2.2. Hamiltonian Systems

- Hamilton function: $H(\vec{q}, \vec{p}, t)$; $\vec{q} = (q_1, \dots, q_N)$, $\vec{p} = (p_1, \dots, p_N)$ — N degrees of freedom
- autonomous: $H = H(\vec{q}, \vec{p})$
- phase space: $(\vec{q}, \vec{p}) \in \mathbb{R}^{2N}$
- Hamilton equations of motion:

$$\boxed{\begin{aligned} \dot{\vec{q}} &= \frac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{q}} \end{aligned}}$$

Does $H(\vec{q}(t), \vec{p}(t))$ depend on time? $\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial t}}_{=0} + \underbrace{\frac{\partial H}{\partial \vec{q}}}_{-\dot{\vec{p}}} \dot{\vec{q}} + \underbrace{\frac{\partial H}{\partial \vec{p}}}_{\dot{\vec{q}}} \dot{\vec{p}} = 0$ constant of motion

How to write as dynamical system?

$$\dot{\vec{x}} = \vec{f}(\vec{x}, t) \qquad \text{with} \quad \vec{x} := \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \quad \vec{f}(\vec{x}, t) := \begin{pmatrix} \frac{\partial H}{\partial \vec{p}} \\ -\frac{\partial H}{\partial \vec{q}} \end{pmatrix} = S_N \frac{\partial H}{\partial \vec{x}}$$

$$S_N := \begin{pmatrix} 0_N & \mathbb{1}_N \\ -\mathbb{1}_N & 0_N \end{pmatrix} \qquad \frac{\partial H}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial H}{\partial \vec{q}} \\ \frac{\partial H}{\partial \vec{p}} \end{pmatrix}$$

a matrix B is symplectic: $B^T S B = S$

- fixed points (autonomous): $\vec{f}(\vec{x}_f) = 0 \Rightarrow \left.\frac{\partial H}{\partial \vec{q}}\right|_{\vec{x}=\vec{x}_f} = 0$ and $\left.\frac{\partial H}{\partial \vec{p}}\right|_{\vec{x}=\vec{x}_f} = 0$

- dynamical matrix A:

$$A = \left.\frac{\partial \vec{f}(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_f} = S_N \underbrace{\left.\frac{\partial^2 H}{\partial \vec{x}^2}\right|_{\vec{x}=\vec{x}_f}}$$

$$=: M = \begin{pmatrix} \frac{\partial^2 H}{\partial \vec{q}^2} & \frac{\partial^2 H}{\partial \vec{q}\,\partial \vec{p}} \\ \frac{\partial^2 H}{\partial \vec{p}\,\partial \vec{q}} & \frac{\partial^2 H}{\partial \vec{p}^2} \end{pmatrix} \quad \text{Hessian}$$

$$\Rightarrow A = SM \quad \text{with} \quad M^T = M$$
$$S^T = -S$$

· consequences for eigenvalues λ_n of A ?

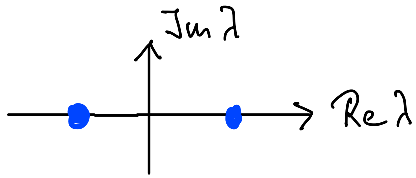
→ exercise 2.1

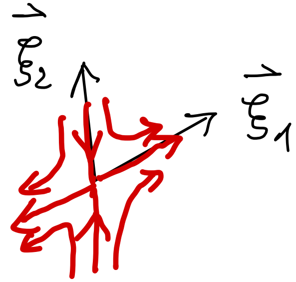
⇒ eigenvalue pairs $\lambda, -\lambda$

from above: A real ⇒ pairs λ, λ^*

cases: 1. trivial: $\lambda = 0$ (along trajectory and perpendicular to energy shell)

