single spin flips cannot change the total angular momentum from $\pm3/2$ to $\mp3/2$.

The total energy of a heavy hole is

$$E^{\mathrm{hh}}_{\mathbf{p}j}=(\gamma_1-2\gamma_2)\frac{p^2}{2m}+g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\cos\theta B_{\mathrm{h}}, \tag{31}$$

where θ is the polar angle of $\mathbf{p}$ with respect to the field direction $\hat{\mathbf{z}}$. Without scattering the equation of motion for the hole density reads

$$\partial_t n_{\mathbf{p}j}=-\{n_{\mathbf{p}j},E^{\mathrm{hh}}_{\mathbf{p}j}\}=g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\cos\theta\nabla_{\mathbf{r}}B_{\mathrm{h}}\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}j}$$
$$-\frac{\mathbf{p}}{m_{\mathrm{hh}}}\cdot\nabla_{\mathbf{r}}n_{\mathbf{p}j}+g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\sin\theta B_{\mathrm{h}}\frac{\hat{\boldsymbol{\theta}}}{\mathbf{p}}\cdot\nabla_{\mathbf{r}}n_{\mathbf{p}j}, \tag{32}$$

where $m_{\mathrm{hh}}=m/(\gamma_1-2\gamma_2)$ is the heavy-hole effective mass. This suggests to define the velocity as

$$\mathbf{v}_{\mathbf{p}}=\frac{\mathbf{p}}{m_{\mathrm{hh}}}-g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\sin\theta\,B_{\mathrm{h}}\frac{\hat{\boldsymbol{\theta}}}{p}. \tag{33}$$

Note that the second term is explicitly of first order. We should use this velocity in the semiclassical equations. However, we find the contribution from the second term to vanish to first order. The reason is essentially that we have to evaluate all other factors in equilibrium due to the explicit B_{h}. This result is proved together with the absence of Berry-phase contributions in Appendix C. We thus drop the second term in Eq. (33).

We now turn to the derivation of the Boltzmann equation for the holes,

$$(\partial_t+\mathbf{v}_{\mathbf{p}}\cdot\nabla_{\mathbf{r}}+\mathbf{F}_{\mathbf{p}j}\cdot\nabla_{\mathbf{p}})n_{\mathbf{p}j}(\mathbf{r})=\mathcal{S}^{\mathrm{dis}}_{\mathbf{p}j}+\sum_m\mathcal{S}^{\mathrm{spin}}_{\mathbf{p}jm} \tag{34}$$

with the force $\mathbf{F}_{\mathbf{p}j}=-g_{\mathrm{h}}\mu_{\mathrm{B}}(j/3)\cos\theta\nabla_{\mathbf{r}}B_{\mathrm{h}}$ for the holes and

$$\partial_t n_{\mathrm{i}}f_m=\int\frac{d^3p}{(2\pi)^3}\sum_j\mathcal{S}^{\mathrm{spin}}_{\mathbf{p}jm} \tag{35}$$

for the impurities. The disorder scattering integral contains the matrix elements ${}_{\mathbf{k}}\langle j|j'\rangle_{\mathbf{k}'}=\langle j|e^{ij^y\theta}e^{ij^z\phi}e^{-ij^z\phi'}e^{-ij^y\theta'}|j'\rangle$. The spin operators are 4×4 matrices in the projected subspace. For heavy holes, explicit evaluation gives the transition probabilities

$$|{}_{\mathbf{k}}\langle j|j'\rangle_{\mathbf{k}'}|^2=\begin{pmatrix}\cos^6\frac{\alpha}{2} & \sin^6\frac{\alpha}{2}\\ \sin^6\frac{\alpha}{2} & \cos^6\frac{\alpha}{2}\end{pmatrix}_{jj'}, \tag{36}$$

where $j,j'=\pm3/2$. Here, α is the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$. The collision integral for disorder scattering of heavy holes reads

$$\mathcal{S}^{\mathrm{dis}}_{\mathbf{p}j}=\int\frac{d^3p'}{(2\pi)^3}\sum_{j'}\frac{1}{N(0)\tau}\delta\bigg(\epsilon_{\mathbf{p}}+g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\cos\theta\,B_{\mathrm{h}}-\epsilon_{\mathbf{p}'}$$
$$-g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j'}{3}\cos\theta'B_{\mathrm{h}}\bigg)|{}_{\mathbf{p}}\langle j|j'\rangle_{\mathbf{p}'}|^2(n_{\mathbf{p}'j'}-n_{\mathbf{p}j}). \tag{37}$$

Note that for forward scattering ($\alpha\sim0$) we get predominantly $j'=j$, whereas for backscattering ($\alpha\sim\pi$) we find predominantly $j'=-j$. Due to the $\mathbf{k}$-dependent quantization axis j is not conserved even by pure disorder scattering due to the Elliott-Yafet mechanism.[46,47]

For the hole-impurity spin scattering we need matrix elements of $\mathbf{s}\cdot\mathbf{S}$. The transition probabilities are

$$P_{\mathbf{p}jm,\mathbf{p}'j'm'} = |{}_{\mathbf{p}}\langle jm|\mathbf{s}\cdot\mathbf{S}|j'm'\rangle_{\mathbf{p}'}|^2 = \frac{1}{9}\Bigg(\frac{1}{4}|{}_{\mathbf{p}}\langle j|j^+|j'\rangle_{\mathbf{p}'}|^2\delta_{m+1,m'} \times[S(S+1)-m(m+1)]+\frac{1}{4}|{}_{\mathbf{p}}\langle j|j^-|j'\rangle_{\mathbf{p}'}|^2\delta_{m-1,m'} \times[S(S+1)-m(m-1)]+|{}_{\mathbf{p}}\langle j|j^z|j'\rangle_{\mathbf{p}'}|^2\delta_{mm'}m^2\Bigg) \tag{38}$$

and the resulting collision integral reads

$$\mathcal{S}^{\text{spin}}_{\mathbf{p}jm} = \int\frac{d^3p'}{(2\pi)^3}\sum_{j'm'}\frac{1}{N(0)\tau_{\text{spin}}}\delta\bigg(\epsilon_{\mathbf{p}}+g_{\text{h}}\mu_{\text{B}}\frac{j}{3}\cos\theta B_{\text{h}} + g_{\text{i}}\mu_{\text{B}}mB_{\text{i}}-\epsilon_{\mathbf{p}'}-g_{\text{h}}\mu_{\text{B}}\frac{j'}{3}\cos\theta' B_{\text{h}}-g_{\text{i}}\mu_{\text{B}}m'B_{\text{i}}\bigg) \times P_{\mathbf{p}jm,\mathbf{p}'j'm'}[n_{\mathbf{p}'j'}(1-n_{\mathbf{p}j})f_{m'}-n_{\mathbf{p}j}(1-n_{\mathbf{p}'j'})f_m]. \tag{39}$$

Since the two collision integrals already include spin relaxation we do not introduce an additional spin-flip term.

We now derive hydrodynamic equations for the hole- and impurity-spin magnetizations

$$\mu_{\text{h}} = -g_{\text{h}}\mu_{\text{B}}\int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\cos\theta\, n_{\mathbf{p}j}, \tag{40}$$

$$\mu_{\text{i}} = -g_{\text{i}}\mu_{\text{B}}n_{\text{i}}\sum_m m f_m \tag{41}$$

and the hole magnetization current

$$\mathbf{j}_\mu = -g_{\text{h}}\mu_{\text{B}}\int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\cos\theta\,\mathbf{v}_{\mathbf{p}}\,n_{\mathbf{p}j}. \tag{42}$$

Some details of the calculations are shown in Appendix B. We start with the impurity spins. In analogy to the conduction-band case we obtain

$$\partial_t\mu_{\text{i}} = \frac{S(S+1)}{18\tau_{\text{spin}}}\frac{g_{\text{i}}}{g_{\text{h}}}\left(\mu_{\text{h}}-\frac{1}{3}\chi_{\text{Pauli}}B_{\text{h}}\right) - \frac{5}{36\tau_{\text{spin}}}\frac{N(0)T}{n_{\text{i}}}(\mu_{\text{i}}-\chi_{\text{Curie}}B_{\text{i}}), \tag{43}$$

where we have again identified the Pauli susceptibility $\chi_{\text{Pauli}}=N(0)g_{\text{h}}^2\mu_{\text{B}}^2/2$ and the Curie susceptibility. Note, however, the factor of 1/3 multiplying the Pauli susceptibility, which is absent for the conduction band. This factor is easily understood by calculating the static, homogeneous spin susceptibility of heavy holes in the absence of impurities. For the static susceptibility we can assume the holes to follow a Fermi distribution, which we expand in B_{h},

$$n_{\mathbf{p}j}\cong n_F(\epsilon_{\mathbf{p}}-\mu)+n_F^{(1)}(\epsilon_{\mathbf{p}}-\mu)g_{\text{h}}\mu_{\text{B}}\frac{j}{3}\cos\theta B_{\text{h}}, \tag{44}$$

where $n_F^{(1)}(E)=n_F(E)[n_F(E)-1]$ is the derivative of the Fermi function. To linear order we then find

$$\mu_{\text{h}} = -g_{\text{h}}^2\mu_{\text{B}}^2\int\frac{d^3p}{(2\pi)^3}\sum_j\left(\frac{j}{3}\right)^2\cos^2\theta B_{\text{h}}\,n_F^{(1)}(\epsilon_{\mathbf{p}}-\mu) = \frac{1}{3}\frac{N(0)g_{\text{h}}^2\mu_{\text{B}}^2}{2}B_{\text{h}} = \frac{1}{3}\chi_{\text{Pauli}}B_{\text{h}}. \tag{45}$$

The extra factor stems from the direction-dependent quantization axis and thus spin-orbit coupling.

