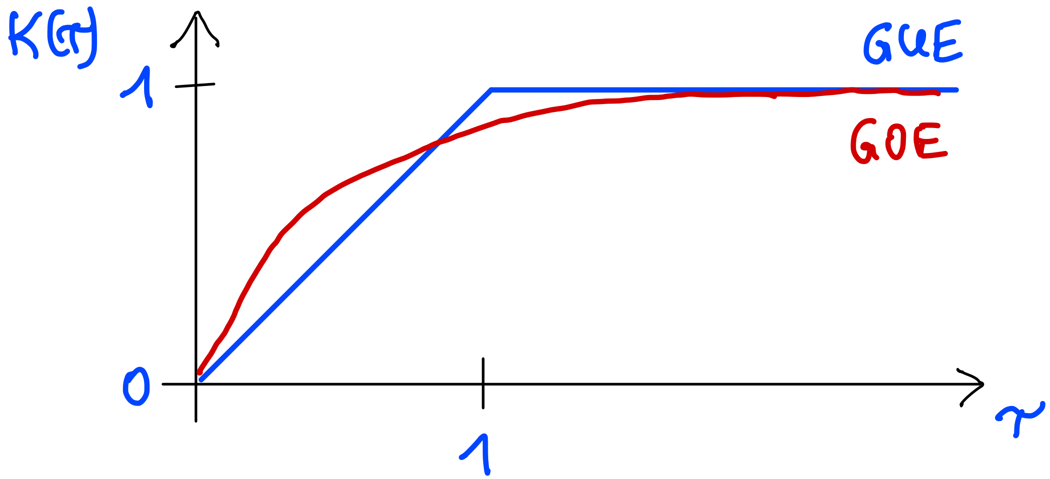
10.6. Spectral statistics

aim: semiclassical derivation of spectral properties using properties of periodic orbits

example: spectral form factor (see 7.7)



random matrix theory:

$\tau < 1$	$\tau > 1$
τ	1
$2\tau - \tau \ln(1+2\tau)$ $= 2\tau - 2\tau^2 + \dots$	$2 - \tau \ln \frac{2\tau+1}{2\tau-1}$

$$K(\tau) = \left\langle \frac{1}{\bar{d}(E)} \int_{-\infty}^{\infty} d\varepsilon \; d_{fl}\left(E + \frac{\varepsilon}{2}\right) d_{fl}\left(E - \frac{\varepsilon}{2}\right) e^{-\frac{i}{\hbar}\varepsilon t} \right\rangle_E$$

average: small E window but many levels

$$\tau = \frac{t}{\tau_H} \qquad \tau_H = \frac{h}{\langle \Delta E \rangle} = h \, \bar{d}(E)$$

use: $$d(E) = \sum_i \delta(E - E_i) = \bar{d}(E) + d_{fe}(E)$$

$$d_{fe}(E) = \frac{1}{\pi\hbar} \sum_{p,n} \frac{T_p}{\left|\det\left(M_p^n - 1\right)\right|^{\frac{1}{2}}} \cos n\left(\frac{S_p(E)}{\hbar} - \frac{\pi}{2}\mu_p\right)$$

$$= \frac{1}{\pi\hbar} \operatorname{Re} \sum_\gamma A_\gamma \, e^{\frac{i}{\hbar} S_\gamma(E)} = \frac{1}{h} \sum_\gamma \left(A_\gamma e^{\frac{i}{\hbar} S_\gamma(E)} + A_\gamma^* e^{-\frac{i}{\hbar} S_\gamma(E)}\right)$$

$$\Rightarrow K(\tau) = \sum_{\gamma,\gamma'} \dots$$

$\leftarrow$ pairs of periodic orbits

terms: $e^{\frac{i}{\hbar}(S_\gamma + S_{\gamma'})} \longrightarrow$ average to zero

terms: $e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \longrightarrow$ relevant if $S_\gamma \approx S_{\gamma'}$

energy dependence: $$S_\gamma\left(E \pm \frac{\varepsilon}{2}\right) = S_\gamma(E) \pm \frac{\varepsilon}{2} \underbrace{\frac{\partial S_\gamma}{\partial E}}_{T_\gamma} + \dots$$

