2.4. Linear maps on torus – Arnold's cat map

- dynamical system discrete in time $\rightarrow$ map
- linear map: linear dynamics not just near fixed point, but everywhere

  Can that be interesting?

  – chaotic

  – general concepts

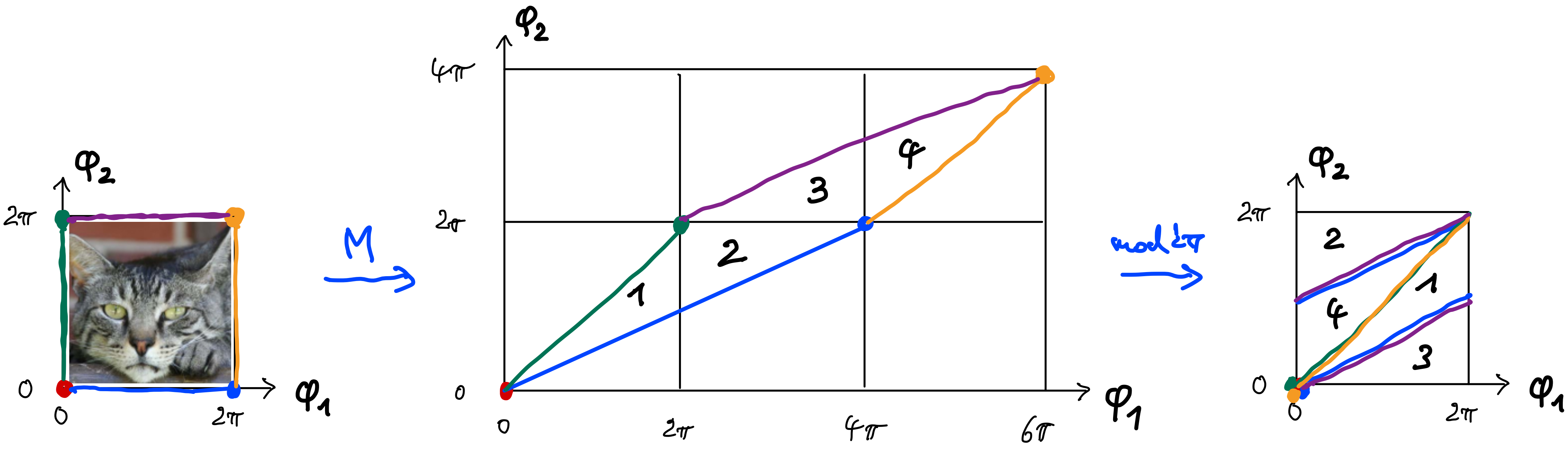
  - invariant manifolds
  - homoclinic points
  - periodic points

1-torus: $\varphi' = m\,\varphi \mod 2\pi$ volume cons.: $|m|=1$

2-torus: $\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \underbrace{\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}}_{M} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \mod 2\pi$ volume cons.: $|\det M| = 1$

example 1: $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotation by $90°$

example 2 : $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ Arnold's cat map (Anosov diffeomorphism)



observation: - disorder, „chaos"

- order after some iterations

16

?

2.4.1. Fixed points and nearby dynamics

Fixed points: $M\vec{x}_f = \vec{x}_f \quad \Rightarrow \quad (M-\mathbb{1})\,\vec{x}_f = 0$

Dynamics near fixed point (at origin):

- discrete in time: $\vec{\varphi}\,' = M\vec{\varphi}$ $\rightsquigarrow$ solve $M\vec{\xi} = \gamma\vec{\xi}$

  How is γ related to λ?

- continuous in time: $\dot{\vec{x}} = A\vec{x}$ $\rightsquigarrow$ $\vec{x}(t) = e^{At}\,\vec{x}(0)$

  $\rightsquigarrow$ $\vec{x}(t+1) = e^{A}\,\vec{x}(t)$

  $\rightsquigarrow$ solve $A\vec{\xi} = \lambda\vec{\xi}$

$$\boxed{\gamma = e^{\lambda}}$$

Cat map:

- fixed points: $M - \mathbb{1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow \vec{x}_f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

· eigenvalues: $\det(M-\gamma \mathbb{1}) = \det\begin{pmatrix} 2-\gamma & 1 \\ 1 & 1-\gamma \end{pmatrix} \overset{!}{=} 0 \Rightarrow \gamma_{1/2} = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$

