10.4. Semiclassical Green function

q.m.: $$G(q,q',E) := \frac{1}{i\hbar}\int_0^\infty dt\, K(q,q',t)\, e^{\frac{i}{\hbar}Et}$$

s.cl.: $$G_{sc}(q,q',E) := \frac{1}{i\hbar}\int_0^\infty dt\, K_{sc}(q,q',t)\, e^{\frac{i}{\hbar}Et} \; ; \quad K_{sc}(q,q',t) = \sum_j A_j\, e^{\frac{i}{\hbar}S_j(q,q',t)}$$

$$= \frac{1}{i\hbar}\int_0^\infty dt \sum_j A_j\, e^{\frac{i}{\hbar}\left(S_j(q,q',t)+Et\right)}$$

cl. paths with arbitrary energy E_j and fixed time t

phase: $W(t) = S_j(q,q',t) + Et$

$$S_j(q,q',t) = \int_{q'}^{q} p_j(q'')\,dq'' - E_j t$$

stationary: $$\dot{W} = \underbrace{\frac{\partial S_j}{\partial t}}_{-E_j(t_s)} + E \overset{!}{=} 0 \;\Rightarrow\; t_s, \quad E_j = E$$

free particle: $$E(t) = \frac{1}{2}m\left(\frac{q-q'}{t}\right)^2$$

$$\Rightarrow W(t_s) = \int_{q'}^{q} p_j(q'')\,dq'' \cancel{- E_j t_s + Et_s} = \int_{q'}^{q} p_j(q'')\,dq'' =: S_j(q,q',E)$$

$$\Rightarrow G_{sc}(q,q',E) = \sum_j B_j\, e^{\frac{i}{\hbar}S_j(q,q',E)}$$

cl. paths with arbitrary t_j and fixed E

10.5. Gutzwiller trace formula

$$d(E) = \sum_n \delta(E - E_n) = -\frac{1}{\pi} \operatorname{Im} \operatorname{Tr} G(q, q', E)$$

$$\operatorname{Tr} G_{sc}(q, q', E) = \int d^f q \, G_{sc}(q, q, E) = \int d^f q \sum_j B_j \, e^{\frac{i}{\hbar} S_j(q, q, E)}$$

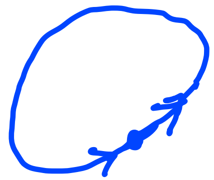
with $q = q'$ closed paths

$q = q'$

phase: $W(q) = S_j(q, q, E)$

stationary: $$\frac{d}{dq} W(q) = \left(\frac{\partial S_j(q, q', E)}{\partial q} + \frac{\partial S_j(q, q', E)}{\partial q'} \right)_{q' = q} \overset{10.3}{=} p - p' \overset{!}{=} 0$$

$\Rightarrow$ $p = p'$ periodic paths



$q = q'$

$p = p'$

remarks: • stationary phase approximation allowed if periodic orbit isolated (fully chaotic system)

• different approach for: – integrable systems Berry, Tabor (1976)

– near-integrable systems Tomsovic, Grinberg, Ullmo (1995)

Paths of length zero: $q' \to q$

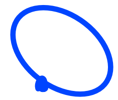
$\Rightarrow$ average density of states (Weyl law): $\bar{N}(E) = \frac{V_r(E)}{h^f}$; $\bar{d}(E) = \frac{d\bar{N}(E)}{dE}$

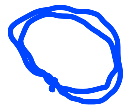
Density of states:

$$d(E) = \sum_i \delta(E - E_i) = \underset{\substack{\uparrow \\ \text{orbits of} \\ \text{length zero}}}{\bar{d}(E)} + \underset{\substack{\uparrow \\ \text{periodic} \\ \text{orbits}}}{d_{fl}(E)} \quad ; \quad N(E) = \bar{N}(E) + N_{fl}(E)$$

$$\boxed{d_{fe}(E) = \frac{1}{\pi\hbar} \sum_{p,n} \frac{T_p}{\left|\det\left(M_p^n - 1\right)\right|^{\frac{1}{2}}} \cos n\left(\frac{S_p(E)}{\hbar} - \frac{\pi}{2}\mu_p\right)}$$

Gutzwiller 1971

$\sum_p$: sum over all primitive periodic orbits 

$\sum_n$: sum over repetitions 

T_p : length of period of primitive orbit

$S_p(E) = \oint p\,dq$ action of primitive orbit

μ_p : Maslov index (reflections, conj. points)

M_p : monodromy matrix : linearized dynamics close to periodic orbit

$$\begin{pmatrix} \delta q \\ \delta p \end{pmatrix} = M_p \begin{pmatrix} \delta q' \\ \delta p' \end{pmatrix}$$

remarks:

- number of periodic orbits increases exponentially with T_p

  prefactor decreases, but slower

  $\Rightarrow$ sum diverges (needs to give δ-functions)

- experiment: long periods often suppressed (dissipation)

  $\Rightarrow$ few contributions relevant

- $d_{fe}(E)$, $N_{fe}(E)$ looks random

- Fourier transformation shows contribution of periodic orbits:

$$e^{\frac{i}{\hbar} S_p(E)} = e^{\frac{i}{\hbar}\left(S_p(E_0) + \underbrace{\frac{\partial S_p}{\partial E}}_{T_p}(E-E_0) + \cdots\right)} \propto e^{\frac{i}{\hbar}(E-E_0)T_p}$$

$\Rightarrow$ F.T. has peak at T_p

Summary Gutzwiller trace formula:

$K(q, q', t)$ Feynman path integral

$\downarrow$ stationary phase approximation

$K_{sc}(q, q', t)$ classical orbits

$\downarrow$ stationary phase approximation

$G_{sc}(q, q', E)$ classical orbits with fixed energy

$\downarrow$ trace, stationary phase approximation

$d(E) = \bar{d}(E) + d_{fl}(E)$ orbits of length zero and periodic orbits

$q = q'$

$p = p'$