We also obtain the equation of motion for the hole magnetization,

$$-\frac{\partial_t\mu_{\text{h}}}{g_{\text{h}}\mu_{\text{B}}}-\frac{\boldsymbol{\nabla}_{\mathbf{r}}\cdot\mathbf{j}_\mu}{g_{\text{h}}\mu_{\text{B}}} = \int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\cos\theta\left(\mathcal{S}^{\text{dis}}_{\mathbf{p}j}+\sum_m\mathcal{S}^{\text{spin}}_{\mathbf{p}jm}\right). \tag{46}$$

Similarly to the calculation in Appendix B, we find

$$\partial_t\mu_{\text{h}}+\boldsymbol{\nabla}_{\mathbf{r}}\cdot\mathbf{j}_\mu = -\left(\frac{1}{5\tau}+\frac{7S(S+1)}{180\tau_{\text{spin}}}\right)\left(\mu_{\text{h}}-\frac{1}{3}\chi_{\text{Pauli}}B_{\text{h}}\right) + \frac{1}{36\tau_{\text{spin}}}\frac{N(0)T}{n_{\text{i}}}\frac{g_{\text{h}}}{g_{\text{i}}}(\mu_{\text{i}}-\chi_{\text{Curie}}B_{\text{i}}). \tag{47}$$

To eliminate the magnetization current $\mathbf{j}_\mu$ we consider its equation of motion with the left-hand side

$$-\frac{\partial_t\mathbf{j}_\mu}{g_{\text{h}}\mu_{\text{B}}}+\int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\cos\theta\,\mathbf{v}_{\mathbf{p}}(\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}n_{\mathbf{p}j}) - \int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\cos\theta\,\mathbf{v}_{\mathbf{p}}\,g_{\text{h}}\mu_{\text{B}}\frac{j}{3}\cos\theta(\boldsymbol{\nabla}_{\mathbf{r}}B_{\text{h}})\cdot\boldsymbol{\nabla}_{\mathbf{p}}n_{\mathbf{p}j}. \tag{48}$$

The first term is again neglected. In the second we have to be more careful because of the explicit angle dependence. For the *conduction* band the factor $v_F^2/3$ is obtained by assuming $n_{\mathbf{p}\sigma}$ to be the equilibrium distribution in a constant Zeeman field. The integral over the direction of $\mathbf{p}$ is then easily performed. Since we obtain a term linear in $\boldsymbol{\nabla}_{\mathbf{r}}\mu_{\text{e}}$, corrections would be of higher order. For the *valence* band we also assume a constant Zeeman field, leading to $n_{\mathbf{p}j}\cong n_{\mathbf{p}}^{(0)}+(j/3)\cos\theta\,\Delta n(p)$. Thus the second term in Eq. (48) becomes

$$v_F^2\int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\cos\theta\hat{\mathbf{p}}\left(\hat{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\frac{j}{3}\cos\theta\Delta n(p)\right) = N(0)\,v_F^2\int d\xi\sum_j\left(\frac{j}{3}\right)^2\begin{pmatrix}1/15 & 0 & 0\\ 0 & 1/15 & 0\\ 0 & 0 & 1/5\end{pmatrix} \times\boldsymbol{\nabla}_{\mathbf{r}}\Delta n[p(\xi)]. \tag{49}$$

In the same approximation we find $-\mu_h/g_h\mu_B = N(0)\int d\xi\, \Sigma_j (j/3)^2 \Delta n[p(\xi)]/3$ so that this term is

$$-v_F^2 \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 3/5 \end{pmatrix} \nabla_{\mathbf{r}} \frac{\mu_h}{g_h\mu_B}. \tag{50}$$

The third term in Eq. (48) is straightforward to evaluate to first order,

$$\frac{N(0)g_h\mu_B}{6} v_F^2 \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 3/5 \end{pmatrix} \nabla_{\mathbf{r}} B_h. \tag{51}$$

Again, the result holds in general due to the Einstein relation. Altogether the equation of motion for the magnetization current is

$$v_F^2 \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 3/5 \end{pmatrix} \nabla_{\mathbf{r}} \left(\mu_h - \frac{1}{3}\chi_{\text{Pauli}} B_h \right) = \int \frac{d^3p}{(2\pi)^3} \sum_j \frac{j}{3} \cos\theta \mathbf{v}_{\mathbf{p}} \left(\mathcal{S}^{\text{dis}}_{\mathbf{p}j} + \sum_m \mathcal{S}^{\text{spin}}_{\mathbf{p}jm} \right). \tag{52}$$

Evaluating the integrals we finally obtain

$$\mathbf{j}_\mu = -D \begin{pmatrix} 3/5 & 0 & 0 \\ 0 & 3/5 & 0 \\ 0 & 0 & 9/5 \end{pmatrix} \nabla_{\mathbf{r}} \left(\mu_h - \frac{1}{3}\chi_{\text{Pauli}} B_h \right), \tag{53}$$

where we have introduced the diffusion constant $D = v_F^2 \tau_{\text{tot}}/3$ with the total relaxation rate $1/\tau_{\text{tot}} = 1/(2\tau) + 5\, S(S+1)/(72\tau_{\text{spin}})$. The spin diffusion in the valence band is thus *anisotropic*. Compared with the result (23) for the conduction band, diffusion along the direction of the effective field is enhanced and diffusion in the transverse directions is suppressed. The origin of this interesting effect again lies in the momentum dependence of the quantization axis: Consider, for example, heavy holes traveling exactly along the x direction. In the Hilbert subspace of these holes all matrix elements of s^z and j^z vanish so that these holes cannot carry any spin magnetization pointing in the z direction. For holes with momentum $\mathbf{p}$ pointing partly in a transverse direction the contribution to spin transport is still suppressed.

Inserting our result for the current into Eq. (47) we obtain the equation of motion for the hole magnetization,

$$\partial_t \mu_h = -\left[\frac{1}{5\tau} + \frac{7S(S+1)}{180\tau_{\text{spin}}} - D\left(\frac{3}{5}\frac{\partial^2}{\partial x^2} + \frac{3}{5}\frac{\partial^2}{\partial y^2} + \frac{9}{5}\frac{\partial^2}{\partial z^2} \right) \right] \times \left(\mu_h - \frac{1}{3}\chi_{\text{Pauli}} B_h \right) + \frac{1}{36\tau_{\text{spin}}} \frac{N(0)T}{n_i} \frac{g_h}{g_i} \times (\mu_i - \chi_{\text{Curie}} B_i). \tag{54}$$

This equation is of the same general form as for the conduction band, the main differences being the reduced Pauli susceptibility and the anisotropic spin diffusion. Fourier transformation yields

$$-i\omega\mu_h = -\left[\frac{1}{5\tau} + \frac{7S(S+1)}{180\tau_{\text{spin}}} + D\frac{3q_x^2 + 3q_y^2 + 9q_z^2}{5} \right] \times \left(\mu_h - \frac{1}{3}\chi_{\text{Pauli}} B_h \right) + \frac{1}{36\tau_{\text{spin}}} \frac{N(0)T}{n_i} \frac{g_h}{g_i} (\mu_i - \chi_{\text{Curie}} B_i), \tag{55}$$

$$-i\omega\mu_i = \frac{S(S+1)}{18\tau_{\text{spin}}} \frac{g_i}{g_h} \left(\mu_h - \frac{1}{3}\chi_{\text{Pauli}} B_h \right) - \frac{5}{36\tau_{\text{spin}}} \frac{N(0)T}{n_i} (\mu_i - \chi_{\text{Curie}} B_i). \tag{56}$$

For $\omega = 0$, $\mathbf{q} = 0$ we find finite solutions at

$$T = T_c = \frac{S(S+1)}{18} N(0)\, J^2\, n_i. \tag{57}$$

The Curie temperature of holes is *reduced* by an extra factor of 1/3 compared to the conduction-band case. This factor stems from the same factor in the Pauli susceptibility and is thus due to spin-orbit coupling. On the other hand, for typical host materials the density of states is much *higher* for the heavy holes than for conduction-band electrons and the exchange integral J is also much larger, enhancing T_c in p-type materials.

We have so far ignored the possible effect of Berry-phase contributions.[26] In Appendix C we show that they vanish to linear order in the effective field. Berry-phase contributions are expected in higher orders, though.

C. Susceptibilities

With the help of the hydrodynamic equations we now derive the linear response of the carrier-spin and impurity-spin magnetizations to external fields coupled to these magnetizations. It is useful to solve the general problem of the Fourier-transformed hydrodynamic equations

$$-i\omega\mu_e = -R_{\mathbf{q}}(\mu_e - \alpha\chi_{\text{Pauli}} B_e) + R_{ei} \frac{N(0)T}{n_i} \frac{g_e}{g_i} (\mu_i - \chi_{\text{Curie}} B_i), \tag{58}$$

$$-i\omega\mu_i = R_{ie} \frac{g_i}{g_e} (\mu_e - \alpha\chi_{\text{Pauli}} B_e) - R_{ii} \frac{N(0)T}{n_i} (\mu_i - \chi_{\text{Curie}} B_i), \tag{59}$$

where

$$B_e = B_e^{\text{ext}} - \frac{J}{g_e g_i \mu_B^2} \mu_i, \tag{60}$$

$$B_{\mathrm{i}}=B_{\mathrm{i}}^{\mathrm{ext}}-\frac{J}{g_{\mathrm{e}}g_{\mathrm{i}}\mu_{\mathrm{B}}^{2}}\,\mu_{\mathrm{e}}, \tag{61}$$

which contain the special cases of the conduction band, Eqs. (26) and (27), and the valence band, Eqs. (55) and (56) (replacing "e" by "h"). The solution reads

$$\begin{pmatrix}\mu_{\mathrm{e}}/g_{\mathrm{e}}\mu_{\mathrm{B}}\\ \mu_{\mathrm{i}}/g_{\mathrm{i}}\mu_{\mathrm{B}}\end{pmatrix}=\frac{N(0)}{\det M}\begin{pmatrix}A_{\mathrm{ee}} & A_{\mathrm{ei}}\\ A_{\mathrm{ie}} & A_{\mathrm{ii}}\end{pmatrix}\begin{pmatrix}g_{\mathrm{e}}\mu_{\mathrm{B}}B_{\mathrm{e}}^{\mathrm{ext}}\\ g_{\mathrm{i}}\mu_{\mathrm{B}}B_{\mathrm{i}}^{\mathrm{ext}}\end{pmatrix} \tag{62}$$

with the determinant of the coefficient matrix

$$\begin{aligned}\det M=&\,6n_{\mathrm{i}}(i\omega)^2+i\omega\,[2\alpha N(0)Jn_{\mathrm{i}}R_{\mathrm{ie}}-6n_{\mathrm{i}}R_{\mathbf{q}}\\ &+2N(0)Jn_{\mathrm{i}}R_{\mathrm{ei}}S(S+1)-6N(0)TR_{\mathrm{ii}}]\\ &+N(0)\,(R_{\mathrm{ei}}R_{\mathrm{ie}}-R_{\mathrm{ii}}R_{\mathbf{q}})\,[\alpha N(0)J^2n_{\mathrm{i}}S(S+1)-6T]\end{aligned} \tag{63}$$

and

$$A_{\mathrm{ee}}=-3\alpha\,[i\omega n_{\mathrm{i}}R_{\mathbf{q}}+N(0)T\,(R_{\mathrm{ei}}R_{\mathrm{ie}}-R_{\mathrm{ii}}R_{\mathbf{q}})], \tag{64}$$

$$A_{\mathrm{ei}}=\alpha n_{\mathrm{i}}[3i\omega R_{\mathrm{ie}}+N(0)J\,(R_{\mathrm{ei}}R_{\mathrm{ie}}-R_{\mathrm{ii}}R_{\mathbf{q}})S(S+1)], \tag{65}$$

$$A_{\mathrm{ie}}=n_{\mathrm{i}}[2i\omega R_{\mathrm{ei}}+\alpha N(0)J\,(R_{\mathrm{ei}}R_{\mathrm{ie}}-R_{\mathrm{ii}}R_{\mathbf{q}})]S(S+1), \tag{66}$$

$$A_{\mathrm{ii}}=-2n_{\mathrm{i}}[R_{\mathrm{ei}}R_{\mathrm{ie}}+R_{\mathrm{ii}}\,(i\omega-R_{\mathbf{q}})]\,S(S+1). \tag{67}$$

