3. Nonlinear systems

3.1. Hartmann-Grobmann theorem

Have results for linear systems any use for nonlinear systems?

no eigenvalue of A has a zero real part

H.-G. thm.: Let $\vec{x}_f$ be a hyperbolic fixed point of $\dot{\vec{x}} = \vec{f}(\vec{x})$.

$\exists$ neighborhood of $\vec{x}_f$ on which the flow $\vec{x}(\vec{x}_0, t)$

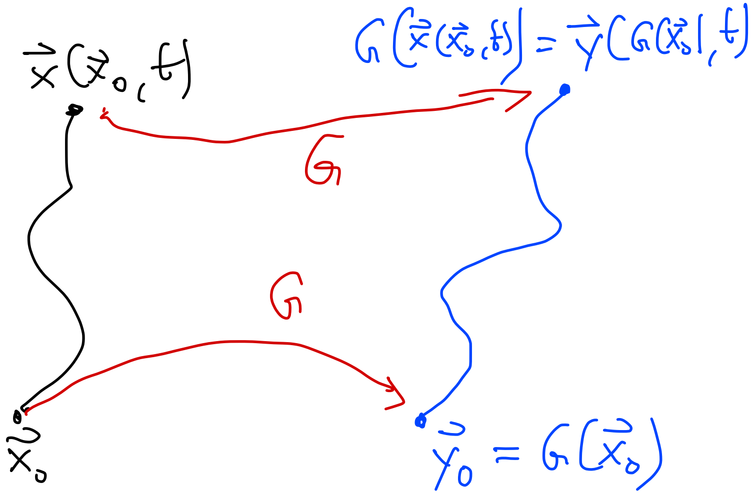
is topologically conjugate to the linear flow $e^{At}\vec{x}_0$.

$$A = \left.\frac{\partial \vec{f}(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_f}$$

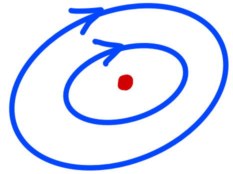
dynamical matrix of linearized dynamics near $\vec{x}_f$

flows $\vec{x}(\vec{x}_0, t)$ and $\vec{y}(\vec{y}_0, t)$ are topologically conjugate if there is a homeomorphism $\vec{y} = \vec{G}(\vec{x})$ s.t. $\forall \vec{x}_0, t$

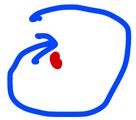
$$G(\vec{x}(\vec{x}_0, t)) = \vec{y}(G(\vec{x}_0), t)$$



i.e. one-to-one correspondence of orbits remark: not true near elliptic fixed point (eigenvalues $\pm i\omega$)



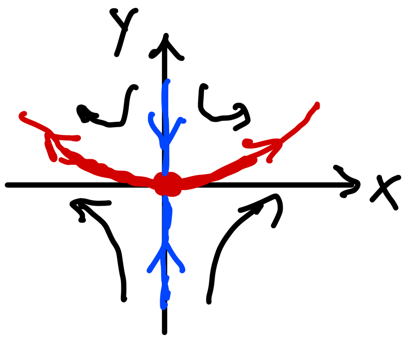
reason: a) nonlinear terms

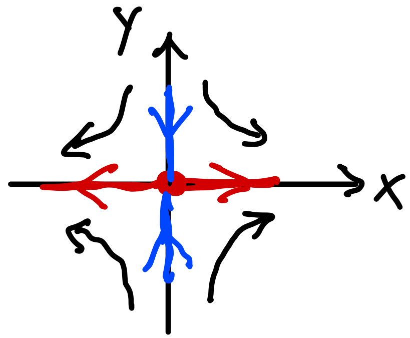


b) Hamiltonian systems: very complex dynamics (KAM theorem, Poincaré–Birkhoff theorem)

example:

<u>nonlinear</u>		<u>linear</u>	
$\dot{x} = x$	fixed point $(0,0)$	$\dot{x} = x$	$x(t) = e^{t} x_0$
$\dot{y} = -y + x^2$	neglect nonlin. terms	$\dot{y} = -y$	$y(t) = e^{-t} y_0$





nonlinear system: determine stable/unstable invariant manifold of $(0,0)$

1. line $x=0$: $\Rightarrow \dot{x}=0,\ \dot{y}=-y$

dynamics : $\left.\begin{aligned} x(t) &= 0 \\ y(t) &= e^{-t} y_0 \end{aligned}\right\}$ stable manifold

