2. Introduction to dynamical systems

Dynamical system : state $\vec{x}$ of system changes

dimension of state space

i) continuous in time $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x},t)$ $t \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^L$ or diff. manifold (circle, torus, ...)

ii) discrete in time $\vec{x}_{t+1} = \vec{f}(\vec{x}_t, t)$ $t \in \mathbb{Z}$

autonomous: $\vec{f} = \vec{f}(\vec{x})$

i) continuous in time $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x},t)$ autonomous: $\vec{f} = \vec{f}(\vec{x})$

Solutions: $\vec{x}(\vec{x}_0, t)$ with initial condition $\vec{x}(\vec{x}_0, t=0) = \vec{x}_0$

„trajectory, orbit, flow, ..."

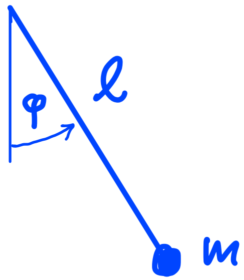
existence and uniqueness

$\Longrightarrow$

- determinism „deterministic chaos"
- trajectories do not intersect in state space

Def.: Fixed point $\vec{x}_f$: $\vec{f}(\vec{x}_f) = 0 \iff \left.\frac{d\vec{x}}{dt}\right|_{\vec{x}=\vec{x}_f} = 0$

example: pendulum



$$m l \ddot{\varphi} = - m g \sin\varphi$$

$$\ddot{\varphi} = - \frac{g}{l} \sin\varphi \qquad 2^{\text{nd}} \text{ order diff. eq.}$$

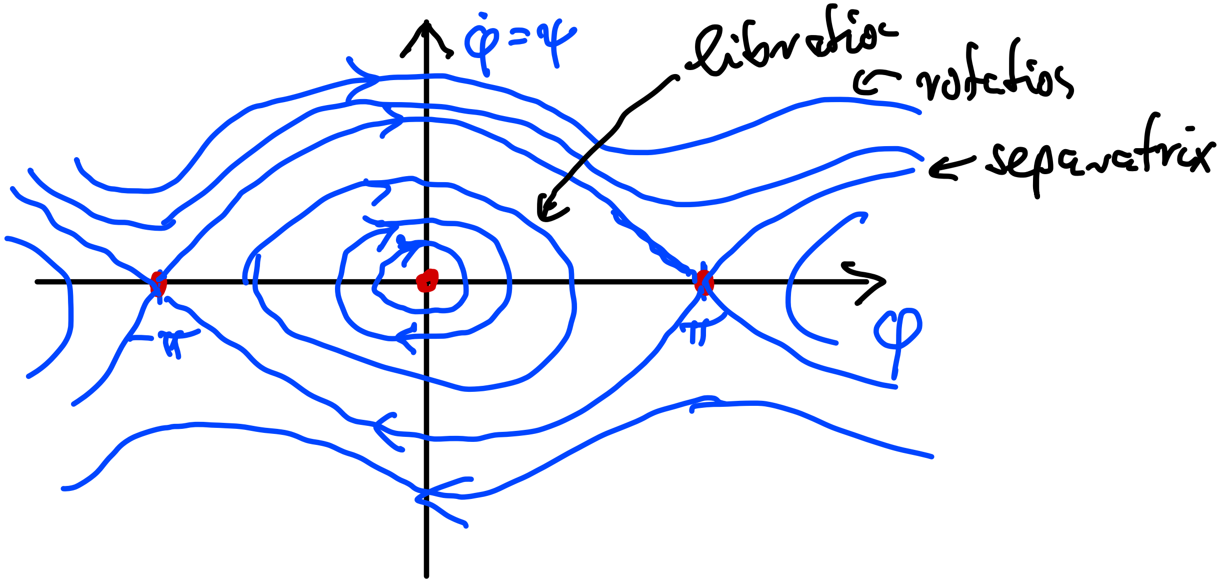
dynamical system? ✓

$$\dot{\varphi} = \psi$$

$$\dot{\psi} = - \frac{g}{l} \sin\varphi$$

$$\vec{x} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \qquad \vec{f}(\vec{x}) = \begin{pmatrix} \psi \\ -\frac{g}{l} \sin\varphi \end{pmatrix}$$

fixed points: $\vec{f}(\vec{x}_f) = 0 \quad \Rightarrow \quad \vec{x}_f = \begin{pmatrix} \varphi_f \\ \psi_f \end{pmatrix} = \begin{pmatrix} z\pi \\ 0 \end{pmatrix} \quad z \in \mathbb{Z}$



observations:

- trajectories cross at $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$ ↯ it takes infinitely long
- dynamics close to fixed point can be quite different

Neighborhood of fixed point: Taylor series

$$\dot{\vec{x}} = \vec{f}(\vec{x}) = \underbrace{\vec{f}(\vec{x}_f)}_{=0} + \underbrace{\frac{\partial \vec{f}(\vec{x})}{\partial \vec{x}}\Bigg|_{\vec{x}=\vec{x}_f}}_{=:\,A \text{ dynamical matrix}} (\vec{x}-\vec{x}_f) + \cdots$$

pendulum:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos\varphi & 0 \end{pmatrix}$$

move origin of coordinate system to $\vec{x}_f$: $\vec{x}-\vec{x}_f \longrightarrow \vec{x}$

$$\Rightarrow \quad \dot{\vec{x}} = A\vec{x} + \cdots$$

2.1. Linear dynamical systems

Neglect nonlinear terms near fixed point: $\dot{\vec{x}} = A\vec{x}$

Dynamics ?

- superposition principle (linear homogenous diff. eq.):

  let $\vec{x}_1(t), \vec{x}_2(t)$ be solutions $\Rightarrow$ $c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$ is solution

- formal solution: $\vec{x}(t) = e^{At}\,\vec{x}_0$

  with $e^{At} = \mathbb{1} + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots$

- solve eigenvalue problem:

$$A\vec{\xi}_n = \lambda_n \vec{\xi}_n$$

eigenvalue

eigenvector

$\rightarrow$ L eigenvalues, eigenvec.

- specific solution: $\vec{x}_n = e^{\lambda_n t}\vec{\xi}_n$

- general solution (assuming no degeneracies, i.e. $\lambda_n \neq \lambda_m$ for $n \neq m$)

$$\vec{x}(t) = \sum_{n=1}^{L} c_n e^{\lambda_n t}\vec{\xi}_n = e^{At}\underbrace{\sum_{n=1}^{L} c_n \vec{\xi}_n}_{\vec{x}_0}$$

A is real $\Rightarrow$

$$A\vec{\xi}_n = \lambda_n \vec{\xi}_n$$
$$\Downarrow$$
$$A\vec{\xi}_n^* = \lambda_n^* \vec{\xi}_n^*$$

$\Rightarrow$ eigenvalue pairs λ, λ^*

Two cases:

- λ_n and $\vec{\xi}_n$ real: $\vec{x}_n(t) = e^{\lambda_n t}\vec{\xi}_n$

  $\lambda_n > 0$: exponential escape from fixed point along $\vec{\xi}_n$

  $\lambda_n < 0$: exponential attraction towards " " " "

• pairs $(\lambda_n, \lambda_m = \lambda_n^*)$ and $(\vec{\xi}_n, \vec{\xi}_m = \vec{\xi}_n^{\,*})$ with $\lambda_n = \alpha + i\omega$

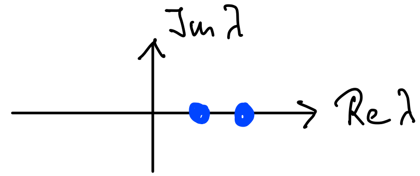
$$\Rightarrow \text{real } \vec{x}_n(t) = c_n e^{\lambda_n t}\vec{\xi}_n + c_n^* e^{\lambda_n^* t}\vec{\xi}_n^{\,*} = 2\,\mathrm{Re}\Big(\underbrace{c_n}_{=|c_n|e^{i\varphi}} e^{(\alpha+i\omega)t}\vec{\xi}_n\Big)$$