The magnetization becomes singular at $T=T_c=(\alpha/6)\,S(S+1)N(0)\,J^2n_{\mathrm{i}}$, in agreement with our earlier results.[48] We now assume ω and $T-T_c$ to be small compared to the rates $R_{\mathbf{q}}$, R_{ei}, R_{ie}, R_{ii} but do not make any assumption about $T-T_c$ vs ω. Then we find

$$\begin{pmatrix}\mu_{\mathrm{e}}\\ \mu_{\mathrm{i}}\end{pmatrix}=\begin{pmatrix}\chi_{\mathrm{ee}} & \chi_{\mathrm{ei}}\\ \chi_{\mathrm{ie}} & \chi_{\mathrm{ii}}\end{pmatrix}\begin{pmatrix}B_{\mathrm{e}}^{\mathrm{ext}}\\ B_{\mathrm{i}}^{\mathrm{ext}}\end{pmatrix} \tag{68}$$

with the susceptibility matrix

$$\begin{aligned}\chi=&\begin{pmatrix}\chi_{\mathrm{ee}} & \chi_{\mathrm{ei}}\\ \chi_{\mathrm{ie}} & \chi_{\mathrm{ii}}\end{pmatrix}\\ =&\,2N(0)\,S(S+1)\,(R_{\mathrm{ei}}R_{\mathrm{ie}}-R_{\mathrm{ii}}R_{\mathbf{q}})\,\mu_{\mathrm{B}}^2\\ &\times\Big(i\omega\,[6R_{\mathbf{q}}-3\alpha N(0)JR_{\mathrm{ie}}-2N(0)JR_{\mathrm{ei}}S(S+1)\\ &+\alpha N^2(0)J^2S(S+1)R_{\mathrm{ii}}]+\alpha N^2(0)J^2S(S+1)\\ &\times(R_{\mathrm{ei}}R_{\mathrm{ie}}-R_{\mathrm{ii}}R_{\mathbf{q}})\frac{T-T_c}{T_c}\Big)^{-1}\\ &\times\begin{pmatrix}g_{\mathrm{e}}^2\left(\frac{\alpha N(0)J}{2}\right)^2 & -g_{\mathrm{e}}g_{\mathrm{i}}\frac{\alpha N(0)J}{2}\\ -g_{\mathrm{e}}g_{\mathrm{i}}\frac{\alpha N(0)J}{2} & g_{\mathrm{i}}^2\end{pmatrix}.\end{aligned} \tag{69}$$

Since the same field acts on the carrier and impurity spins, the physical susceptibility of the carrier spins is $\chi_{\mathrm{ee}}+\chi_{\mathrm{ei}}$, while the susceptibility of the impurity spins is $\chi_{\mathrm{ie}}+\chi_{\mathrm{ii}}$. The total susceptibility describing the response of the total magnetization is $\chi_{\mathrm{tot}}=\chi_{\mathrm{ee}}+\chi_{\mathrm{ei}}+\chi_{\mathrm{ie}}+\chi_{\mathrm{ii}}$. Note that this physical susceptibility is always paramagnetic since the components of the matrix factor in Eq. (69) combine to $[g_{\mathrm{e}}\alpha N(0)J/2-g_{\mathrm{i}}]^2$. In the static case $\omega=0$ all four components are of Curie form. We already see that the dimensionless parameter $-\alpha N(0)J/2$ has a special meaning: It is the ratio between the average electron spin and the average impurity spin in an applied field, regardless of whether the field acts on the electrons or on the impurities.

We now consider the special case of *conduction-band electrons*. Inserting the appropriate factors from Eqs. (26) and (27), we obtain the susceptibility matrix

$$\begin{aligned}\chi=&\,N(0)\,S(S+1)\,\mu_{\mathrm{B}}^2\\ &\times\left\{-i\omega\left[6\tau_{\mathrm{spin}}+2S(S+1)\frac{[1-N(0)J/2]^2}{2/\tau_{\mathrm{flip}}+Dq^2}\right]\right.\\ &\left.+2S(S+1)\left(\frac{N(0)J}{2}\right)^2\frac{T-T_c}{T_c}\right\}^{-1}\\ &\times\begin{pmatrix}g_{\mathrm{e}}^2\left(\frac{N(0)J}{2}\right)^2 & -g_{\mathrm{e}}g_{\mathrm{i}}\frac{N(0)J}{2}\\ -g_{\mathrm{e}}g_{\mathrm{i}}\frac{N(0)J}{2} & g_{\mathrm{i}}^2\end{pmatrix}.\end{aligned} \tag{70}$$

This susceptibility describes the linear response of an n-type DMS. The same result would be obtained for a simple model of spin-1/2 *holes*, which is sometimes employed in the literature.

Note that the only $\mathbf{q}$ dependence appears in the coefficient of ω. This is quite different from the standard Ornstein-Zernicke form[24,25] of the susceptibility. We discuss this point further below. The only typical length scale in χ is $\xi_{\mathrm{e}}=\sqrt{D\tau_{\mathrm{flip}}/2}$. This is the relaxation length of the *total* spin. In the semiclassical approximation ξ_{e} does not show any critical behavior at T_c.

The susceptibility also describes the magnetic excitations. Their dispersion is obtained by equating the denominator to zero and solving for ω. We see that these modes are diffusive with relaxation rates

$$\lambda=i\omega=\frac{2S(S+1)\left(\frac{N(0)J}{2}\right)^2}{6\tau_{\mathrm{spin}}+2S(S+1)\frac{[1-N(0)J/2]^2}{2/\tau_{\mathrm{flip}}+Dq^2}}\,\frac{T-T_c}{T_c}. \tag{71}$$

The rate λ is smallest for $\mathbf{q}=0$. The $\mathbf{q}$ dependence is controlled by the total-spin relaxation length ξ_{e}. In the semiclassical approximation λ goes to zero for $T\to T_c$ for all $\mathbf{q}$ simultaneously, but see the discussion below.

We now consider the case of *valence-band holes*. Inserting the appropriate parameter values from Eqs. (55) and (56) we obtain

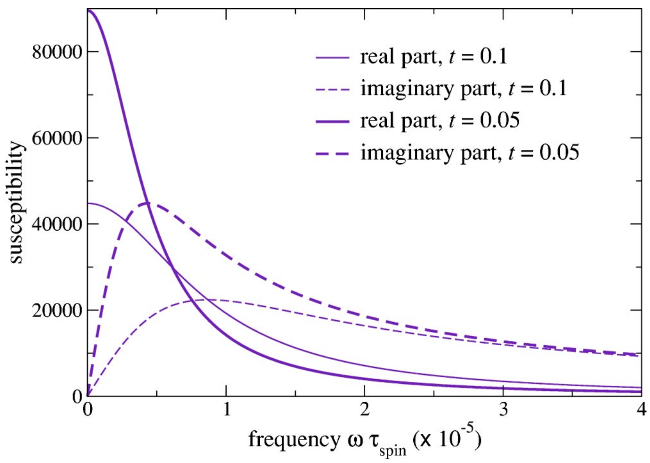

FIG. 1. Real and imaginary parts of the dimensionless impurity-impurity susceptibility $\chi_{\mathrm{ii}}/[N(0)g_{\mathrm{i}}^2\mu_{\mathrm{B}}^2]$ at zero momentum for two values of the reduced temperature $t\equiv(T-T_c)/T_c$. We have used $S=5/2$, $N(0)J/6=0.0061$, and $\tau_{\mathrm{spin}}/\tau=10$.

$$\chi=\frac{5}{18}S(S+1)\,N(0)\,\mu_{\mathrm{B}}^2 \times\left\{-i\omega\left[6\tau_{\mathrm{spin}}+\frac{S(S+1)}{15}\frac{[1-5N(0)J/6]^2}{\tilde{R}_{\mathbf{q}}}\right] +\frac{5}{3}S(S+1)\left(\frac{N(0)J}{6}\right)^2\frac{T-T_c}{T_c}\right\}^{-1} \times\begin{pmatrix} g_{\mathrm{e}}^2\left(\frac{N(0)J}{6}\right)^2 & -g_{\mathrm{e}}g_{\mathrm{i}}\frac{N(0)J}{6} \\ -g_{\mathrm{e}}g_{\mathrm{i}}\frac{N(0)J}{6} & g_{\mathrm{i}}^2 \end{pmatrix} \tag{72}$$

with

$$\tilde{R}_{\mathbf{q}}=\frac{1}{5\tau}+\frac{S(S+1)}{36\tau_{\mathrm{spin}}}+D\frac{3q_x^2+3q_y^2+9q_z^2}{5} =\frac{2}{5\tau_{\mathrm{tot}}}+D\frac{3q_x^2+3q_y^2+9q_z^2}{5}. \tag{73}$$

This susceptibility applies to *p*-type DMS. Compared to the conduction band the only differences except for simple rescaling are that not the parameter $-N(0)J/6$ itself but $-5N(0)J/6$ appears in the **q**-dependent term and that the diffusion is anisotropic.

To exhibit the frequency dependence, Fig. 1 shows the dimensionless impurity-impurity susceptibility $\chi_{\mathrm{ii}}/[N(0)g_{\mathrm{i}}^2\mu_{\mathrm{B}}^2]$ for $\mathbf{q}=0$. The other components of the matrix χ only differ by constant factors. At $\omega=0$ we obtain the Curie law $\chi_{\mathrm{ii}}\propto 1/t$, where $t\equiv(T-T_c)/T_c$, and χ_{ii} is purely real. As a function of frequency, Re χ_{ii} decreases while Im χ_{ii} initially increases and there is a crossover to a predominantly imaginary susceptibility at a characteristic frequency $\omega\sim\lambda\propto t$ given below.

For both the conduction band and the valence band the susceptibilities depend on **q** only through the coefficient of the frequency ω. The *static* susceptibility ($\omega=0$) is thus independent of **q** in our approximation. This would mean that the instability appears simultaneously at all **q**. The tendency of the system to become ferromagnetic is not found within the semiclassical Boltzmann approach since the Boltzmann equation does not incorporate physics at large momenta $q\sim k_F$. We expect the most important effect for $q\sim k_F$ to be the **q** dependence of the Pauli susceptibility.[37] Inserting this dependence by hand, we obtain an additional term of the order of $+q^2/k_F^2$ in the denominator, which makes the instability first appear at $\mathbf{q}=0$, leading to ferromagnetism. A rigorous evaluation of the susceptibility at all momenta requires a fully quantum-mechanical calculation, which we leave as work for the future.