2. $x \neq 0$: which function $y(x)$ follows dynamics :

$$\frac{dy}{dx} \overset{!}{=} \frac{\dot{y}}{\dot{x}} = \frac{-y+x^2}{x}$$

$$\underbrace{x \cdot \frac{dy}{dx} + y}_{\frac{d(xy)}{dx}} = x^2$$

$$xy = \frac{1}{3}x^3 + c$$

$$y(x) = \frac{1}{3}x^2 + \frac{c}{x}$$

manifold starts at $(0,0) \Rightarrow c=0$

dynamics:

$$x(t) = e^{t} x_0$$

$$y(t) = \frac{1}{3} e^{2t} x_0^2 \longleftarrow$$

unstable manifold

homeomorphism (from lin. to nonlin.): $\vec{G}(x,y) = (x, y + \frac{1}{3}x^2)$

Invariant manifold theorem:

Let $\vec{x}_f$ be a hyperbolic fixed point of $\dot{\vec{x}} = \vec{f}(\vec{x})$.

At $\vec{x}_f$ stable and unstable manifolds of the nonlinear system

are tangential

to the stable and unstable manifolds of the linear system.

3.2. Poincaré section and Poincaré map

<u>Poincaré section:</u>

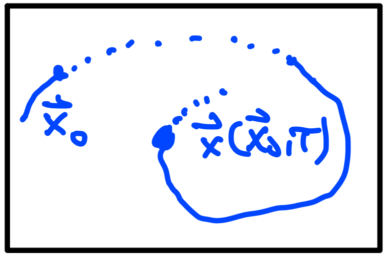
phase space $(\vec{q},\vec{p})$: $2N$ dimensional

energy surface $H(\vec{q},\vec{p}) = E = \text{const}$: $2N-1$ dimensional

Poincaré section Σ : $2N-2$ dimensional

hyperplane or manifold with crossing direction

Σ



e.g. $q_N \equiv 0$, $p_N > 0$

orbits should cross Σ non-tangential (globally not always possible)

example: particle in 2D potential: $H(\vec{q},\vec{p}) = \dfrac{p_x^2 + p_y^2}{2m} + V(x,y)$

phase space : x, y, p_x, p_y

energy surface: $p_y = \pm\sqrt{2(E - V(x,y))m - p_x^2}$

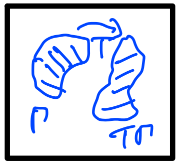
Poincaré section: e.g. $y = 0$, $p_y > 0$

coordinates on Σ: x, p_x

Poincaré map $T: \Sigma \to \Sigma$

$$\vec{x}_0 \to \vec{x}(\vec{x}_0, \tau(\vec{x}_0)) =: T(\vec{x}_0)$$

$\uparrow$ return time for consecutive crossing of Σ in the same direction



remarks:

- T is diffeomorphism
- return time $\tau(\vec{x}_0)$ depends on $\vec{x}_0$
- Poincaré-Cartan-Integral theorem

  $\Rightarrow$ area conservation of T

$$\oint_{\Gamma} \vec{p}\, d\vec{q} + H\, dt = \oint_{T\Gamma} \vec{p}\, d\vec{q} + H\, dt$$

$$\oint_{\Gamma} \vec{p}\, d\vec{q} = \oint_{T\Gamma} \vec{p}\, d\vec{q}$$

- Poincaré map of time continuous Hamiltonian system

  $\to$ time discrete Hamiltonian system

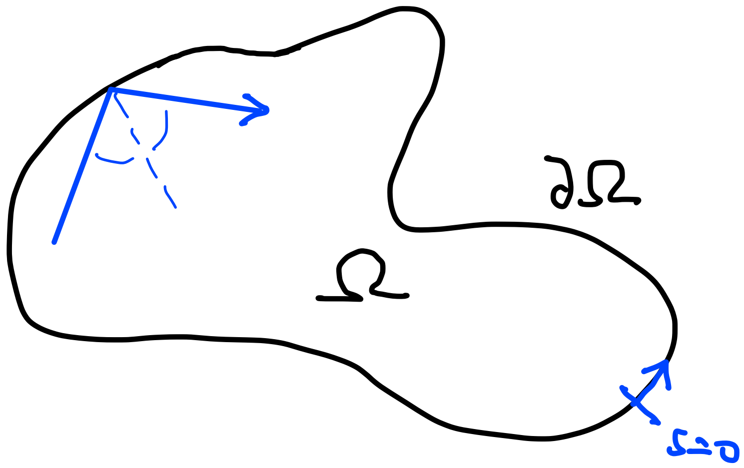
- periodic orbit with n intersections

  $\to$ periodic points $\vec{x}_1, \dots, \vec{x}_n$ of T with period n

$$T^n(\vec{x}_j) = \vec{x}_j \quad \forall j$$

3.3. Billiards

- Dynamics along straight lines (no diff. eq. to be solved)
- Reflection at boundary $\partial\Omega$