2. real pair: $\pm \lambda$

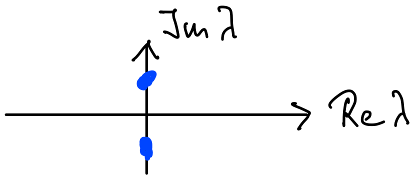


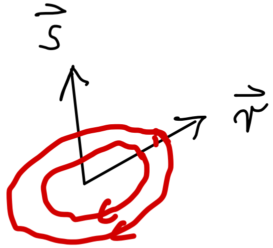


saddle

hyperbolic

3. imaginary pair: $\pm i\omega$



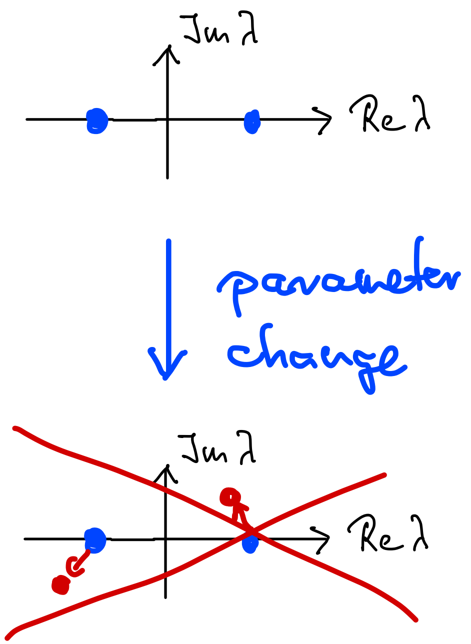


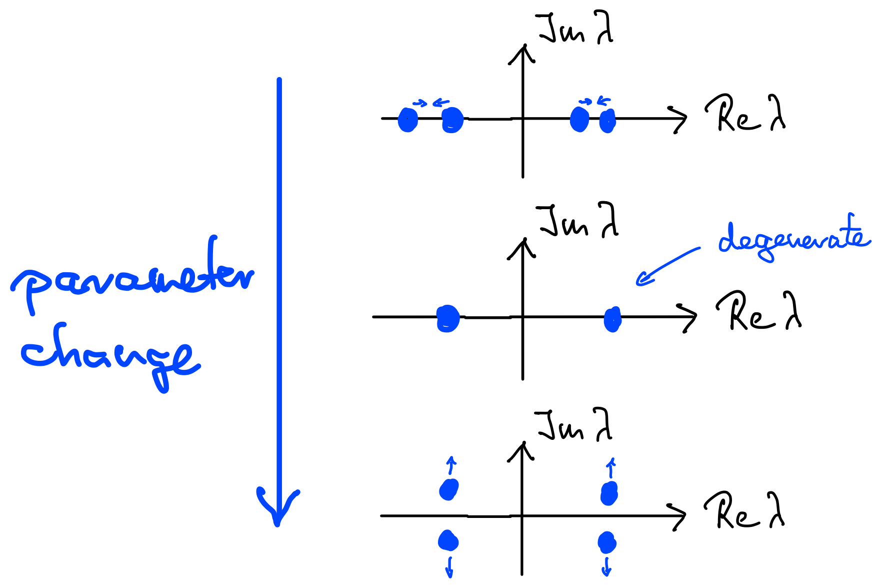
center

elliptic

4. complex quartet: $\pm \alpha \pm i\omega$ (needs ≥ 3 d.o.f.)

- structural stability:





- normal forms:

$\pm\lambda$: $\quad H = -\lambda\, q_1 p_1$

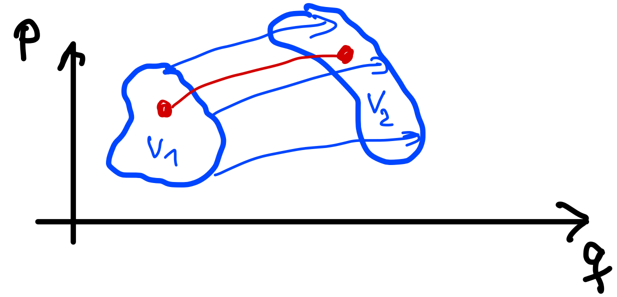
$\pm i\omega$: $\quad H = \frac{\omega}{2}\left(q_1^2 + p_1^2\right)$

$\pm\alpha \pm i\omega$: $\quad H = -\alpha\left(q_1 p_1 + q_2 p_2\right) + \omega\left(q_2 p_1 - p_2 q_1\right)$

General properties:

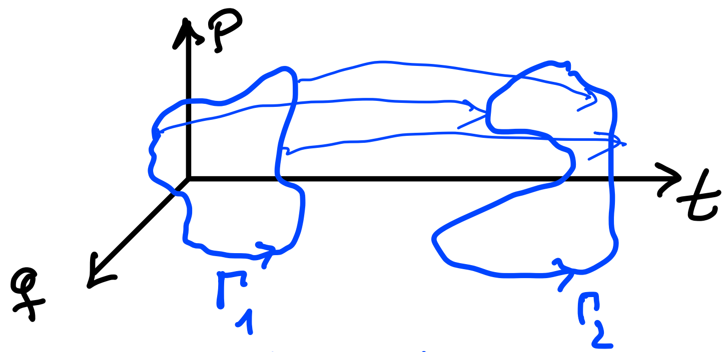
1. Trajectories do not intersect in phase space (autonomous).