One could think that a ferromagnetic interaction between the carriers themselves introduces a new length scale and might therefore introduce a q^2 term into the denominator of χ. In our approach such a ferromagnetic coupling between the carriers, say holes, leads to an additional term in the effective field,

$$B_{\mathrm{h}}=B_{\mathrm{h}}^{\mathrm{ext}}-\frac{J}{g_{\mathrm{h}}g_{\mathrm{i}}\mu_{\mathrm{B}}^2}\mu_{\mathrm{i}}+\frac{K}{g_{\mathrm{h}}^2\mu_{\mathrm{B}}^2}\mu_{\mathrm{h}} \tag{74}$$

with $K>0$. The derivation can be carried through. The resulting susceptibility for the valence band reads

$$\chi=\frac{5}{18}S(S+1)\,N(0)\,\mu_{\mathrm{B}}^2 \times\left\{-i\omega\left[6\tau_{\mathrm{spin}}+\frac{S(S+1)}{15}\frac{[1-5N(0)J_\kappa/6]^2}{\tilde{R}_{\mathbf{q}}}\right] +\frac{5}{3}S(S+1)\left(\frac{N(0)J_\kappa}{6}\right)^2(1-\kappa)\frac{T-T_c^\kappa}{T_c^\kappa}\right\}^{-1} \times\begin{pmatrix} g_{\mathrm{e}}^2\left(\frac{N(0)J_\kappa}{6}\right)^2 & -g_{\mathrm{e}}g_{\mathrm{i}}\frac{N(0)J_\kappa}{6} \\ -g_{\mathrm{e}}g_{\mathrm{i}}\frac{N(0)J_\kappa}{6} & g_{\mathrm{i}}^2 \end{pmatrix} \tag{75}$$

with $\kappa=N(0)K/6$, $J_\kappa=J/(1-\kappa)$, and[4,40]

$$T_c^\kappa=\frac{S(S+1)}{18}\frac{N(0)\,J^2n_{\mathrm{i}}}{1-\kappa}. \tag{76}$$

The Curie temperature is enhanced by the *Stoner factor* $(1-\kappa)^{-1}$.[49] The same result is obtained by introducing an appropriate Landau parameter $F_0^a=-\kappa$ into Fermi-liquid theory.[4,18] We see that a carrier-carrier ferromagnetic exchange interaction does not change the functional form of the susceptibility. In particular, it does not introduce an Ornstein-Zernicke-type q^2 term.

Let us estimate the parameter $N(0)J/6$: For a parabolic band $N(0)=m_{\mathrm{hh}}/(2\pi^2)(3\pi^2 n)^{1/3}$, where n is the carrier density. For $Ga_{1-x}Mn_xAs$ with $x=0.05$ and $p=0.3$ holes per manganese atom we get $N(0)\approx 7.26\times 10^{-4}$ meV^{-1} nm^{-3}. On the other hand,[19,50] $J\approx(50\pm 5)$ meV nm^3 so that $N(0)J/6\approx 0.0061$. The parameter is thus *small*. However, we emphasize that the derivation is valid for general $N(0)J$. The small value explains why the hole contribution to the magnetization is small compared to the manganese one.[19,51] Also, for antiferromagnetic coupling J is positive so that the hole- and impurity-spin magnetizations are opposite in sign, in agreement with experiments.[19,50–52]

The above estimate of $N(0)$ relies on the spherical approximation and on the omission of the light-hole band, which are not well justified at the hole concentration used here. A realistic Slater-Koster tight-binding description of the unperturbed valence band[53] gives a density of states per spin direction of 1.18×10^{-3} eV^{-1} Å^{-3}. The dimensionless parameter $N(0)J/6\approx 0.0099$ is thus somewhat increased by assuming a realistic band structure.

Equation (73) shows that the typical length scale of χ is $\xi_{\rm h}=\sqrt{5D\tau_{\rm tot}/2}$, which corresponds to $\xi_{\rm e}$ in the conduction-band case. The time appearing in $\xi_{\rm h}$ should thus be the relaxation time of the total magnetization.

The magnetic excitations are again diffusive modes. The relaxation rates of diffusive spin-wave modes with polarization along z are

$$\lambda = i\omega = \frac{\frac{5}{3}S(S+1)\left(\frac{N(0)J}{6}\right)^2}{6\tau_{\rm spin}+\frac{S(S+1)}{15}\frac{[1-5N(0)J/6]^2}{\tilde{R}_{\mathbf{q}}}}\,\frac{T-T_c}{T_c}. \tag{77}$$

To illustrate the momentum dependence, Fig. 2 shows the rate λ as a function of $q_x\xi_{\rm h}$ and $q_z\xi_{\rm h}$. The relaxation of the collective modes is of course much slower than the microscopic time scale $\tau_{\rm spin}$ of spin scattering, with which it is here compared. The dispersion in λ is most pronounced for strong spin scattering and vanishes for $\tau_{\rm spin}/\tau\to\infty$. The anisotropy is also apparent from Fig. 2: λ rises faster in the longitudinal (z) direction. For $T\to T_c$ from above, all rates λ scale towards zero as $T-T_c$ in our semiclassical approach.

We propose to measure the magnetic susceptibility in the paramagnetic phase at small $\mathbf{q}$ and ω for various DMS. In particular, such an experiment should look for the anisotropic spin diffusion in p-type DMS. Studying samples with similar concentrations of magnetic impurities but different concentrations of *nonmagnetic* scatterers introduced by codoping[54] would allow to change the scattering rate $1/\tau$ while holding $1/\tau_{\rm spin}$ and the mean-field T_c nearly fixed.

III. ANOMALOUS HALL-VOLTAGE NOISE

In this section we apply the semiclassical theory to the derivation of the voltage noise in the transverse direction in the paramagnetic phase. The average anomalous Hall voltage vanishes for $T>T_c$ due to the vanishing average magnetization. However, *fluctuations* in the magnetization are present and are in fact critically enhanced as T_c is approached. This leads to fluctuations in the anomalous Hall voltage, which

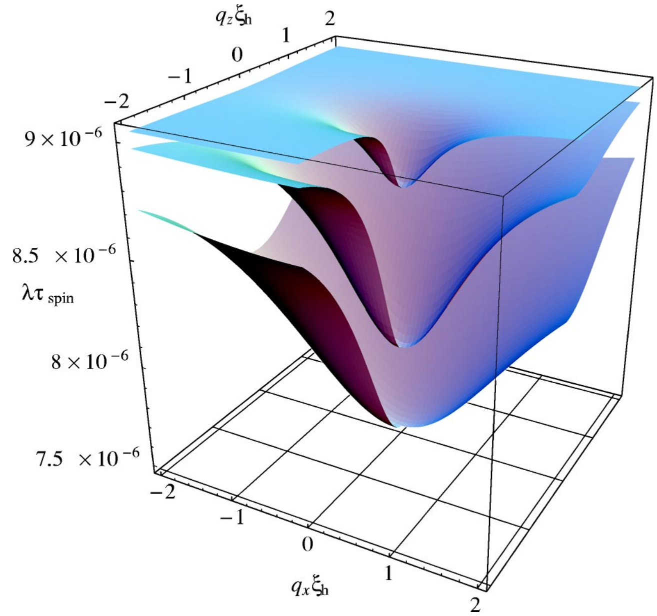

FIG. 2. Relaxation rate λ of diffusive collective spin-wave modes with polarization along z, in natural units of the spin-scattering rate $1/\tau_{\rm spin}$, as a function of $q_x\xi_{\rm h}$ and $q_z\xi_{\rm h}$, where $\xi_{\rm h}=\sqrt{5D\tau_{\rm tot}/2}$ is the typical length scale of the susceptibility. λ is the typical rate in the magnetic susceptibility of p-type DMS. The three sheets are for $\tau_{\rm spin}/\tau=1,2,10$ (from bottom to top). We have used $S=5/2$, $N(0)J/6=0.0061$, and $t=(T-T_c)/T_c=0.1$ (note $\lambda\propto t$).

we derive in the following. Following Ref. 26, we consider the Berry-phase contribution to the anomalous Hall effect for a p-type DMS.

The fluctuations in the Hall voltage are governed by the correlation function of the effective magnetic field acting on the hole spins. This correlation function is closely related to the impurity-impurity spin susceptibility $\chi_{\rm ii}$ evaluated above. Typical Hall-bar samples are much larger than the spin-relaxation length $\xi_{\rm h}$. Hence, we can restrict ourselves to the homogeneous component $\mathbf{q}=0$. Fluctuations with nonzero $\mathbf{q}$ cancel out in the macroscopic voltage measurement. On the other hand, the frequency dependence of the $\chi_{\rm ii}$ is important since ω can become larger than $T-T_c$ close to the transition.

We describe a p-type DMS in the metallic regime by the Hamiltonian

$$H=H_{\rm kin}+Jn_{\rm i}\mathbf{s}\cdot\mathbf{S}-e\mathbf{E}\cdot\mathbf{r}, \tag{78}$$

where $\mathbf{s}$ is the hole-spin operator, $\mathbf{S}$ is the *averaged* ($\mathbf{q}=0$) impurity spin, and $\mathbf{E}$ is a homogeneous, static external electric field. The external magnetic field vanishes. The kinetic Hamiltonian is given in Eq. (29).

The anomalous Hall conductivity has been derived by Jungwirth *et al.*[26] The derivation is similar to the one in Appendix C. The exchange and electric-field terms are treated as small perturbations. The equation of motion of $\mathbf{r}$, Eq. (C1), can be rewritten as[26]

$$\dot{\mathbf{r}}=\nabla_{\mathbf{p}}E_{\mathbf{p}j}-e\mathbf{E}\times\mathbf{\Omega}+2\,{\rm Im}\langle\nabla_{\mathbf{p}}u|\partial_t u\rangle \tag{79}$$

with $\mathbf{\Omega}={\rm Im}\langle\nabla_{\mathbf{p}}u|\times|\nabla_{\mathbf{p}}u\rangle$ and the heavy-hole energy

$$E_{\mathbf{p}j}=\frac{p^2}{2m}(\gamma_1-2\gamma_2)+\frac{Jn_{\rm i}}{3}j\,\hat{\mathbf{p}}\cdot\mathbf{S}-e\mathbf{E}\cdot\mathbf{r}, \tag{80}$$

up to first order in $\mathbf{E}$ and $\mathbf{S}$.

The charge response is derived from the Boltzmann equation $(\partial_t+\dot{\mathbf{r}}\cdot\boldsymbol{\nabla}_{\mathbf{r}}+\dot{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{p}})n_{\mathbf{p}j}=\mathcal{S}^{\rm dis}_{\mathbf{p}j}$. We restrict ourselves to nonmagnetic disorder scattering, assuming the disorder scattering rate $1/\tau$ to be large compared to the spin-scattering rate $1/\tau_{\rm spin}$, since the latter would only complicate the notation without introducing new physics. For the anomalous Hall effect we are concerned with the *charge* density $\rho=e\int d^3p/(2\pi)^3\Sigma_j n_{\mathbf{p}j}$ and current density $\mathbf{j}=e\int d^3p/(2\pi)^3\Sigma_j \mathbf{v}_{\mathbf{p}j}n_{\mathbf{p}j}$.