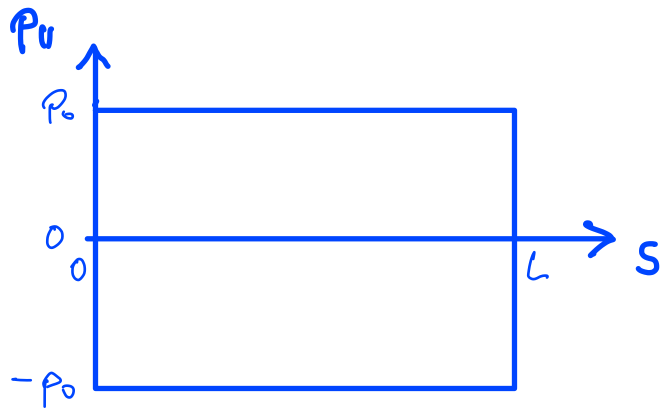


Phase space $\Gamma = \{ (\vec{q}, \vec{p}) \mid \vec{q} \in \Omega, \vec{p} \in \mathbb{R}^2 \}$

Dynamics independent of E : fix $|\vec{p}| = p_0 = 1$

Birkhoff coordinates $(s, p_\parallel)$:

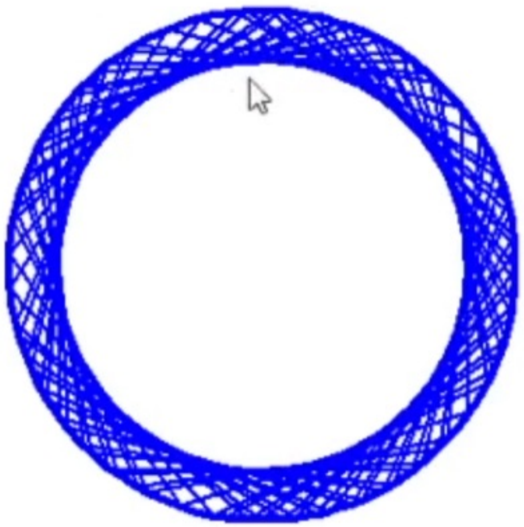
- $s \in [0, L)$ position along $\partial\Omega$ of length L
- $p_\parallel \in [-p_0, p_0]$ momentum parallel to $\partial\Omega$ (does not change by reflection)

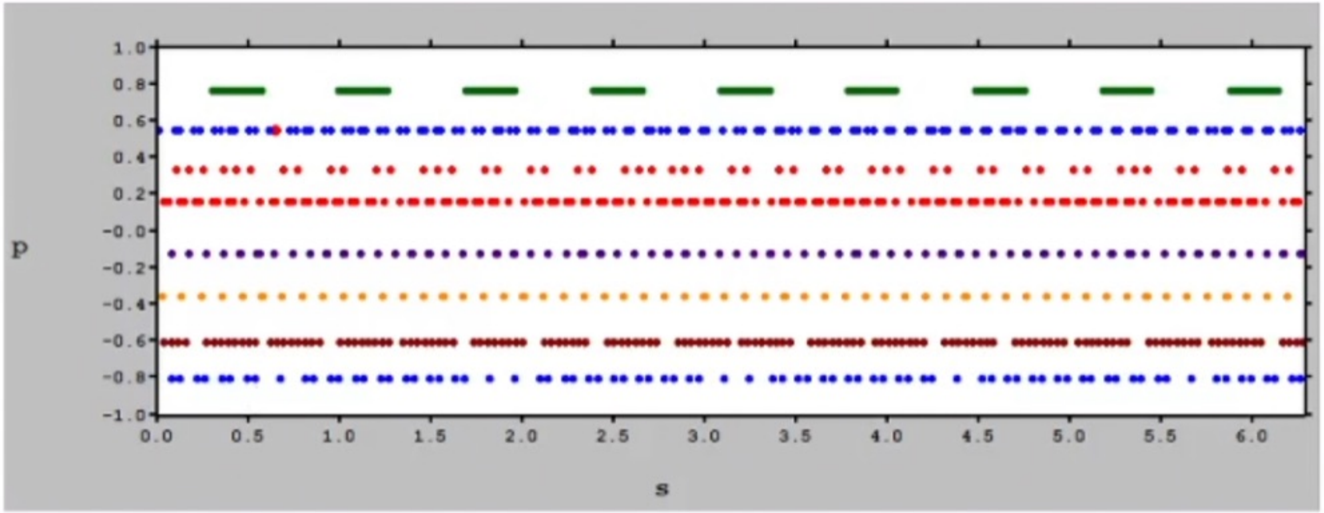


Poincaré section with Σ infinitesimal inwards from boundary

Billiard examples with $N=2$ d.o.f.