2. Time evolved volume in phase space encloses always the same trajectories



3. Liouville theorem: invariance of volume in phase space: $V_1 = V_2$

4. Poincaré-Cartan integral theorem



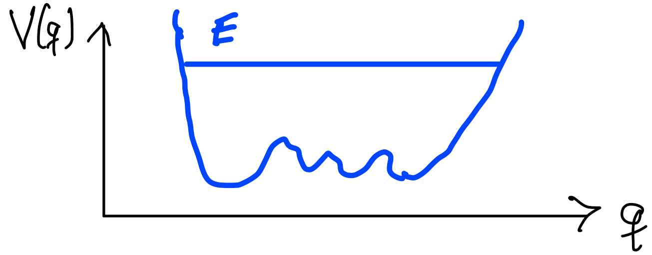
not at fixed t

$$\oint_{\Gamma_1} (\vec{p}\, d\vec{q} - H\, dt) = \oint_{\Gamma_2} \cdots$$

5. Poincaré recurrence theorem:

Almost all trajectories leaving an initial volume Γ_0 in phase space will return to Γ_0.

assumption: finite phase space volume at fixed energy surface $H(\vec{q},\vec{p})=E$



proof:

- Γ_τ is time evolved Γ_0 after time τ. Assume $\Gamma_\tau \cap \Gamma_0 = \emptyset$
- volumes $\Gamma_0, \Gamma_\tau, \Gamma_{2\tau}, \dots$ are equal, available volume is finite

$\Rightarrow \exists m,n$: $\Gamma_{m\tau} \cap \Gamma_{n\tau}$ has nonzero measure

$\Rightarrow$ $\Gamma_{(m-n)\tau} \cap \Gamma_0$ " " " (first overlap)

$\Rightarrow$ subset of Γ_0 with " " has returned to Γ_0

- What about rest of Γ_0?

  assume subset $\gamma_0 \subset \Gamma_0$ never returns to Γ_0

  apply above proof to γ_0 $\Rightarrow$ subset of γ_0 returns to $\gamma_0 \subset \Gamma_0$ ↯

Comments:

- no details of dynamics required
- no statement about individual trajectory
- distribution of return times is open problem!

2.3. Integrable systems

Def.: A hamiltonian system with N degrees of freedom is integrable, if there exist N independent integrals (constants) of motion, $I_1, I_2, \dots, I_N$ (including H), which are in involution, i.e. $\{I_i, I_j\} = 0$.

For integrable systems there exists a canonical transformation from generalized coordinates and momenta to action-angle variables:

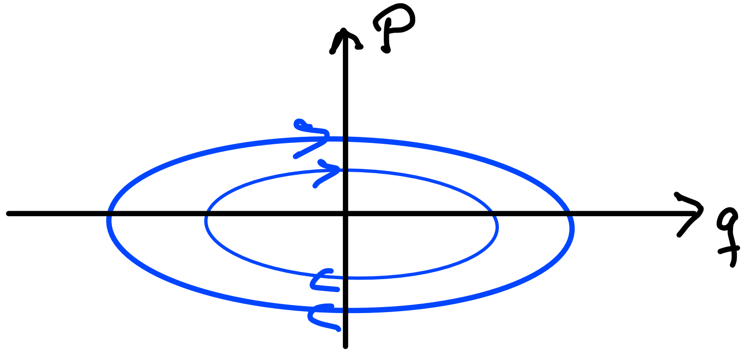
$$\begin{matrix} q_1, \dots, q_N \\ p_1, \dots, p_N \end{matrix} \longrightarrow \begin{matrix} I_1, \dots, I_N \\ \varphi_1, \dots, \varphi_N \end{matrix}$$

with
- $\tilde{H} = \tilde{H}(I_1, \dots, I_N)$ $\Rightarrow$ equations of motion:
- φ_j cyclic variables $\qquad \dot{I}_j = -\dfrac{\partial \tilde{H}}{\partial \varphi_j} = 0 \qquad \Rightarrow I_j = \text{const}$
- $I_j = \dfrac{1}{2\pi} \oint\limits_{\varphi_j \in [0, 2\pi]} \vec{p}\, d\vec{q} \qquad \dot{\varphi}_j = \dfrac{\partial \tilde{H}}{\partial I_j} =: \omega_j(I_1, \dots, I_N) = \text{const} \qquad \Rightarrow \varphi_j = \omega_j t + \varphi_j^0$

frequencies

Example: Harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$



canonical transformation ($G_1(q,\varphi) = -\frac{1}{2} m\omega q^2 \tan\varphi$)

$$q = \sqrt{\frac{2I}{m\omega}} \cos\varphi \qquad \Longleftrightarrow \qquad \varphi = \arctan \frac{-p}{m\omega q}$$