The leading contribution to the Hall current is found at first order in $\mathbf{S}$,

$$\mathbf{j}_{\rm AH}=\frac{e^2Jn_{\rm i}}{4\pi^2k_F}\mathbf{E}\times\mathbf{S}\left(\frac{1}{\gamma_1-2\gamma_2}-\frac{2}{3\gamma_2}\right)m, \tag{81}$$

where k_F is the Fermi wave number in the heavy-hole band. A homogeneous charge distribution has been assumed to obtain this result. In the limit of large heavy-hole/light-hole mass ratio $m_{\rm hh}/m_{\rm lh}\gg1$ the Kohn-Luttinger parameters satisfy $\gamma_1-2\gamma_2\ll\gamma_2$ and we obtain the simpler result $\mathbf{j}_{\rm AH}=\tilde{\sigma}\,\mathbf{E}\times\mathbf{S}$ with[26]

$$\tilde{\sigma}=\frac{\sigma_{\rm AH}}{S}=\frac{e^2m_{\rm hh}Jn_{\rm i}}{4\pi^2k_F}. \tag{82}$$

Note that the first-order contribution is purely transverse. We see that in the paramagnetic phase the average anomalous Hall current vanishes. However, its fluctuations $\langle\mathbf{j}_{\rm AH}\cdot\mathbf{j}_{\rm AH}\rangle$ do not. We write

$$\langle\mathbf{j}_{\rm AH}\cdot\mathbf{j}_{\rm AH}\rangle=\tilde{\sigma}^2\langle(\mathbf{E}\times\mathbf{S})(\mathbf{E}\times\mathbf{S})\rangle$$
$$=\tilde{\sigma}^2\left(E^2\langle\mathbf{S}\cdot\mathbf{S}\rangle-\sum_{\alpha\beta}E_\alpha E_\beta\langle S^\alpha S^\beta\rangle\right). \tag{83}$$

In the paramagnetic phase this gives

$$\langle\mathbf{j}_{\rm AH}(t)\cdot\mathbf{j}_{\rm AH}(0)\rangle=2\tilde{\sigma}^2E^2\,\langle S^z(t)\,S^z(0)\rangle. \tag{84}$$

The time-dependent correlation function can be expressed by the *impurity-impurity* part $\chi_{\rm ii}$ of the susceptibility in the p-type case with the help of the fluctuation-dissipation theorem,[37]

$$\int dt\,e^{-i\omega t}\langle S^z(\mathbf{r},t)\,S^z(0,0)\rangle\cong\frac{2T\,{\rm Im}\,\chi^{zz}_{\rm ii}(\mathbf{r},\omega)}{g_{\rm i}^2\mu_{\rm B}^2n_{\rm i}^2\omega} \tag{85}$$

for $\omega\ll T$.

We now evaluate the correlation function of the anomalous Hall voltage $U_{\rm AH}$ between the front and back sides of the relevant Hall-bar region shown in Fig. 3. Since Coulomb interaction suppresses charge fluctuations the current density is assumed to be homogeneous; deviations are only expected to occur at frequencies of the order of the plasma frequency. We can then write the anomalous Hall voltage as $U_{\rm AH}=L_yj^y_{\rm AH}/\sigma_D$, where $\sigma_D=e^2n_{\rm h}\tau/m_{\rm hh}$ is the Drude conductivity. If the electric field is applied in the x direction, the anomalous Hall current density in the y direction is $j^y_{\rm AH}=-\tilde{\sigma}ES^z$. Due to homogeneity we can average the current over the sample volume. The voltage correlation function is then

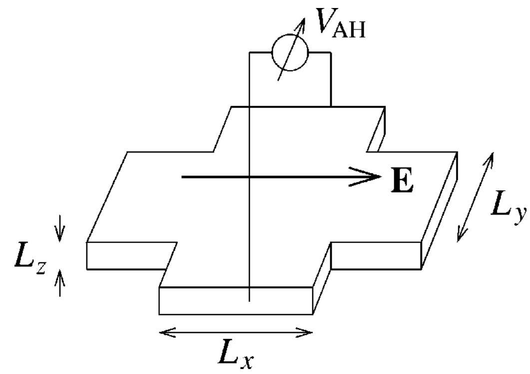

FIG. 3. Geometry of the relevant section of the Hall bar.

$$\langle U_{\rm AH}(t)\,U_{\rm AH}(0)\rangle=\frac{\tilde{\sigma}^2}{\sigma_D^2}\frac{L_y}{L_xL_z}E^2\int d^3r\langle S^z(\mathbf{r},t)S^z(0,0)\rangle. \tag{86}$$

Taking the Fourier transform, expressing the electric field by the voltage applied to the relevant sample region, $E=U/L_x$, and inserting Eq. (85) we obtain

$$\langle U_{\rm AH}U_{\rm AH}\rangle_\omega=\int dt\,e^{-i\omega t}\langle U_{\rm AH}(t)\,U_{\rm AH}(0)\rangle$$
$$=\frac{\tilde{\sigma}^2}{\sigma_D^2}\frac{L_y}{L_x^3L_z}U^2\frac{2\,{\rm Im}\,\chi^{zz}_{\rm ii}(\mathbf{q}=0,\omega)}{g_{\rm i}^2\mu_{\rm B}^2n_{\rm i}^2(1-e^{-\omega/t})}. \tag{87}$$

Assuming $1/\tau\gg1/\tau_{\rm spin}$ and $N(0)J\ll1$, Eq. (72) gives, to leading order in ω,

$${\rm Im}\,\chi^{zz}_{\rm ii}(0,\omega)\cong\omega\frac{3N(0)g_{\rm i}^2\mu_{\rm B}^2\,\tau_{\rm spin}}{5\,S(S+1)\left(\frac{N(0)J}{6}\right)^4\left(\frac{T-T_c}{T_c}\right)^2}. \tag{88}$$

This leads to

$$\frac{\langle U_{\rm AH}U_{\rm AH}\rangle_\omega}{U^2}\cong\frac{\tilde{\sigma}^2}{\sigma_D^2}\frac{L_y}{L_x^3L_z}T\frac{6N(0)\,\tau_{\rm spin}}{5\,S(S+1)\,n_{\rm i}^2\left(\frac{N(0)J}{6}\right)^4\left(\frac{T-T_c}{T_c}\right)^2} \tag{89}$$

so that the noise spectrum is independent of ω for small ω. Close to the Curie temperature the integrated noise $\langle U^2_{\rm AH}\rangle=\langle U_{\rm AH}U_{\rm AH}\rangle_\omega\Delta\omega$ with the detector bandwidth $\Delta\omega=2\pi\Delta f$ satisfies

$$\frac{\langle U^2_{\rm AH}\rangle}{U^2}\cong\frac{\tilde{\sigma}^2}{\sigma_D^2}\frac{L_y}{L_x^3L_z}\frac{12\Delta\omega\,\tau_{\rm spin}}{5n_{\rm i}\left(\frac{N(0)J}{6}\right)^2\left(\frac{T-T_c}{T_c}\right)^2}. \tag{90}$$

The ratio of conductivities is

$$\frac{\tilde{\sigma}}{\sigma_D} = \frac{3}{2}\frac{n_\mathrm{i}}{n_\mathrm{h}}\frac{N(0)J}{6}\frac{1}{E_F\tau} \tag{91}$$

with the Fermi energy $E_F = k_F^2/2m_\mathrm{hh}$. The factor $n_\mathrm{i}/n_\mathrm{h}$ lies in the range $1,\ldots,10$, the ubiquitous factor $N(0)J/6$ drops out of the final result, and $1/E_F\tau$ has to be reasonably small for our metallic picture to apply. The final dimensionless expression for the integrated noise is

$$\frac{\langle U_\mathrm{AH}^2\rangle}{U^2} \cong \frac{27}{5}\left(\frac{n_\mathrm{i}}{n_\mathrm{h}}\frac{1}{E_F\tau}\right)^2 \frac{L_y}{L_x^3L_z}\frac{1}{n_\mathrm{i}}\left(\frac{T_c}{T-T_c}\right)^2 \Delta\omega\,\tau_\mathrm{spin}. \tag{92}$$

This contribution to the noise is critically enhanced as the Curie temperature is approached. In a homogeneous system it should diverge at T_c but real DMS are, by their very nature, disordered and the transition is broadened by macroscopic inhomogeneity of T_c. Furthermore, the effect strongly depends on the length L_x of the relevant region of the Hall bar in the electric-field direction, being large for small L_x. It is more weakly enhanced by a small sample thickness L_z and by a *large* sample width L_y across which the voltage is measured. The effect is also increased by strong compensation ($n_\mathrm{h} \ll n_\mathrm{i}$) and in samples showing bad metallic behavior (small $E_F\tau$).

The anomalous Hall-voltage noise is in competition with the *thermal* (Johnson-Nyquist) voltage noise,[55] which in integrated form is $\langle U_\mathrm{th}^2\rangle = 2TR\,\Delta\omega/\pi = 2T(L_y/\sigma_D L_x L_z)\Delta\omega/\pi$. The two contributions can be experimentally distinguished by their different temperature and voltage dependences. The anomalous Hall-voltage noise $\langle U_\mathrm{AH}^2\rangle$ is proportional to the applied voltage squared, whereas the thermal voltage noise is independent of voltage.

Besides being an interesting physical effect, measurement of the anomalous Hall-voltage noise would provide an independent approach to the impurity-spin susceptibility and to important experimental parameters, such as the compensation fraction $n_\mathrm{h}/n_\mathrm{i}$ with respect to the density of magnetically active impurities. The Hall-voltage noise would also provide a new way to determine the Curie temperature. More generally, such experiments would test the applicability of the semiclassical theory to DMS.[26] It may also be interesting to study the anomalous Hall-voltage noise in conventional itinerant ferromagnets such as iron.

IV. CONCLUSIONS

A semiclassical approach based on Boltzmann equations for electrons or holes and impurity spins has been used to derive hydrodynamic equations of motion and spin susceptibilities of DMS in the paramagnetic phase. This theory gives the leading frequency and wave-vector dependence at small ω and $\mathbf{q}$. Our results apply to *p*-type and *n*-type DMS, to III-V, II-VI, and group-IV host semiconductors, arbitrary impurity spin quantum number *S*, and ferromagnetic or antiferromagnetic exchange coupling *J* of carrier and impurity spins. While the form of the equations of motion is easy to understand, the susceptibility has a nonstandard $\mathbf{q}$ dependence, which only appears in the frequency-dependent term. Thus the semiclassical diffusive dynamics does not lead to any $\mathbf{q}$ dependence of the *static* susceptibility. Such terms are expected to be introduced by physics at the much larger momentum scale of the Fermi momentum k_F.

