In-class problems

Problem 1

Consider a system described by the time-independent Hamiltonian H. The *Matsubara* Green function is defined as

$$\mathcal{G}(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') = -\langle T_\tau \Psi_\sigma(\mathbf{r}, \tau)\Psi^\dagger_{\sigma'}(\mathbf{r}', \tau')\rangle,$$

where for any operator

$$A(\tau) = e^{H\tau/\hbar}\, A\, e^{-H\tau/\hbar},$$

and T_τ is the time-ordering directive

$$T_\tau A(\tau)B(\tau') = \begin{cases} A(\tau)B(\tau') & \text{for } \tau > \tau', \\ \pm B(\tau')A(\tau) & \text{for } \tau < \tau', \end{cases}$$

where the upper (lower) sign holds for bosonic (fermionic) operators. Since H is time independent the Green function only depends on the difference $\tau - \tau'$, which we denote by τ from now on. One can then show that the resulting Green function $\mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', \tau)$ is defined only for $\tau \in [-\hbar\beta, \hbar\beta]$ and satisfies

$$\mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', \tau - \hbar\beta) = \pm\mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', \tau).$$

The Fourier transformation between time and frequency space is given by

$$\mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', i\omega_n) = \int_0^{\hbar\beta} d\tau\, e^{i\omega_n\tau}\, \mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', \tau),$$

$$\mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', \tau) = \frac{1}{\hbar\beta}\sum_{i\omega_n} e^{-i\omega_n\tau}\, \mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', i\omega_n).$$

(a) Show that only the discrete (Matsubara) frequencies

$$\omega_n = \begin{cases} \dfrac{2n\pi}{\hbar\beta}, & n \in \mathbb{Z}, \quad \text{for bosons,} \\ \dfrac{(2n+1)\pi}{\hbar\beta}, & n \in \mathbb{Z}, \quad \text{for fermions} \end{cases}$$

occur.

(b) Assume that the system in non-interacting and non-magnetic. Then the Hamiltonian takes the form

$$H = \int d^3r \sum_\sigma \Psi^\dagger_\sigma(\mathbf{r})\, \tilde{H}\, \Psi_\sigma(\mathbf{r}),$$

where $\tilde{H}$ is the usual first-quantized single-particle Hamiltonian. With

$$\Psi_\sigma(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}\sigma},$$

where V is the volume, H is diagonalized,

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}}\, a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}.$$

Evaluate the Fourier-transformed Green function

$$\mathcal{G}_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}(i\omega_n) = \frac{1}{V}\int d^3r\, d^3r'\, e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'}\, \mathcal{G}(\mathbf{r}\sigma, \mathbf{r}'\sigma', i\omega_n).$$

Hints: Commutation relations of bosonic/fermionic annihilation and creation operators? Thermal averages of bosonic/fermionic particle number operators?

Homework (due December 19, 2019)

Problem 2

Consider a one-dimensional superfluid (think of helium in a capillary or a superconducting wire, neglecting the electromagnetic field).

(a) Vortices are not stable in a one-dimensional superfluid. Briefly explain why.

(b) Adapt the discussion of superfluid thin films from the lecture to the one-dimensional case. As seen in part (a), vortices can be disregarded. Obtain the correlation function of phase fluctuations, $\langle[\phi(x) - \phi(0)]^2\rangle$, and, from this, the correlation function of the order parameter, $\langle\psi^*(x)\psi(0)\rangle$. Discuss the type of order. *Note*: If you can argue that results pertaining to the two-dimensional case carry over to one dimension, you do not have to repeat their derivations.

Problem 3

The Poisson equation in three dimensions reads, in Gaussian units,

$$\nabla^2\phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}).$$

This implies (why?) that the Fourier transform of the scalar potential of a point charge Q reads

$$\hat{\phi}(\mathbf{q}) = 4\pi\,\frac{Q}{q^2}.$$

(a) We generalize this result to the *Yukawa potential*

$$\hat{\phi}(\mathbf{q}) = 4\pi\,\frac{Q}{\kappa^2 + q^2},$$

where $\kappa > 0$ is a constant. Find the Yukawa potential in real space, $\phi(\mathbf{r})$. Discuss the limit for $\kappa \to 0$.

(b) Perform the same analysis as in (a) and (b) for two-dimensional space. Note that the Poisson equation now reads $\nabla^2\phi(\mathbf{r}) = -2\pi\rho(\mathbf{r})$, since the prefactor is ultimately the integral over all angles, which is 4π in three dimensions but 2π in two dimensions. (In SI units, this factor is not included explicitly so that the Poisson equation reads $\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0$ in any number of dimensions.) You may need an integral table.

Remark: The results are useful both for the (BKT) theory of superfluid and superconducting films and for the microscopic theory of superconductivity.