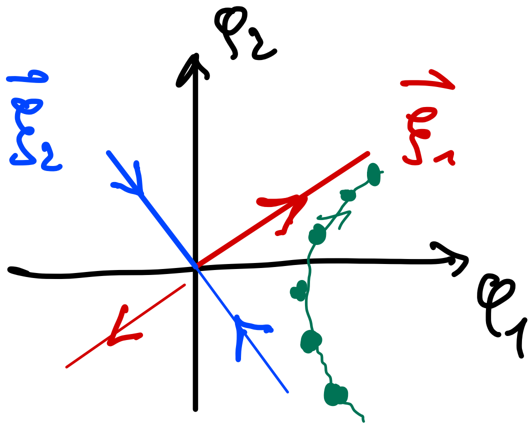
$\Rightarrow \quad \gamma_1 > 1 \qquad (\lambda_1 > 0) \quad$ expanding

$\gamma_2 = \frac{1}{\gamma_1} < 1 \qquad (\lambda_2 = -\lambda_1 < 0) \quad$ contracting

$\Rightarrow$ hyperbolic fixed point (no $\mathrm{Re}\,\lambda = 0 \Leftrightarrow$ no $|\gamma| = 1$)

· eigenvectors:

$$\vec{\xi}_{1/2} = \sqrt{\frac{2}{5 \pm \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{\pm\sqrt{5}-1}{2} \end{pmatrix}$$



(here: orthogonal)

· dynamics near fixed point:

$$\vec{\varphi}\,' = M\vec{\varphi} = M\left(\alpha\vec{\xi}_1 + \beta\vec{\xi}_2\right) = \gamma_1\alpha\vec{\xi}_1 + \gamma_2\beta\vec{\xi}_2$$

2.4.2. Invariant manifolds

Which set of points approaches hyperbolic fixed points?

Stable invariant manifold $W^s := \left\{ \vec{x} \,\middle|\, \lim_{n\to\infty} T^n \vec{x} = \vec{x}_f \right\}$

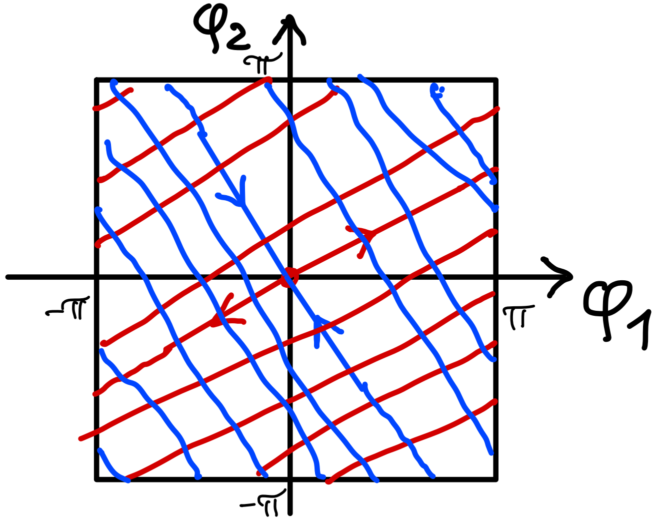
Which set of points approaches hyperbolic fixed points in negative time?

Unstable invariant manifold $W^u := \left\{ \vec{x} \,\middle|\, \lim_{n\to\infty} T^{-n} \vec{x} = \vec{x}_f \right\}$

Defs.:

- <u>invariant</u> : a set W is invariant under a map T if $TW = W$
- <u>manifold</u> of dimension N : topological space locally homeomorph to $\mathbb{R}^N$

Cat map:



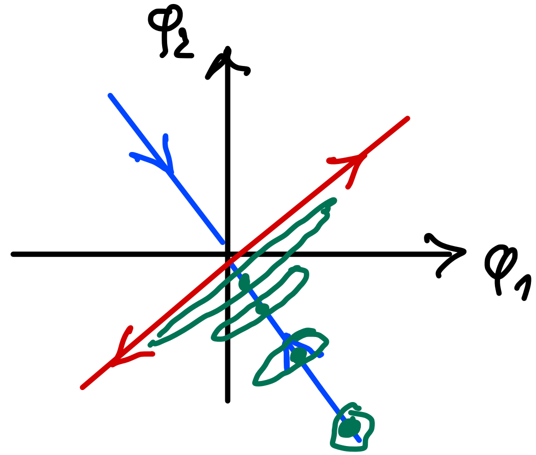
- W^u is straight line along $\vec{\xi}_1$
- W^s is straight line along $\vec{\xi}_2$

(general stable/unstable manifolds are not straight lines, just close to fixed point)

- slope of W^u, W^s is irrational $\Rightarrow$ W^u, W^s are dense on torus (Jacobi thm.)