Spin-orbit coupling in the valence band leads to qualitative differences in the susceptibility of holes compared to electrons. The first difference is a suppression of the mean-field Curie temperature of *p*-type DMS compared to *n*-type DMS by a factor of 1/3, which can be traced back directly to the momentum dependence of the spin quantization axis in the presence of spin-orbit coupling. On the other hand, the Curie temperature in *p*-type DMS is enhanced by the typically larger density of states and exchange coupling. The second difference is the anisotropic spin diffusion in the valence band, which is apparent in the equation of motion of the hole magnetization and also makes the $\mathbf{q}$ dependence of the susceptibilities anisotropic. The anisotropic diffusion is due to the fact that holes moving in a direction perpendicular to the magnetization or effective field have vanishing expectation value of the spin in the magnetization direction and thus do not contribute to its transport.

The results have been applied to evaluate the *noise* in the anomalous Hall voltage in DMS, which is governed by the impurity-spin susceptibility at small frequencies and momentum $\mathbf{q}\to 0$. Unlike the *average* anomalous Hall voltage this quantity does not vanish in the paramagnetic phase and is even critically enhanced close to T_c. The noise gives an independent experimental approach to the impurity-spin susceptibility. We have derived the detailed dependence of the signal on the impurity and hole concentrations and on the sample geometry.

ACKNOWLEDGMENTS

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APPENDIX A: HYDRODYNAMIC EQUATIONS, CONDUCTION BAND

In this appendix we collect a number of calculations pertaining to the conduction-band case. The derivation of the hydrodynamic equations in Sec. II A requires the evaluation of various integrals over the collision terms $\mathcal{S}_{\mathbf{p}\sigma}^\mathrm{dis}$, $\mathcal{S}_{\mathbf{p}\sigma m}^\mathrm{spin}$, and $\mathcal{S}_{\mathbf{p}\sigma}^\mathrm{flip}$. We do not show all evaluations but only present a few to clarify the method and approximations used here.

The first integral we need is

$$\int \frac{d^3p}{(2\pi)^3}\sum_{\sigma m} m\,\mathcal{S}_{\mathbf{p}\sigma m}^\mathrm{spin}, \tag{A1}$$

which appears in the equation of motion (16) of the impurity-spin magnetization. We divide the collision integral into three terms,

$$\mathcal{S}^{\rm spin}_{\mathbf{p}\sigma m}=\mathcal{S}^{\rm spin,0}_{\mathbf{p}\sigma m}+\mathcal{S}^{\rm spin,+1}_{\mathbf{p}\sigma m}+\mathcal{S}^{\rm spin,-1}_{\mathbf{p}\sigma m}, \tag{A2}$$

corresponding to $m'=m$ (no spin flip), $m'=m+1$, and $m'=m-1$, respectively. The first contribution is

$$\int\frac{d^3p}{(2\pi)^3}\sum_{\sigma m} m\,\mathcal{S}^{\rm spin,0}_{\mathbf{p}\sigma m}=\frac{\sum_m m^3 f_m}{4N(0)\tau_{\rm spin}}\int\frac{d^3p\,d^3p'}{(2\pi)^6}\times\sum_{\sigma m}\delta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{p}'})(n_{\mathbf{p}'\sigma}-n_{\mathbf{p}\sigma})=0, \tag{A3}$$

as can be seen by renaming $\mathbf{p}\leftrightarrow\mathbf{p}'$ in the term with $n_{\mathbf{p}\sigma}$. The other two contributions can be treated together as

$$\int\frac{d^3p}{(2\pi)^3}\sum_{\sigma m} m\,\mathcal{S}^{\rm spin,\pm1}_{\mathbf{p}\sigma m}=\frac{1}{4N(0)\tau_{\rm spin}}\int\frac{d^3p\,d^3p'}{(2\pi)^6}\sum_m\delta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{p}'}\pm g_e\mu_B B_e\mp g_i\mu_B B_i)m\,[S(S+1)-m(m\pm1)]\times[n_{\mathbf{p}'\mp}(1-n_{\mathbf{p}\pm})f_{m\pm1}-n_{\mathbf{p}\pm}(1-n_{\mathbf{p}'\mp})f_m]. \tag{A4}$$

We write $f_m=1/(2S+1)+\Delta f_m$, where $\Sigma_m\Delta f_m=0$, and divide the integral into terms of zero and first order in Δf_m,

$$\int\frac{d^3p}{(2\pi)^3}\sum_{\sigma m} m\,\mathcal{S}^{\rm spin,\pm1}_{\mathbf{p}\sigma m}=\Sigma^{(0)}+\Sigma^{(1)}. \tag{A5}$$

In the zero-order term we expand the δ function in B_e, B_i, and write all terms strictly in first order. This allows to perform the integrals,

$$\Sigma^{(0)}\cong\frac{1}{4\tau_{\rm spin}}\sum_m\frac{\mp m^2}{2S+1}\Bigg[\int\frac{d^3p'}{(2\pi)^3}n_{\mathbf{p}'\mp}-\int\frac{d^3p}{(2\pi)^3}n_{\mathbf{p}\pm}+N(0)\int d\xi\,d\xi'\,\delta^{(1)}(\xi-\xi')[n_F(\xi')-n_F(\xi)]\times(\pm g_e\mu_B B_e\mp g_i\mu_B B_i)\Bigg]=-\frac{S(S+1)}{6\tau_{\rm spin}}\frac{\mu_e}{g_e\mu_B}+\frac{N(0)\,S(S+1)}{12\tau_{\rm spin}}\times(g_e\mu_B B_e-g_i\mu_B B_i). \tag{A6}$$

We have used partial integration in the last term. The term $\Sigma^{(1)}$ is explicitly of first order in Δf_m so that all other factors are to be evaluated in field-free equilibrium,

$$\Sigma^{(1)}=\frac{N(0)T}{4\tau_{\rm spin}}\sum_m m[S(S+1)-m(m\pm1)](\Delta f_{m\pm1}-\Delta f_m). \tag{A7}$$

In the sum we replace m by $m\mp1$ in the term containing $f_{m\pm1}$. If we still sum over m from $-S$ to S, we expect additional contributions at both ends, but these vanish due to the factor $S(S+1)-m(m\pm1)$. Thus we obtain $\Sigma^{(1)}=N(0)T/4\tau_{\rm spin}\Sigma_m[-m\pm3m^2\mp S(S+1)]\Delta f_m$. This expression obviously simplifies when the contributions from $\mathcal{S}^{\rm spin,+1}$ and $\mathcal{S}^{\rm spin,-1}$ are added. The contribution from $\mathcal{S}^{\rm spin,0}$ vanishes anyway. Consequently, the result for the full integral is

$$\int\frac{d^3p}{(2\pi)^3}\sum_{\sigma m} m\,\mathcal{S}^{\rm spin}_{\mathbf{p}\sigma m}=-\frac{S(S+1)}{3\tau_{\rm spin}}\frac{\mu_e}{g_e\mu_B}+\frac{N(0)T}{2\tau_{\rm spin}}\frac{\mu_i}{g_e\mu_B n_i}+\frac{N(0)S(S+1)}{6\tau_{\rm spin}}(g_e\mu_B B_e-g_i\mu_B B_i). \tag{A8}$$

Note that we have expressed this result in terms of μ_i instead of the occupation fractions f_m. This can be done in all our results so that a closed set of equations for the two magnetizations μ_e and μ_i is obtained.

The integrals required for the equation of motion of μ_e are quite similar. In the equation for the magnetization current $\mathbf{j}_\mu$ we need integrals such as

$$\int\frac{d^3p}{(2\pi)^3}\sum_\sigma\sigma\mathbf{v}_{\mathbf{p}}\mathcal{S}^{\rm dis}_{\mathbf{p}\sigma}=\frac{1}{N(0)\tau}\int\frac{d^3p\,d^3p'}{(2\pi)^6}\sum_\sigma\sigma\mathbf{v}_{\mathbf{p}}\,\delta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{p}'})\,(n_{\mathbf{p}'\sigma}-n_{\mathbf{p}\sigma})=-\frac{1}{\tau}\int\frac{d^3p}{(2\pi)^3}\sum_\sigma\sigma\mathbf{v}_{\mathbf{p}}\,n_{\mathbf{p}\sigma}=\frac{1}{\tau}\frac{\mathbf{j}_\mu}{g_e\mu_B}, \tag{A9}$$

where the term with $n_{\mathbf{p}'\sigma}$ vanishes since it is odd in $\mathbf{p}$. Similar evaluations are required for $\mathcal{S}^{\rm flip}$ and $\mathcal{S}^{\rm spin}$.

APPENDIX B: HYDRODYNAMIC EQUATIONS, VALENCE BAND

Even though we restrict ourselves to the heavy-hole band, the angular integrals are much more complicated than in the conduction-band case since the explicit expression (40) for the hole magnetization μ_h, the transition probabilities, and the Zeeman energies now all depend on the direction in momentum space. As noted above, the analytical expressions for the transition probabilities are rather complicated. We use MATHEMATICA to analytically perform the angular integrals of the form

$$\int\frac{d\Omega}{4\pi}\cos^n\theta\,|{}_{\mathbf{p}}\langle j|A|j'\rangle_{\mathbf{p}'}|^2 \tag{B1}$$

with $n=0,1,2$ and $A=1,j^z,j^+,j^-$, resulting in expressions such as

$$\int \frac{d\Omega}{4\pi}\cos^2\theta |{}_{\mathbf{p}}\langle j|j^{\pm}|j'\rangle_{\mathbf{p}'}|^2 = \frac{3}{40}\begin{cases}(7-6\cos\theta'+\cos 2\theta')\cos^2\frac{\theta'}{2} & \text{for } j'=\mp 3/2\\ (7+6\cos\theta'+\cos 2\theta')\sin^2\frac{\theta'}{2} & \text{for } j'=\pm 3/2.\end{cases} \tag{B2}$$

Here, θ and θ' are polar angles of $\mathbf{p}$ and $\mathbf{p}'$, respectively.