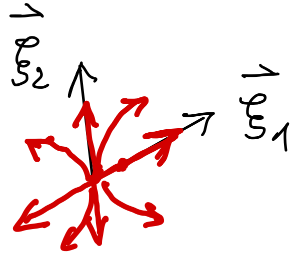
$$= 2|c_n|e^{\alpha t}\Big(\cos(\omega t+\varphi)\,\mathrm{Re}\,\vec{\xi}_n - \sin(\omega t+\varphi)\,\mathrm{Im}\,\vec{\xi}_n\Big)$$

$$\left.\begin{array}{l} \alpha>0: \text{ exponential expansion} \\ \alpha<0: \text{ exponential contraction} \\ \\ \alpha=0: \end{array}\right\} + \text{periodic oscillation}$$

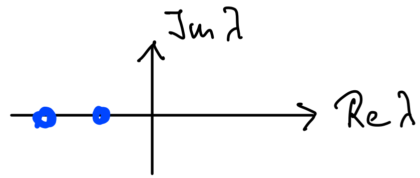
$L = 2$: 2 eigenvalues

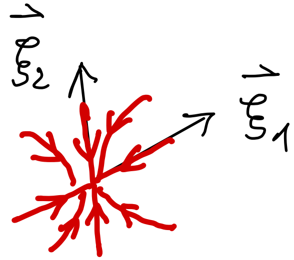
- both real :



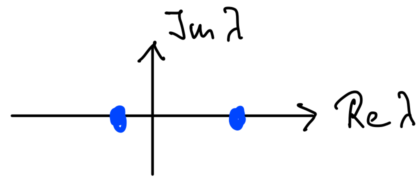


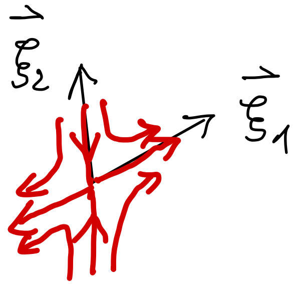
unstable node





stable node



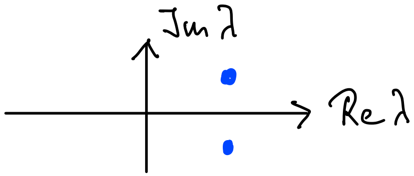


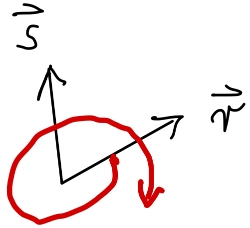
saddle

$$\vec{\xi} = \vec{r} + i\vec{s}$$

- both complex:

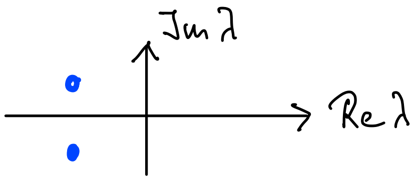
$\alpha > 0$

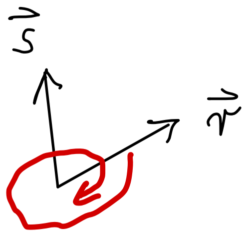




unstable focus

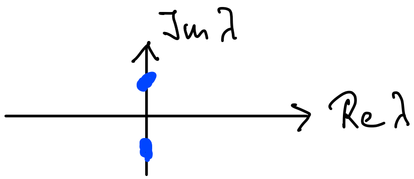
$\alpha < 0$

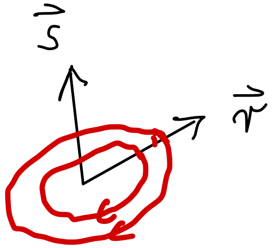




stable focus

$\alpha = 0$





center

first 5 fixed points : hyperbolic (no eigenvalue has zero real part)

last fixed point : elliptic (it is neutrally / marginally stable)