- Circle billiard



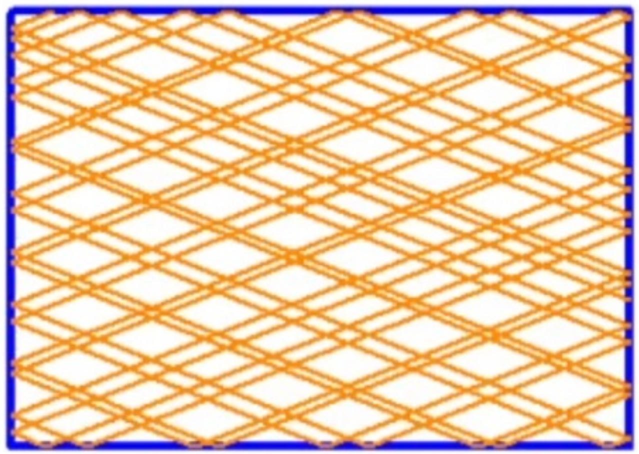


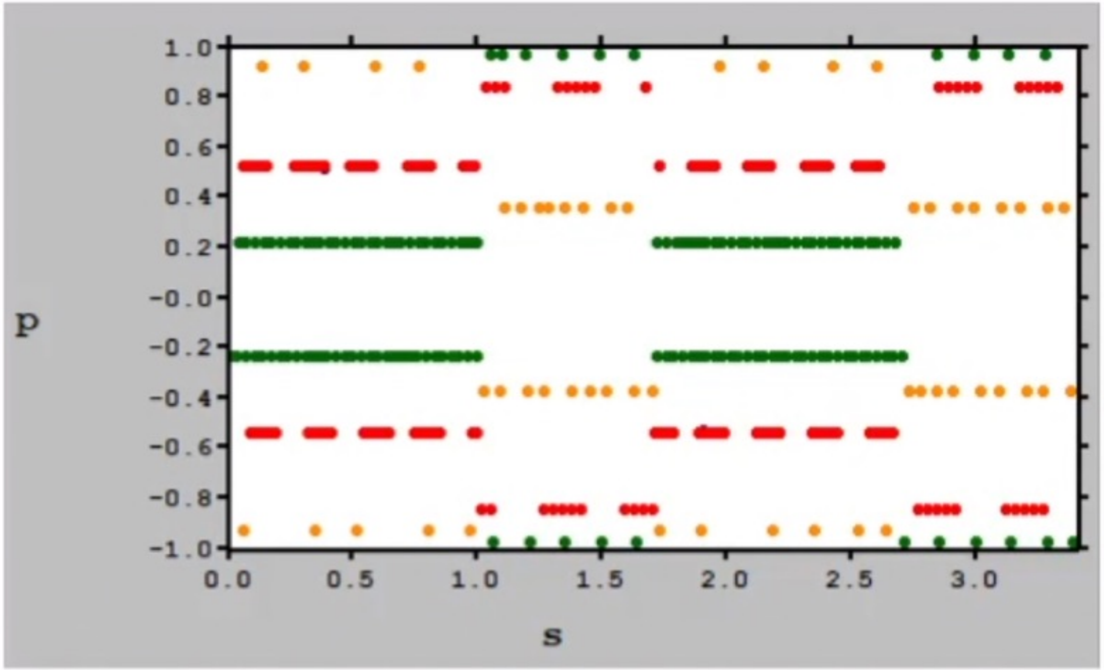
2 constants of motion:
- energy
- angular momentum