$$p = -\sqrt{2m\omega I} \sin\varphi \qquad\qquad I = \frac{p^2}{2m\omega} + \frac{1}{2} m\omega q^2$$

$$\Rightarrow \quad \tilde{H}(I) = \omega I$$

$$\Rightarrow \quad \dot{I} = -\frac{\partial \tilde{H}}{\partial \varphi} = 0$$

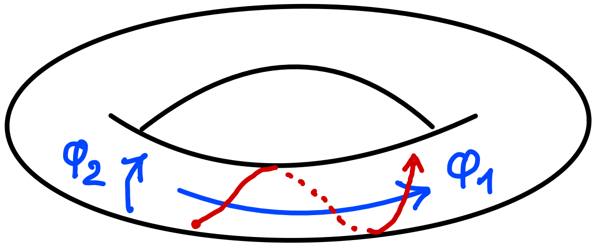
$$\dot{\varphi} = \frac{\partial \tilde{H}}{\partial I} = \omega$$

Dependence on number N of d.o.f.:

- $N=1$: $\quad I = \text{const} \Rightarrow$ motion on circle with $\quad \varphi = \omega t + \varphi_0$

- $N=2$: $\quad \vec{I} = \text{const} \Rightarrow$ motion on product of two circles ($\hat{=}$ 2-torus) in 4D phase space $\quad \vec{\varphi} = \vec{\omega} t + \vec{\varphi}_0$

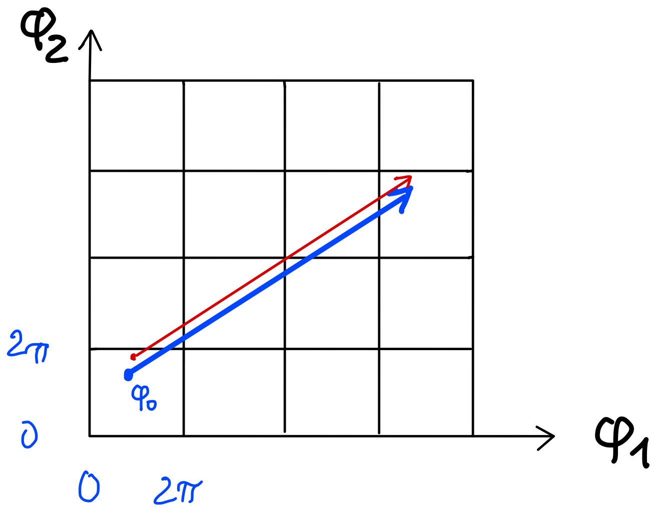
torus:



$\vec{I}$ selects torus

$\vec{\varphi}$ determines point on torus

plane:



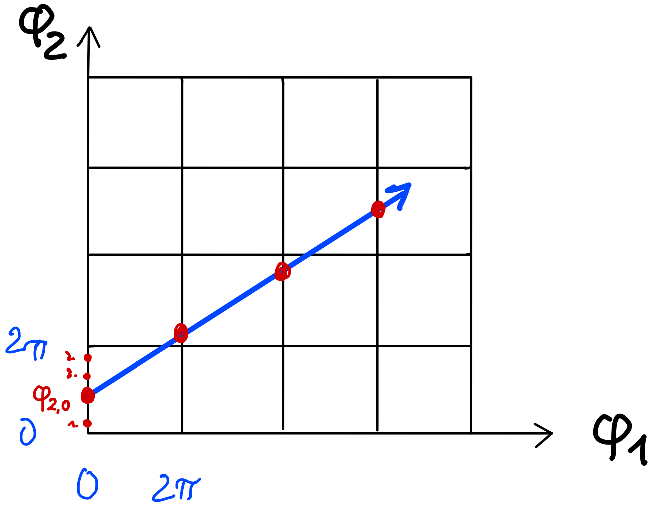
different $\vec{\varphi}_0$ give parallel lines

Poincaré section (more general def. later):

take φ_2 whenever $\varphi_1 = 0 \mod 2\pi$

$$\varphi_1 = k \cdot 2\pi = \omega_1 t$$

$$\varphi_2 = \omega_2 t + \varphi_{2,0}$$



$$\varphi_{2,k} := \varphi_{2,0} + 2\pi k \frac{\omega_2}{\omega_1} \mod 2\pi$$

2 cases: a) $\frac{\omega_2}{\omega_1} = \frac{r}{s}$ rational $(r, s \in \mathbb{Z})$

$$\Rightarrow \varphi_{2,s} = \varphi_{2,0} + 2\pi \underbrace{s \frac{\omega_2}{\omega_1}}_{r} = \varphi_{2,0} \mod 2\pi$$