T_γ length of period

$$\Rightarrow K(\tau) = \frac{1}{h^2 \bar{d}(E)} \left\langle \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'}\, e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \underbrace{\int_{-\infty}^{\infty} d\varepsilon\, e^{\frac{i}{\hbar}\varepsilon\left(\frac{T_\gamma}{2} + \frac{T_{\gamma'}}{2} - t\right)}}_{2\pi\hbar\, \delta\left(t - \frac{T_\gamma + T_{\gamma'}}{2}\right)} \right\rangle_E$$

$$K(\tau) = \frac{1}{\tau_H} \left\langle \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'}\, e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})}\, \delta\left(t - \frac{T_\gamma + T_{\gamma'}}{2}\right) \right\rangle_E$$

10.6.1. Diagonal Approximation

Berry 1985

assume: $\gamma \neq \gamma' \Rightarrow S_\gamma \neq S_{\gamma'} \Rightarrow$ average to zero

$$\sum_{\gamma,\gamma'} \longrightarrow \sum_{\gamma=\gamma'}' \Rightarrow K(\tau) = \frac{1}{T_H} \sum_\gamma' |A_\gamma|^2 \delta(t - T_\gamma)$$

use. sum rule by Hannay and Ozorio de Almeida:

principle of uniformity in ergodic systems:

long periodic orbits explore energy surface uniformly, just as typical orbits

$\Rightarrow \cdot\cdot \Rightarrow \sum_\gamma' |A_\gamma|^2 \delta(t - T_\gamma) = t$ for $t \gg T_0$ (shortest periodic orbit)

$$\Rightarrow K(\tau) = \begin{cases} \tau & \text{GUE} \\ 2\tau & \text{GOE} \end{cases}$$

GOE: $\gamma' = \gamma$ and $\gamma' = \gamma$ time reversed, $S_{\gamma'} = S_\gamma$

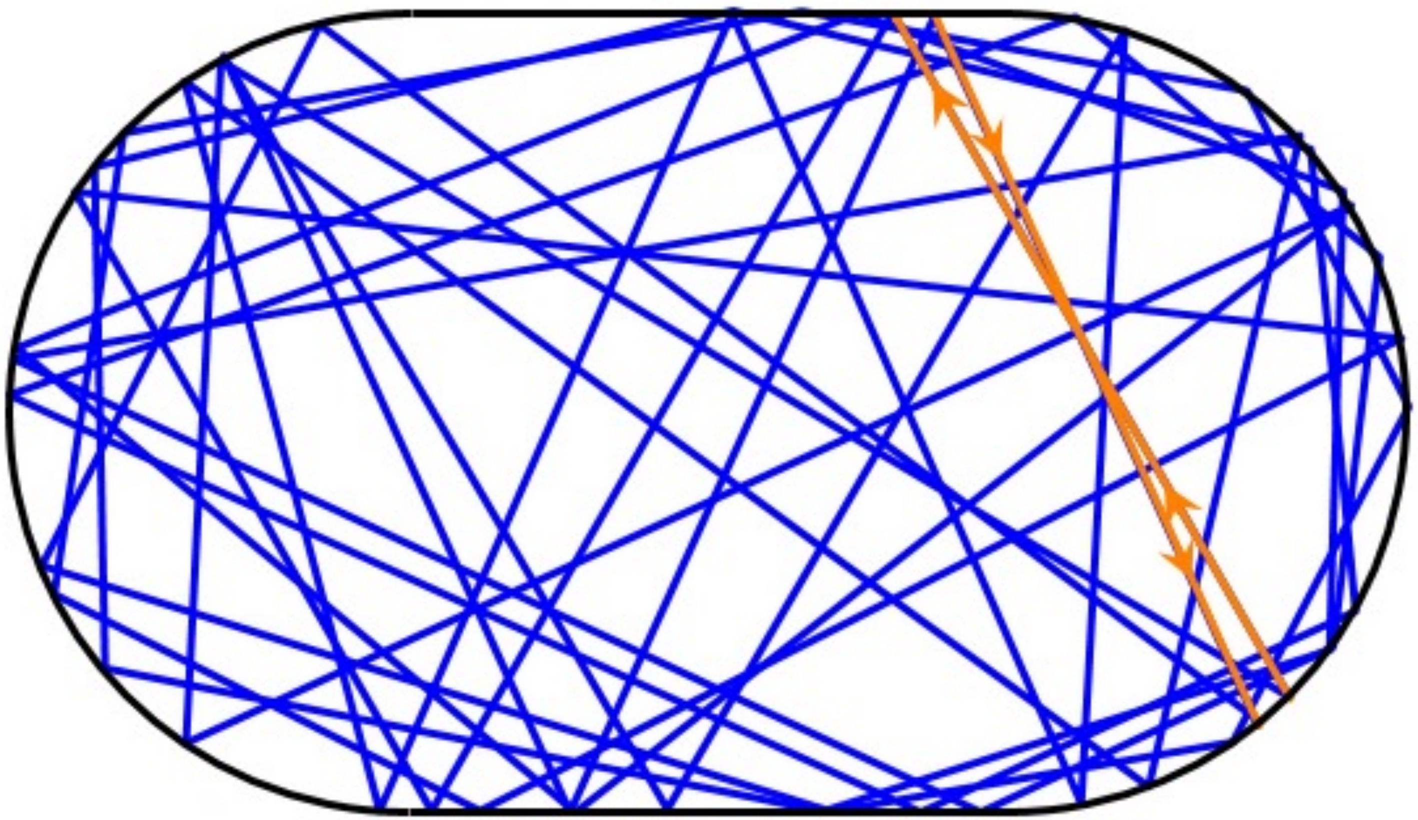
10.6.2. Off-diagonal contributions: pairs and bunches