As an example, we here evaluate the integral

$$\int \frac{d^3p}{(2\pi)^3}\sum_{jm} m\,\mathcal{S}^{\text{spin}}_{\mathbf{p}jm}, \tag{B3}$$

which corresponds to the one considered in Appendix A. The collision integral is again divided into $\mathcal{S}^{\text{spin},0}+\mathcal{S}^{\text{spin},+1}+\mathcal{S}^{\text{spin},-1}$. The contribution from $\mathcal{S}^{\text{spin},0}$ vanishes in analogy with Eq. (A3). The other terms are expanded in Δf_m up to linear order,

$$\int \frac{d^3p}{(2\pi)^3}\sum_{jm} m\mathcal{S}^{\text{spin},\pm 1}_{\mathbf{p}jm}=\Sigma^{(0)}+\Sigma^{(1)}. \tag{B4}$$

In $\Sigma^{(0)}$ the δ function is expanded in B_{h}, B_{i} and the term is then divided into $\Sigma^{(0)}_{\text{noflip}}+\Sigma^{(0)}_{\text{flip}}$, where in the first (second) term $j'=j$ ($j'=-j$). The first term is evaluated similarly to the conduction-band case, taking the more complicated angular integrals (B1) into account,

$$\Sigma^{(0)}_{\text{noflip}}=-\frac{S(S+1)}{108\tau_{\text{spin}}}\left[\frac{3}{2}\frac{\mu_{\text{h}}}{g_{\text{h}}\mu_{\text{B}}}+\frac{N(0)}{4}g_{\text{h}}\mu_{\text{B}}B_{\text{h}} - \frac{5}{4}N(0)\,g_{\text{i}}\mu_{\text{B}}B_{\text{i}}\right]. \tag{B5}$$

Also, writing out $\Sigma^{(0)}_{\text{flip}}$ and renaming $j\leftrightarrow -j$ in the first term one can see that $\Sigma^{(0)}_{\text{flip}}=\Sigma^{(0)}_{\text{noflip}}$. $\Sigma^{(1)}$ can also be evaluated similarly to the conduction-band case,

$$\Sigma^{(1)}=\frac{N(0)T}{36\tau_{\text{spin}}}\frac{5}{2}\sum_m [-m\pm 3m^2\mp S(S+1)]\,\Delta f_m, \tag{B6}$$

which simplifies under summation over the three contributions,

$$\int \frac{d^3p}{(2\pi)^3}\sum_{jm} m\,\mathcal{S}^{\text{spin}}_{\mathbf{p}jm} = -\frac{S(S+1)}{18\tau_{\text{spin}}}\frac{\mu_{\text{h}}}{g_{\text{h}}\mu_{\text{B}}}+\frac{N(0)\,S(S+1)}{108\tau_{\text{spin}}}g_{\text{h}}\mu_{\text{B}}B_{\text{h}} + \frac{5N(0)T}{36\tau_{\text{spin}}}\frac{\mu_{\text{i}}}{g_{\text{i}}\mu_{\text{B}}n_{\text{i}}}-\frac{5N(0)\,S(S+1)}{108\tau_{\text{spin}}}g_{\text{i}}\mu_{\text{B}}B_{\text{i}}. \tag{B7}$$

In the integrals pertaining to the hole magnetization and magnetization current we obtain some terms in which the occupation fractions f_m cannot be reduced to μ_m. These terms cancel in the final equations of motion so that again a closed set of equations for μ_{h} and μ_{i} is obtained.

APPENDIX C: ABSENCE OF BERRY-PHASE CONTRIBUTIONS

In the present appendix we show that Berry-phase corrections do not contribute to the hydrodynamic equations to linear order. In the framework of semiclassical theory they have been discussed in detail by Sundaram and Niu.[27] If one considers a wave packet made up of electrons of a single band, with narrow spread in real and momentum space, and with center-of-mass position $\mathbf{r}$ and mean momentum $\mathbf{p}$, then the semiclassical equations of motion for these quantities are, in the absence of scattering,[27]

$$\begin{aligned}\dot{\mathbf{r}}=&\nabla_{\mathbf{p}}\tilde{E}_{\mathbf{p}\sigma}-i\dot{p}_\alpha(\langle\nabla_{\mathbf{p}}u|\nabla^\alpha_{\mathbf{p}}u\rangle-\langle\nabla^\alpha_{\mathbf{p}}u|\nabla_{\mathbf{p}}u\rangle)\\&-i\dot{r}_\alpha(\langle\nabla_{\mathbf{p}}u|\nabla^\alpha_{\mathbf{r}}u\rangle-\langle\nabla^\alpha_{\mathbf{r}}u|\nabla_{\mathbf{p}}u\rangle)\\&-i(\langle\nabla_{\mathbf{p}}u|\partial_t u\rangle-\langle\partial_t u|\nabla_{\mathbf{p}}u\rangle),\end{aligned} \tag{C1}$$

$$\begin{aligned}\dot{\mathbf{p}}=&-\nabla_{\mathbf{r}}\tilde{E}_{\mathbf{p}\sigma}+i\dot{p}_\alpha(\langle\nabla_{\mathbf{r}}u|\nabla^\alpha_{\mathbf{p}}u\rangle-\langle\nabla^\alpha_{\mathbf{p}}u|\nabla_{\mathbf{r}}u\rangle)\\&+i\dot{r}_\alpha(\langle\nabla_{\mathbf{r}}u|\nabla^\alpha_{\mathbf{r}}u\rangle-\langle\nabla^\alpha_{\mathbf{r}}u|\nabla_{\mathbf{r}}u\rangle)\\&+i(\langle\nabla_{\mathbf{r}}u|\partial_t u\rangle-\langle\partial_t u|\nabla_{\mathbf{r}}u\rangle).\end{aligned} \tag{C2}$$

Summation over α is implied. $|u\rangle=|u_{\mathbf{p}\sigma}\rangle$ is the periodic part of the Bloch wave function and $\tilde{E}_{\mathbf{p}\sigma}$ is the wave-packet energy with a Berry-phase correction,[27]

$$\tilde{E}_{\mathbf{p}\sigma}=E_{\mathbf{p}\sigma}-\text{Im}\,\langle\nabla_{\mathbf{r}}u_{\mathbf{p}\sigma}|\cdot(E_{\mathbf{p}\sigma}-H_c)|\nabla_{\mathbf{p}}u_{\mathbf{p}\sigma}\rangle, \tag{C3}$$

where H_c is the local Hamiltonian for the wave-packet center and momentum and $E_{\mathbf{p}\sigma}$ is the corresponding eigenenergy. In the hole case σ should be replaced by j. Note that the spatial gradient $\nabla_{\mathbf{r}}$ acts on the *center-of-mass* vector, on which the states $|u\rangle$ depend parametrically.

For the conduction band we can immediately see that Berry-phase effects are absent: In field-free equilibrium all spatial and temporal derivatives vanish. The $\mathbf{p}$ gradients also vanish since for the Hamiltonian $H^{(0)}=p^2/(2m_{\text{cb}})$ the periodic part $|u^{(0)}\rangle$ of the Bloch wave function is constant and the spin part $|\pm 1/2\rangle$ is also independent of $\mathbf{p}$. This is not changed by the Zeeman term since it commutes with the kinetic energy in the absence of spin-orbit coupling. Thus all terms in Eqs. (C1)–(C3) vanish.

For the valence band in the spherical approximation, Eq. (29), the spin part of the Bloch wave function is given by Eq. (30). The $\mathbf{p}$ gradient is then

$$\nabla_{\mathbf{p}}|j\rangle_{\mathbf{p}}=-i\left(j^z\frac{\hat{\boldsymbol{\phi}}}{p\sin\theta}e^{-ij^z\phi}e^{-ij^y\theta}+e^{-ij^z\phi}j^y\frac{\hat{\boldsymbol{\theta}}}{p}e^{-ij^y\theta}\right)|j\rangle. \tag{C4}$$

Furthermore, the Zeeman term does not commute with the kinetic energy so that we expect contributions from the perturbation. We use a perturbation expansion in the effective field to obtain the terms appearing in Eqs. (C1) and (C2). The hole Hamiltonian in the spherical approximation reads

$$H=\underbrace{\frac{1}{2m}[(\gamma_1+5\gamma_2/2)\,p^2-2\gamma_2(\mathbf{p}\cdot\mathbf{j})^2]}_{H_0}+\underbrace{g_{\mathrm{h}}\mu_{\mathrm{B}}\,\mathbf{s}\cdot\hat{\mathbf{z}}\,B_{\mathrm{h}}}_{H_1}. \tag{C5}$$

The unperturbed eigenenergies are

$$\epsilon_{\mathbf{p}j}^{(0)}=\frac{p^2}{2m}\begin{cases}(\gamma_1-2\gamma_2) & \text{for } j=\pm 3/2\\ (\gamma_1+2\gamma_2) & \text{for } j=\pm 1/2\end{cases} \tag{C6}$$

and the eigenstates are $|u_{\mathbf{p}j}^{(0)}\rangle$, where only the spin part has a nontrivial $\mathbf{p}$ dependence. Assuming an effective field in the z direction, the first-order perturbation is

$$\epsilon_{\mathbf{p}j}^{(1)}=g_{\mathrm{h}}\mu_{\mathrm{B}}\,\tfrac{1}{3}\,\langle j|e^{ij^y\theta}e^{ij^z\phi}\,j^z\,e^{-ij^z\phi}e^{-ij^y\theta}|j\rangle\,B_{\mathrm{h}}. \tag{C7}$$

Restricted to heavy holes, $\epsilon_{\mathbf{p}j}^{(1)}=g_{\mathrm{h}}\mu_{\mathrm{B}}(j/3)\cos\theta B_{\mathrm{h}}$. Degenerate perturbation theory yields the perturbations to the states,

$$|u_{\mathbf{p},\pm 3/2}^{(1)}\rangle=g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{1}{3}B_{\mathrm{h}}\left(\frac{\langle u_{\mathbf{p},1/2}^{(0)}|\,j^z\,|u_{\mathbf{p},\pm 3/2}^{(0)}\rangle}{\epsilon_{\mathbf{p},\pm 3/2}^{(0)}-\epsilon_{\mathbf{p},1/2}^{(0)}}|u_{\mathbf{p},1/2}^{(0)}\rangle+\frac{\langle u_{\mathbf{p},-1/2}^{(0)}|\,j^z\,|u_{\mathbf{p},\pm 3/2}^{(0)}\rangle}{\epsilon_{\mathbf{p},\pm 3/2}^{(0)}-\epsilon_{\mathbf{p},-1/2}^{(0)}}|u_{\mathbf{p},-1/2}^{(0)}\rangle\right). \tag{C8}$$

Introducing the difference between heavy- and light-hole energies, $g_{\mathbf{p}}=-2\gamma_2 p^2/m$, we obtain

$$|u_{\mathbf{p},\pm 3/2}^{(1)}\rangle=-\frac{g_{\mathrm{h}}\mu_{\mathrm{B}}\sin\theta\,B_{\mathrm{h}}}{2\sqrt{3}\,g_{\mathbf{p}}}|u_{\mathbf{p},\pm 1/2}^{(0)}\rangle. \tag{C9}$$

Simplifying the notation by writing only the spin part of the wave function, this gives

$$|u_{\mathbf{p},\pm 3/2}^{(1)}\rangle=-\frac{g_{\mathrm{h}}\mu_{\mathrm{B}}\sin\theta\,B_{\mathrm{h}}}{2\sqrt{3}\,g_{\mathbf{p}}}e^{-ij^z\phi}e^{-ij^y\theta}\left|\pm\frac{1}{2}\right\rangle. \tag{C10}$$