- periodic with periods

- $\{\varphi_{2,k}\}$ contains s equally spaced points in $[0, 2\pi]$

b) $\frac{\omega_2}{\omega_1}$ irrational ($\nexists r,s \in \mathbb{Z}$)

$\Rightarrow \forall k \quad \varphi_{2,k} \neq \varphi_{2,0} \mod 2\pi$

$\Rightarrow \{\varphi_{2,k}\}$ is dense in $[0, 2\pi]$

Def. dense: A is dense in B, if in ε-neighborhood of any point of B one finds an element of A

proof: • divide $[0,2\pi]$ into k intervals of length $\varepsilon = \frac{2\pi}{k}$

• take $k+1$ points $\varphi_{2,0}, \dots, \varphi_{2,k}$ mod 2π

$\Rightarrow$ 2 are in same interval, e.g. $\varphi_{2,p}$ and $\varphi_{2,q}$

$\Rightarrow |\varphi_{2,p} - \varphi_{2,q}| < \varepsilon$

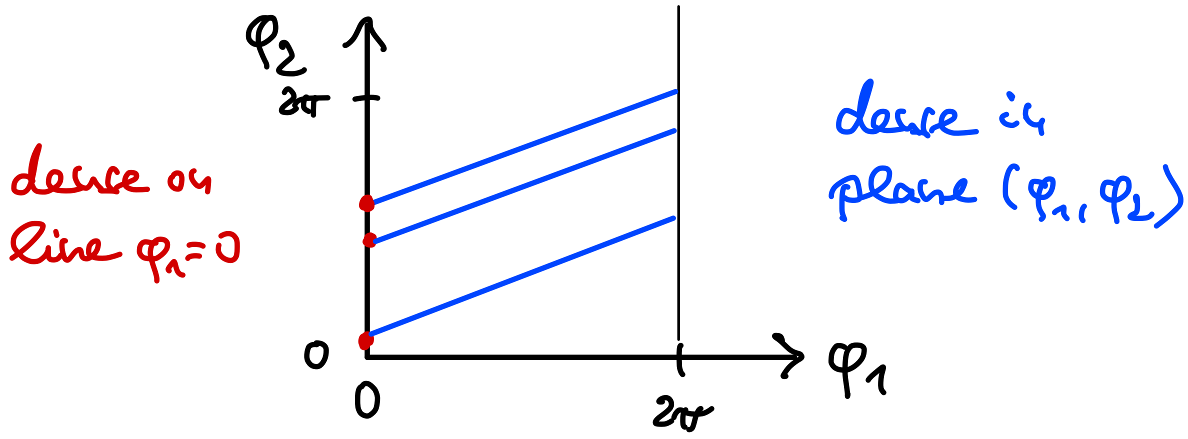
$\Rightarrow |\varphi_{2,p-q} - \varphi_{2,0}| < \varepsilon$

• for $j = 0, 1, 2, \dots$: $\varphi_{2,j(p-q)}$ makes steps $< \varepsilon$

$\Rightarrow$ comes to ε-neighborhood of any φ $\square$

$\Rightarrow$ Jacobi theorem (1835)

Every orbit with irrational $\frac{\omega_2}{\omega_1}$ is dense on torus



- $N \geq 3$ (generalization of rational vs. irrational):

N frequencies ω_j are incommensurate (rationally independent)

if for $k_j \in \mathbb{Z}$:

$$k_1 \omega_1 + \ldots + k_N \omega_N = 0 \quad \Leftrightarrow \quad k_1 = \ldots = k_N = 0$$

$N = 2$: $k_1 \omega_1 + k_2 \omega_2 = 0$ rational: $\frac{\omega_1}{\omega_2} = -\frac{k_2}{k_1}$ $\Rightarrow$ commensurate

irrational: incommensurate

Def. ergodicity :

time average = phase space average for any function $f(\vec{\varphi})$ on torus $\vec{I}$

$$\langle f(\vec{\varphi}) \rangle_t := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\vec{\varphi}(\vec{\varphi}_0, t))\, dt$$

$$\shortparallel$$

$$\langle f(\vec{\varphi}) \rangle_{torus} := \frac{1}{(2\pi)^N} \int f(\vec{\varphi})\, d\varphi^N$$

Theorem: Integrable dynamics $\vec{\varphi}(t) = \vec{\varphi}_0 + \vec{\omega} t$ on incommensurate torus is ergodic.

proof: exercise 2.2

$\Rightarrow$ orbit is dense on N-torus

$\Rightarrow$ time spent in region D is proportional to area of D