- check Poincaré recurrence theorem:

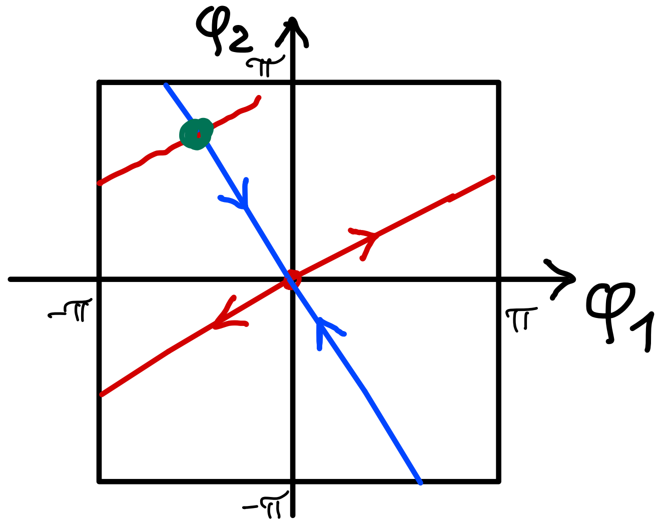
all points on W^s converge to fixed point
– they never return to their neighborhood
– this set is dense
but of measure zero



2.4.3. Homoclinic points

Def.: Intersection of stable and unstable manifold of the same hyperbolic fixed point $\longrightarrow$ <u>homoclinic</u> point

(of different hyperbolic fixed points $\longrightarrow$ <u>heteroclinic</u> point)



remark:

existence of one homoclinic point

$\Downarrow$ (later, homoclinic tangle)

chaotic dynamics

Cat map:

- chaotic dynamics (sensitive dependence on initial condition)

  - choose $\vec{\varphi}_\alpha$, $\vec{\varphi}_\beta$ with $|\vec{\varphi}_\alpha - \vec{\varphi}_\beta| \ll 2\pi$

  - $\vec{\varphi}_\alpha = \alpha_1 \vec{\xi}_1 + \alpha_2 \vec{\xi}_2 \Rightarrow \vec{\varphi}_\alpha^{(n)} = M^n \vec{\varphi}_\alpha = \alpha_1 \gamma_1^n \vec{\xi}_1 + \alpha_2 \gamma_2^n \vec{\xi}_2 \mod 2\pi$

    $\vec{\varphi}_\beta = \beta_1 \vec{\xi}_1 + \beta_2 \vec{\xi}_2 \Rightarrow \vec{\varphi}_\beta^{(n)} = \quad = \beta_1 \ldots, \quad \beta_2 \ldots$

$$\Rightarrow \vec{\varphi}_\alpha^{(n)} - \vec{\varphi}_\beta^{(n)} = (\alpha_1 - \beta_1) \underbrace{\gamma_1^n}_{\downarrow \atop \infty} \vec{\xi}_1 + (\alpha_2 - \beta_2) \underbrace{\gamma_2^n}_{\downarrow \atop 0} \vec{\xi}_2 \mod 2\pi$$

$$\Rightarrow |\vec{\varphi}_\alpha^{(n)} - \vec{\varphi}_\beta^{(n)}| \overset{n \gg 1}{\approx} |\alpha_1 - \beta_1| \gamma_1^n$$

exponential growth of initial distance along $\vec{\xi}_1$ until $O(2\pi)$

- $\exists$ ∞-many homoclinic points. They are dense for cat map.

proof:
- $\exists$ one, as $\vec{e}_{\xi_1}, \vec{e}_{\xi_2}$ are not parallel
- iteration gives new h.p.
- no periodicity, since convergence to fixed point

$\Rightarrow$ ∞-many $\square$

- W^u and W^s of cat map are dense and intersect ($90°$)

$\Rightarrow$ h.p. are dense $\square$

2.4.4 Periodic points

Def.: $\vec{\varphi}_0$ is a periodic point with period p if

$$T^p \vec{\varphi}_0 = \vec{\varphi}_0$$

remark: A periodic point is a fixed point of the p-fold map.

$\longrightarrow$ exercise 2.3