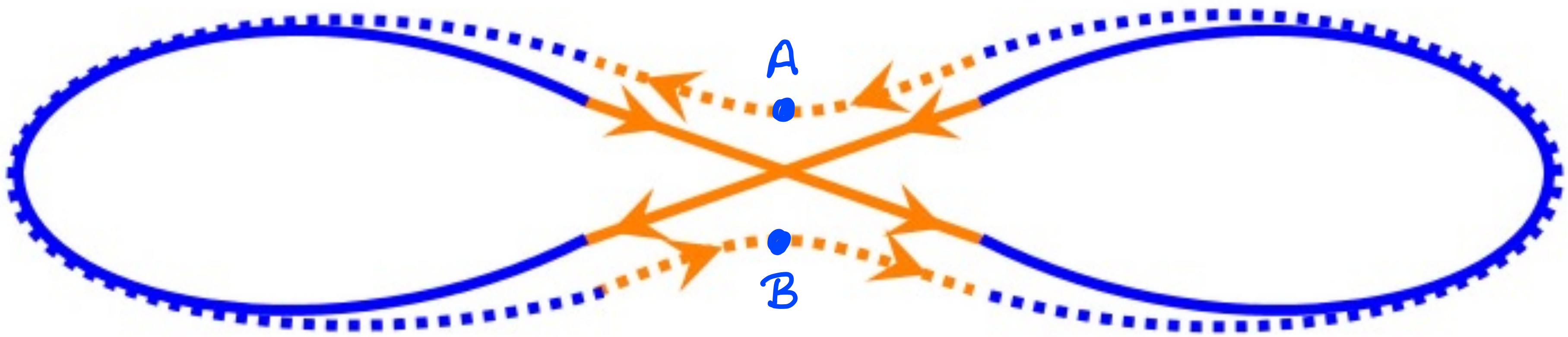
There are bunches of long periodic orbits, which have almost identical actions.

Their contribution is relevant in $\sum_{\gamma,\gamma'}$

A) Pairs   Sieber, Richter (2001)

ergodicity $\Rightarrow$
- long orbit comes close to every point in ph.sp.
- the same holds for long periodic orbit
- in particular: same position, almost opposite direction (self-encounter)



