In-class problems

Problem 1

(a) For type-I superconductors, the phase boundary in the magnetic-field–temperature phase diagram is given by $H_c(T)$. Along the phase boundary, the Gibbs free energy of the normal and superconducting phases is equal,

$$G_n(T,H) = G_s(T,H),$$

since the two phases are in equilibrium. Apply this identity and the relation

$$dG = -S\,dT - 4\pi\,VM\,dH$$

(S is the entropy, V is the volume, M is the magnetization) to two points (H,T) and $(H+\delta H, T+\delta T)$ on the phase boundary. Use this to show that

$$-S_n\,\delta T - 4\pi\,VM_n\,\delta H = -S_s\,\delta T - 4\pi\,VM_s\,\delta H.$$

Use $M_n = 0$ and $M_s = -H/4\pi$ (why?) to show that the latent heat per volume, $w_l = T(S_n - S_s)/V$, is given by

$$w_l = -TH_c\,\frac{dH_c(T)}{dT}.$$

Discuss the order of the phase transition, also at T_c. *Remark*: This result is analogous to the Clausius-Clapeyron equation for the liquid-gas transition.

(b) Obtain a more explicit result for the latent heat w_l close to T_c using Ginzburg-Landau theory.

Problem 2

We have seen that, for the three-dimensional neutral superfluid above the critical temperature, Gaussian fluctuations lead to a divergence of the specific heat of the form

$$c_V \equiv \frac{C_V}{V} \sim \frac{1}{\sqrt{T-T_c}}.$$

Generalize the derivation within Ginzburg-Landau theory to d spatial dimensions. Up to a numerical factor, the full result for $d < 4$ and the contribution from small momenta, i.e., from long-wavelength fluctuations, for $d > 4$ can be obtained by dimensional analysis. The case $d = 4$ requires an additional thought.

Hint: Separate the momentum integral into a one-dimensional radial integral and an integral over angles. The latter gives the surface area S_d of the d-dimensional unit hypersphere. For the solution, the explicit form of S_d is not needed.

Remark: Ginzburg-Landau theory becomes exact close to T_c for $d \geq d_c$, where d_c is its upper critical dimension. Unfortunately, d_c equals 4.

(Please turn over)

Homework (due November 28, 2019)

Problem 3

Time-dependent Ginzburg-Landau theory for neutral superfluids is based on the action $\mathcal{S} = \int dt\, d^3r\, \mathcal{L}$ with the Lagrange density

$$\mathcal{L} = i\hbar\, \psi^* \frac{\partial \psi}{\partial t} - \alpha\, |\psi|^2 - \frac{\beta}{2}\, |\psi|^4 - \frac{\hbar^2}{2m^*}\, |\nabla\psi|^2.$$

(a) Show that the time-dependent Ginzburg-Landau equation

$$i\hbar\, \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m^*}\, \nabla^2\psi + \alpha\, \psi + \beta\, |\psi|^2\psi \tag{1}$$

follows from the variational principle $\delta\mathcal{S} = 0$. You can either do this by considering small deviations from the equilibrium state, as done for the static case in the lecture, or use functional differentiation (see lecture notes for Quantum Theory 2).

(b) Consider the uniform case of Eq. (1) for the superconducting phase ($\alpha < 0$). Split off the equilibrium value,

$$\psi(t) = \sqrt{-\frac{\alpha}{\beta}}\, f(t),$$

and derive a compact equation for $f(t)$. Introducing the time scale $\tau := -\hbar/\alpha > 0$ is useful. Solve the equation with the help of the ansatz

$$f(t) = g(t)\, e^{i\phi(t)} \quad \text{with } g, \phi \in \mathbb{R}.$$

Show that the solutions conserve g.

(c) Based on part (b), consider the case of suddenly switching on superconducting pairing at time $t = 0$. This is described by the initial condition $\psi(t = 0) = 0$. Show that there is no nontrivial solution for $\psi(t)$.

(d) Based on part (b), consider a sudden change in the parameter α at time $t = 0$, specifically a jump from $\tilde{\alpha}$ to α, where $\tilde{\alpha}, \alpha < 0$. The parameter β is unchanged. The initial condition is then $\psi(t = 0) = \sqrt{-\tilde{\alpha}/\beta}$. Find the special solution for $\psi(t)$ with this initial condition. Describe the result.