In-class problem

Problem 1

We define Cooper-pair creation and annihilation operators

$$p_{\mathbf{k}}^\dagger := c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k},\downarrow}^\dagger,$$
$$p_{\mathbf{k}} := c_{-\mathbf{k},\downarrow} c_{\mathbf{k}\uparrow}.$$

(a) Show that the commutators

$$[p_{\mathbf{k}}, p_{\mathbf{k}'}^\dagger], \quad [p_{\mathbf{k}}, p_{\mathbf{k}'}], \quad [p_{\mathbf{k}}^\dagger, p_{\mathbf{k}'}^\dagger]$$

vanish for $\mathbf{k}' \neq \mathbf{k}$.

(b) Evaluate $p_{\mathbf{k}}^2$ and $(p_{\mathbf{k}}^\dagger)^2$.

(c) Evaluate the commutator $[p_{\mathbf{k}}, p_{\mathbf{k}}^\dagger]$. Thereby, show that for the same $\mathbf{k}$ the pair operators do *not* satisfy the bosonic commutation relation. Interpret the results.

(d) Show that the BCS ground state can be written as

$$|\Psi_{\text{BCS}}\rangle \propto \prod_{\mathbf{k}} \exp(\alpha_{\mathbf{k}}\, p_{\mathbf{k}}^\dagger)\, |0\rangle.$$

Relate the coefficients $\alpha_{\mathbf{k}}$ to the amplitudes $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ introduced in the lecture.

(e) Evaluate

$$\langle \Psi_{\text{BCS}}|\, p_{\mathbf{q}}\, |\Psi_{\text{BCS}}\rangle.$$

Homework (due January 28, 2020)

Problem 2

Consider the BCS Hamiltonian in the form

$$H_{\text{BCS}} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}}\, c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}}^*\, c_{-\mathbf{k},\downarrow} c_{\mathbf{k}\uparrow} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}}\, c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k},\downarrow}^\dagger + \text{const},$$

where the gap function can be written as

$$\Delta_{\mathbf{k}} = |\Delta_{\mathbf{k}}|\, e^{i\phi_{\mathbf{k}}}.$$

Obtain the time derivative of the number operator $n_{\mathbf{k}\uparrow} := c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow}$ in the Heisenberg picture from the Heisenberg equation

$$\frac{dn_{\mathbf{k}\uparrow}}{dt} = \frac{i}{\hbar}\,[H_{\text{BCS}}, n_{\mathbf{k}\uparrow}]\,.$$

Compare the result to the derivative of H_{BCS} with respect to the phase of the gap, $\partial H_{\text{BCS}}/\partial \phi_{\mathbf{k}}$. (The results for $n_{-\mathbf{k},\downarrow}$ are analogous.) What does the result suggest with regard to canonically conjugate variables?

(Please turn over)

Problem 3

The Hubbard model with negative U, i.e., with a local *attractive* interaction, serves as a simple model for a superconductor. The purpose of this problem is for you to work through a solvable case, which is also of high relevance. The Hamiltonian reads

$$H = -\sum_{ij\sigma} t_{ij}\, c_{i\sigma}^\dagger c_{j\sigma} + \sum_i U\, c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow}$$

with $U < 0$. Note that the chemical potential term $-\mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma}$ has been absorbed into the hopping term.

(a) Derive the Hamiltonian H in momentum space, with the normal-state dispersion described by $\xi_\mathbf{k}$. You cannot hope to obtain $\xi_\mathbf{k}$ explicitly since the hopping amplitudes t_{ij} are not given! Decouple the interaction term in mean-field approximation under the assumption that $\langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k},\downarrow}^\dagger \rangle$ and $\langle c_{-\mathbf{k},\downarrow} c_{\mathbf{k}\uparrow} \rangle$ are non-zero, while all other averages of products of two fermionic operators vanish. Introduce

$$\Delta := -\frac{U}{N} \sum_{\mathbf{k}'} \langle c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}'\uparrow} \rangle.$$

Δ is independent of $\mathbf{k}$ by construction. Derive the resulting mean-field Hamiltonian H_{BCS}.

(b) Express H_{BCS} in terms of the new operators $\gamma_{\mathbf{k}\sigma}$, $\gamma_{\mathbf{k}\sigma}^\dagger$, which are defined by the Bogoliubov transformation

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u_\mathbf{k}\, \gamma_{\mathbf{k}\uparrow} + v_\mathbf{k}\, \gamma_{-\mathbf{k},\downarrow}^\dagger, \\ c_{-\mathbf{k},\downarrow}^\dagger &= -v_\mathbf{k}^*\, \gamma_{\mathbf{k}\uparrow} + u_\mathbf{k}^*\, \gamma_{-\mathbf{k},\downarrow}^\dagger. \end{aligned}$$

(c) Show that the new operators $\gamma_{\mathbf{k}\sigma}$, $\gamma_{\mathbf{k}\sigma}^\dagger$ satisfy the fermionic anticommutation relations if

$$|u_\mathbf{k}|^2 + |v_\mathbf{k}|^2 = 1.$$

(d) Determine how $u_\mathbf{k}$ and $v_\mathbf{k}$ have to be chosen to make H_{BCS} diagonal in $\gamma_{\mathbf{k}\sigma}$ and $\gamma_{\mathbf{k}\sigma}^\dagger$.

(e) Derive the energy spectrum of the Bogoliubov quasiparticles.