The Berry-phase correction for the energy of heavy holes, given in Eq. (C3), is, to first order,

$$\begin{aligned}\Delta\epsilon_{\mathbf{p}j}&=-\mathrm{Im}\,\langle\nabla_{\mathbf{r}}u_{\mathbf{p}j}^{(1)}|\cdot(\epsilon_{\mathbf{p}j}^{(0)}-H_c^{(0)})|\nabla_{\mathbf{p}}u_{\mathbf{p}j}^{(0)}\rangle\\&=\mathrm{Im}\,\frac{g_{\mathrm{h}}\mu_{\mathrm{B}}\sin\theta\,\nabla_{\mathbf{r}}B_{\mathrm{h}}}{2\sqrt{3}\,g_{\mathbf{p}}}\cdot\langle u_{\mathbf{p},j/3}^{(0)}|(\epsilon_{\mathbf{p}j}^{(0)}-H_0)|\nabla_{\mathbf{p}}u_{\mathbf{p}j}^{(0)}\rangle.\end{aligned} \tag{C11}$$

Using that $|u_{\mathbf{p},j/3}^{(0)}\rangle$ is an eigenstate of H_0 we obtain

$$\begin{aligned}\Delta\epsilon_{\mathbf{p}j}&=\frac{g_{\mathrm{h}}\mu_{\mathrm{B}}\sin\theta\,\nabla_{\mathbf{r}}B_{\mathrm{h}}}{2\sqrt{3}}\cdot\mathrm{Im}\left[i\frac{\hat{\boldsymbol{\phi}}}{p\sin\theta}\frac{\sqrt{3}}{2}\sin\theta-i\frac{\hat{\boldsymbol{\theta}}}{p}\left(\pm i\frac{\sqrt{3}}{2}\right)\right]\\&=\frac{g_{\mathrm{h}}\mu_{\mathrm{B}}}{4p}(\hat{\mathbf{z}}\times\hat{\mathbf{p}})\cdot\nabla_{\mathbf{r}}B_{\mathrm{h}}.\end{aligned} \tag{C12}$$

This correction evidently diverges for small $\mathbf{p}$. The origin is the breakdown of perturbation theory as the energy difference $g_{\mathbf{p}}$ between heavy and light holes goes to zero. This divergence is not crucial here since states deep inside the Fermi sea do not contribute to the response.

The energy entering the semiclassical equations of motion is, to first order, $\tilde{E}_{\mathbf{p}j}=\epsilon_{\mathbf{p}j}^{(0)}+\epsilon_{\mathbf{p}j}^{(1)}+\Delta\epsilon_{\mathbf{p}j}$. Thus Eq. (C2) reads, to first order,

$$\begin{aligned}\dot{\mathbf{p}}&\cong-\nabla_{\mathbf{r}}\tilde{E}_{\mathbf{p}\sigma}\\&\cong-g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\cos\theta\,\nabla_{\mathbf{r}}B_{\mathrm{h}}-\frac{g_{\mathrm{h}}\mu_{\mathrm{B}}}{4p}\nabla_{\mathbf{r}}[(\hat{\mathbf{z}}\times\hat{\mathbf{p}})\cdot\nabla_{\mathbf{r}}B_{\mathrm{h}}].\end{aligned} \tag{C13}$$

Thus we find an additional force which is proportional to a second derivative of the field but independent of the spin direction, i.e., an *orbital* contribution. Then Eq. (C1) becomes, dropping subscripts $\mathbf{p}$, j,

$$\begin{aligned}\dot{\mathbf{r}}&\cong\nabla_{\mathbf{p}}\tilde{E}-i\dot{p}_\alpha(\langle\nabla_{\mathbf{p}}u^{(0)}|\nabla_{\mathbf{p}}^\alpha u^{(0)}\rangle-\langle\nabla_{\mathbf{p}}^\alpha u^{(0)}|\nabla_{\mathbf{p}}u^{(0)}\rangle)\\&\quad-i\nabla_{\mathbf{p}}^\alpha\epsilon^{(0)}(\langle\nabla_{\mathbf{p}}u^{(0)}|\nabla_{\mathbf{r}}^\alpha u\rangle-\langle\nabla_{\mathbf{r}}^\alpha u|\nabla_{\mathbf{p}}u^{(0)}\rangle)\\&\quad-i(\langle\nabla_{\mathbf{p}}u^{(0)}|\partial_t u\rangle-\langle\partial_t u|\nabla_{\mathbf{p}}u^{(0)}\rangle).\end{aligned} \tag{C14}$$

The term multiplying $\dot{p}_\alpha$ can be evaluated explicitly and is found to vanish for the heavy holes. Thus

$$\begin{aligned}\dot{\mathbf{r}}&\cong\frac{\mathbf{p}}{m_{\mathrm{hh}}}-g_{\mathrm{h}}\mu_{\mathrm{B}}\frac{j}{3}\sin\theta B_{\mathrm{h}}\frac{\hat{\boldsymbol{\theta}}}{p}\\&\quad-\mathrm{Im}\,\nabla_{\mathbf{p}}\langle\nabla_{\mathbf{r}}u|\cdot(\epsilon_{\mathbf{p}j}^{(0)}-H_c^{(0)})|\nabla_{\mathbf{p}}u^{(0)}\rangle\\&\quad-i\frac{p_\alpha}{m_{\mathrm{hh}}}(\langle\nabla_{\mathbf{p}}u^{(0)}|\nabla_{\mathbf{r}}^\alpha u\rangle-\langle\nabla_{\mathbf{r}}^\alpha u|\nabla_{\mathbf{p}}u^{(0)}\rangle)\\&\quad-i(\langle\nabla_{\mathbf{p}}u^{(0)}|\partial_t u\rangle-\langle\partial_t u|\nabla_{\mathbf{p}}u^{(0)}\rangle),\end{aligned} \tag{C15}$$

cf. Eq. (33). To first order, the Boltzmann equation reads

$$\partial_t n_{\mathbf{p}j}+\frac{\mathbf{p}}{m_{\mathrm{hh}}}\cdot\nabla_{\mathbf{r}}n_{\mathbf{p}j}+\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}}^{(0)}=\mathcal{S}_{\mathbf{p}j}^{\mathrm{dis}}+\sum_m\mathcal{S}_{\mathbf{p}jm}^{\mathrm{spin}} \tag{C16}$$

since $\nabla_{\mathbf{r}}n_{\mathbf{p}j}$ and $\dot{\mathbf{p}}$ are both linear in the perturbation. Thus the correction terms in Eq. (C15) drop out here and the only new term on the left-hand side comes from the orbital force in Eq. (C13). The equation of motion of μ_{h} is obtained by multiplying the Boltzmann equation with $-g_{\mathrm{h}}\mu_{\mathrm{B}}(j/3)\cos\theta$ and summing over $\mathbf{p}$, j. The orbital-force term drops out since it contains $\Sigma_j j=0$.

The right-hand side of Eq. (C16) also has to be multiplied with $-g_{\mathrm{h}}\mu_{\mathrm{B}}(j/3)\cos\theta$ and summed over $\mathbf{p}$, j. The Berry-phase correction $\Delta\epsilon_{\mathbf{p}j}$ to the energy appears in the δ functions implementing energy conservation. If we evaluate the resulting integrals by expanding this δ function as in Appendix B, all terms multiplied with $\Delta\epsilon_{\mathbf{p}j}$ should be evaluated to order zero. Then the only j, j' dependence comes from the explicit factor $j/3$ and from the transition probabilities. However, explicit evaluation in the 4×4 spin space shows that $\Sigma_{jj'}(j/3)|_{\mathbf{p}}\langle j|j'\rangle_{\mathbf{p}'}|^2=0$, $\Sigma_{jj'}(j/3)|_{\mathbf{p}}\langle j|j^z|j'\rangle_{\mathbf{p}'}|^2=0$, $\Sigma_{jj'}(j/3)(|_{\mathbf{p}}\langle j|j^+|j'\rangle_{\mathbf{p}'}|^2+|_{\mathbf{p}}\langle j|j^-|j'\rangle_{\mathbf{p}'}|^2)=0$ so that all these terms vanish. Thus there is no contribution to the equation of motion for the hole magnetization.

The equation of motion for the magnetization *current* $\mathbf{j}_\mu$ contains an additional factor of $\dot{\mathbf{r}}$ in the integrand, which should be calculated to linear order, see Eq. (C15). For the left-hand side we obtain

$$-\frac{\partial_t \mathbf{j}_\mu}{g_h\mu_B}+\int\frac{d^3p}{(2\pi)^3}\sum_j\frac{j}{3}\times\cos\theta\,\dot{\mathbf{r}}\left(\frac{\mathbf{p}}{m_{hh}}\cdot\nabla_{\mathbf{r}}n_{\mathbf{p}j}-g_h\mu_B\frac{j}{3}\cos\theta\nabla_{\mathbf{r}}B_h\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}}^{(0)}-\frac{g_h\mu_B}{4p}\nabla_{\mathbf{r}}[\hat{\mathbf{p}}\cdot(\nabla_{\mathbf{r}}\times\mathbf{B}_h)]\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}}^{(0)}\right). \tag{C17}$$

Since all terms multiplied by $\dot{\mathbf{r}}$ are already of first order we replace $\dot{\mathbf{r}}$ by $\mathbf{p}/m_{hh}$. Then the first two terms in the parentheses are identical to the ones calculated above and the third vanishes due to $\Sigma_j j=0$. Thus the left-hand side of the equation of motion is not changed by Berry-phase contributions. On the right-hand side we have to multiply the collision integrals by $(j/3)\cos\theta\dot{\mathbf{r}}$ with $\dot{\mathbf{r}}\cong\mathbf{p}/m_{hh}+\Delta\mathbf{v}$ from Eq. (C15). $\Delta\mathbf{v}$ contains the term from the $\mathbf{p}$ dependence of the Zeeman energy as well as the Berry-phase corrections. The contribution from $\mathbf{p}/m_{hh}$ is what we have calculated in Sec. II B except for the additional Berry-phase correction $\Delta\epsilon_{\mathbf{p}j}$ in the δ functions. This correction is irrelevant, however, by the argument of the previous paragraph.

In the second contribution, $\Delta\mathbf{v}$ is of first order so that the collision integrals should be evaluated to order zero. But these are of course zero since there is no net scattering in equilibrium. Consequently, the linear contributions to the velocity $\dot{\mathbf{r}}$ drop out of the equation of motion for $\mathbf{j}_\mu$. In conclusion, we have shown that the hydrodynamic equations for the valence-band case are unaffected by Berry phases to linear order. The results of Sec. II B are thus correct.

*Electronic address: timm@physik.fu-berlin.de

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