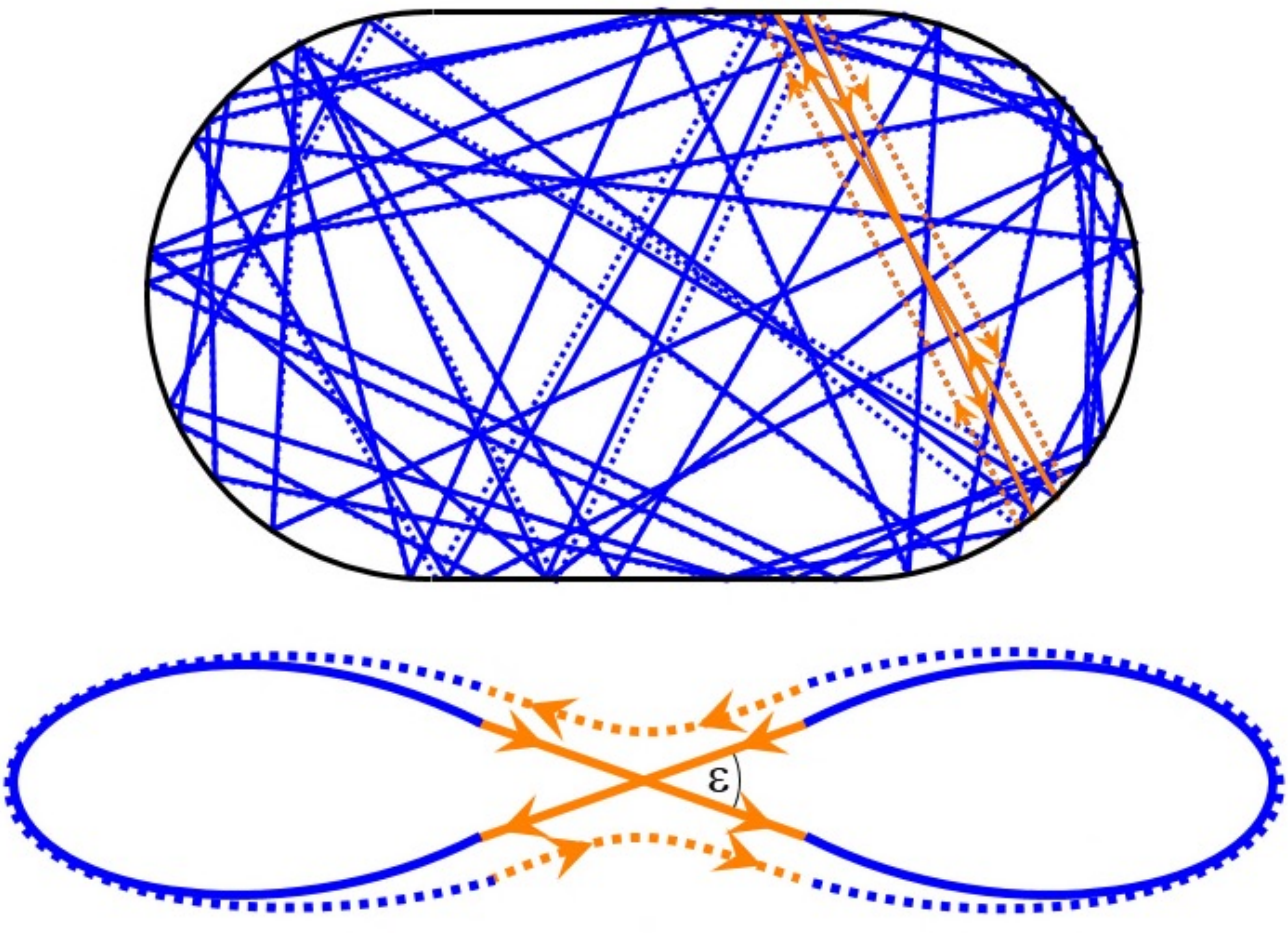


cartoon: loop stands for long part of periodic orbit

Linearized dynamics around periodic orbit

$\Rightarrow \exists$ another periodic orbit with:

- one loop almost identical
- other loop almost identical, but time-reversed
- unique (4 linear eq. with 4 unknowns)
- A close to unstable manifold of periodic orbit
- A close to stable manifold of time-reversed periodic orbit
- B close to stable manifold of periodic orbit
- B close to unstable manifold of time-reversed periodic orbit


ε

furthermore needed:

