In-class problems

Problem 1

The Maxwell equations in vacuum read, in Gaussian units,

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 4\pi\rho, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.\end{aligned}$$

To describe electromagnetism in superconductors, we add the two London equations

$$\begin{aligned}\frac{\partial \mathbf{j}}{\partial t} &= \frac{c^2}{4\pi\lambda_L^2}\,\mathbf{E}, \\ \nabla \times \mathbf{j} &= -\frac{c}{4\pi\lambda_L^2}\,\mathbf{B},\end{aligned}$$

where λ_L is the London penetration depth and we have assumed that the whole current is a supercurrent.

The goal of this problem is to convince yourself that electromagnetic waves can penetrate into a superconductor but with an altered dispersion relation $\omega(\mathbf{k})$. Assume that the charge density vanishes everywhere, $\rho \equiv 0$.

(a) Write the above six coupled equations in Fourier space, which essentially amounts to substituting $\nabla \to i\mathbf{k}$, $\partial/\partial t \to -i\omega$ (why?). Convince yourself that the solutions are *transverse* waves.

(b) Express $\mathbf{B}$ in terms of $\mathbf{E}$ and use this to eliminate the $\mathbf{B}$ field.

(c) Express the supercurrent density $\mathbf{j}$ in terms of the electric field $\mathbf{E}$ in two ways. Obtain the dispersion relation $\omega(\mathbf{k})$ from comparing the two expressions. Sketch and discuss the result. How does it fit together with the Meißner-Ochsenfeld effect?

Hint: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Remark: The plasma frequency in a superconductor is defined as $\omega_p = c/\lambda_L$, which should appear in your results. We have here simplified things by using the vacuum Maxwell equations. In real superconductors, the dielectric function at the wave's frequency, $\epsilon(\omega)$, would also enter.

Problem 2

The Boltzmann equation reads

$$\left(\frac{\partial}{\partial t} + \frac{\hbar\mathbf{k}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{F}}{\hbar} \cdot \frac{\partial}{\partial \mathbf{k}}\right)\rho = -\mathcal{S}[\rho].$$

Consider the relaxation-time approximation $\mathcal{S} = (\rho - \rho_0)/\tau$, where $\rho_0(\mathbf{k})$ is the equilibrium Fermi-Dirac distribution function.

(a) Show that, in the absence of an applied force, $\rho = \rho_0$ is indeed a stationary solution.

(b) Show that the deviations $\eta(\mathbf{r}, \mathbf{k}, t) = \rho(\mathbf{r}, \mathbf{k}, t) - \rho_0(\mathbf{k})$ decay for $t \to \infty$ in the absence of a force. To that end, consider the differential equation for η. It can be solved by Fourier transformation in $\mathbf{r}$ and t or, equivalently, by making the ansatz

$$\eta(\mathbf{r}, \mathbf{k}, t) = \eta_0(\mathbf{k})\, e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)},$$

where $\mathbf{q}$ is different from $\mathbf{k}$. One could say that $\mathbf{q}$ is the Fourier complement of the variable $\mathbf{r}$. (*Hint*: obtain the dispersion $\omega(\mathbf{k})$.) What happens in a superconductor?

(Please turn over)

Homework (due November 5, 2019)

Problem 3

Read the chapter on Bose-Einstein condensation (Ch. 3) in the lecture notes. Extend the derivations to the case of the *two-dimensional* ideal Bose gas, as outlined in the following.

(a) Derive expressions for the logarithm of the partition function, $\ln \mathcal{Z}$, and for the particle number N in terms of integrals over energy. Instead of the volume, the area A of the system should appear. Why would it be incorrect to split off the contribution from the ground state in this case?

(b) Obtain the fugacity $y = e^{\beta\mu}$ as a function of N. This can be done in analytical form (*hint*: $g_1(y)$ is an elementary function). Use the result to write $\ln \mathcal{Z}$ as a function of the particle number. Show that there is no phase transition.

(c) Derive an expression for the two-dimensional pressure $p = -\partial\Phi/\partial A$. Show that its limiting behavior at low temperatures reads

$$p \cong \frac{\pi^3}{3h^2}\, m\, (k_B T)^2.$$

Compare the result to the three-dimensional Bose-Einstein condensate and to the classical ideal gas.

Remarks: The function $g_2(y)$ is also denoted by $\mathrm{Li}_2(y)$ and called the "dilogarithm." Its derivative is

$$\frac{d}{dy}\,\mathrm{Li}_2(y) = -\frac{\ln(1-y)}{y}$$

and its leading behavior for y approaching unity reads

$$\mathrm{Li}_2(y) \cong \frac{\pi^2}{6} - (1-y)\big[1 - \ln(1-y)\big].$$