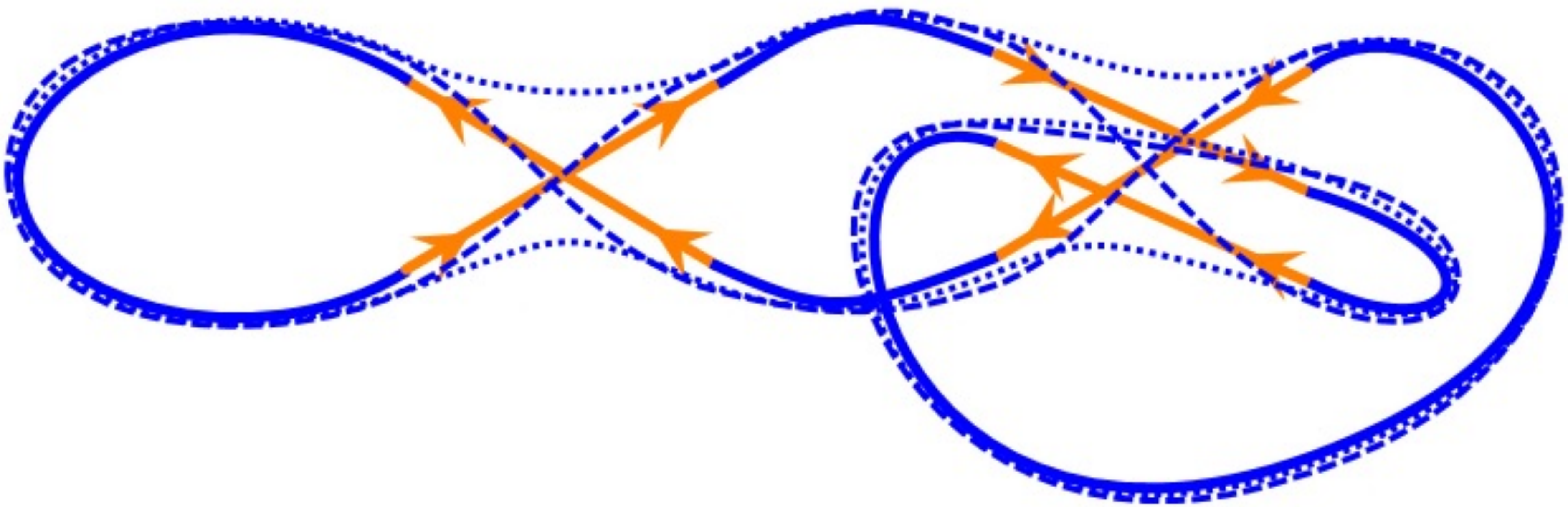
- action almost identical $|S_\gamma - S_{\gamma'}| \sim \varepsilon^2$ with ε crossing angle
- relevant if $|S_\gamma - S_{\gamma'}| < \hbar$
- prob. distr. of ε
- instability of dynamics

$\Rightarrow K(\tau)$: $-2\tau^2$ term ; GOE

cancellation ; GUE

B) Bunches   Haake et al 2004

Long periodic orbits have many self-encounters, also same region



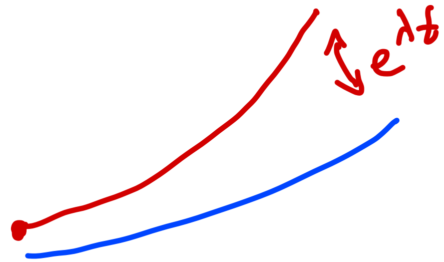
combinatorics $\Rightarrow \dots \Rightarrow$ agreement with GUE, GOE, GSE for $\tau < 1$

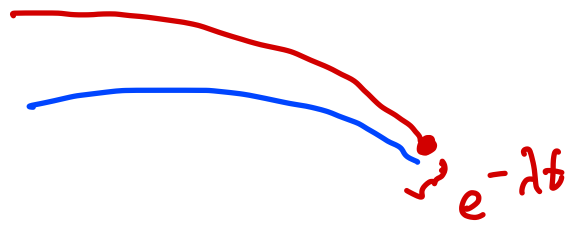
remarks: • Q.m. insight from classical understanding of bunches of periodic orbits with almost identical action

• existence of partner orbit from exponential sensitivity in chaotic systems

instability of
<u>initial value problem</u>

stability of
<u>boundary value problem</u>





• conjecture: level statistics of classically chaotic systems
(7.4.) is well described by random matrix theory (RMT) ✓

Casati, Valz-Gris, Guarneri 1980; Bohigas, Giannoni, Schmit 1983