$\Big\}\Rightarrow$ integrable

$$\vec{L} = \vec{r} \times \vec{p} = d\, p_0\, \vec{e}_z$$

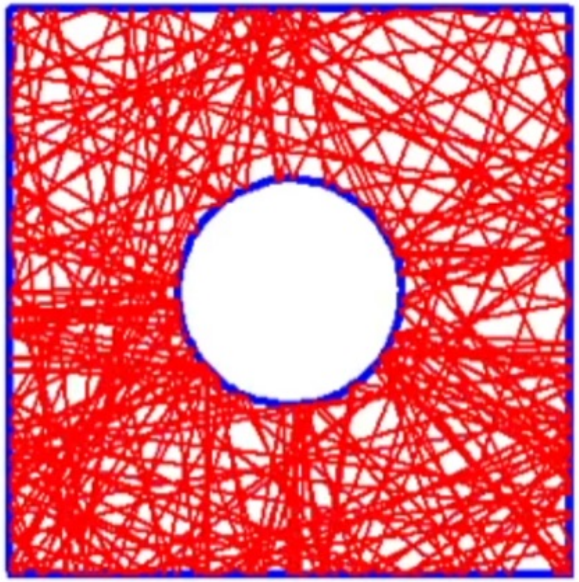
- Rectangle billiard

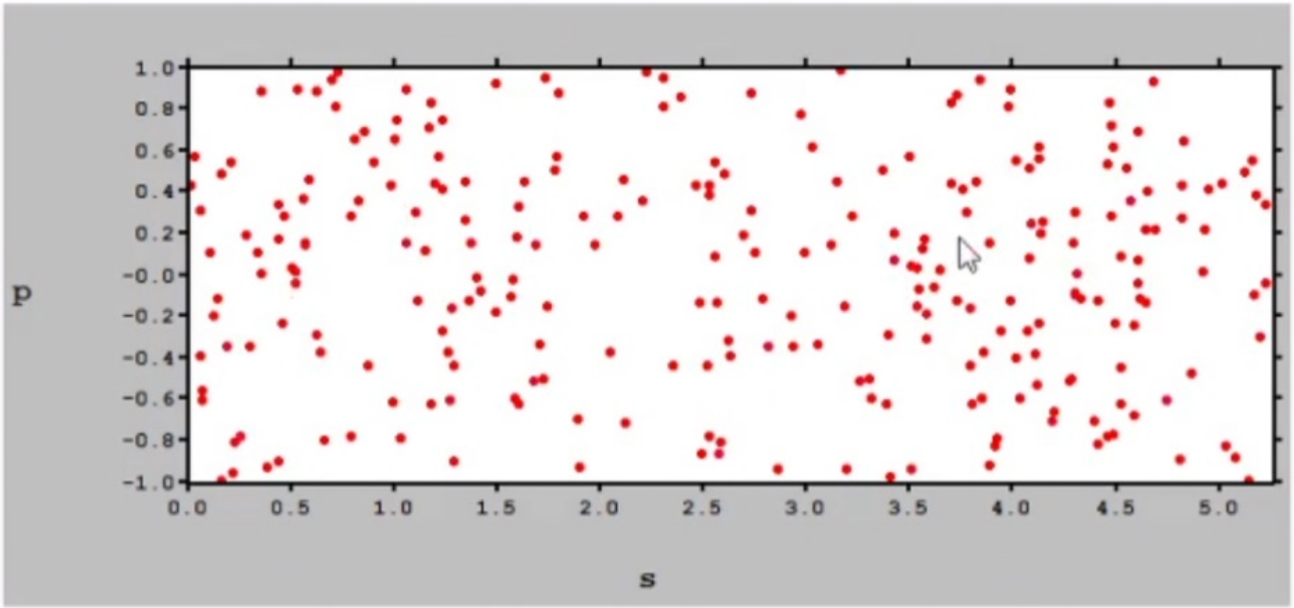




2 constants of motion : $\left.\begin{array}{l} \bullet\ T_x = \dfrac{p_x^2}{2} \\[2ex] \bullet\ T_y = \dfrac{p_y^2}{2} \end{array}\right\} \Rightarrow$ integrable

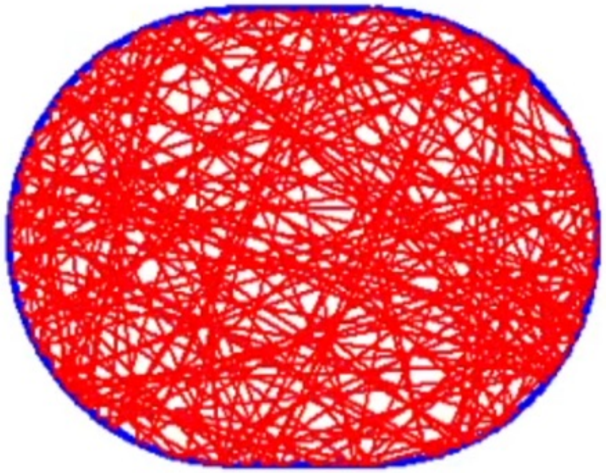
- Sinai billiard

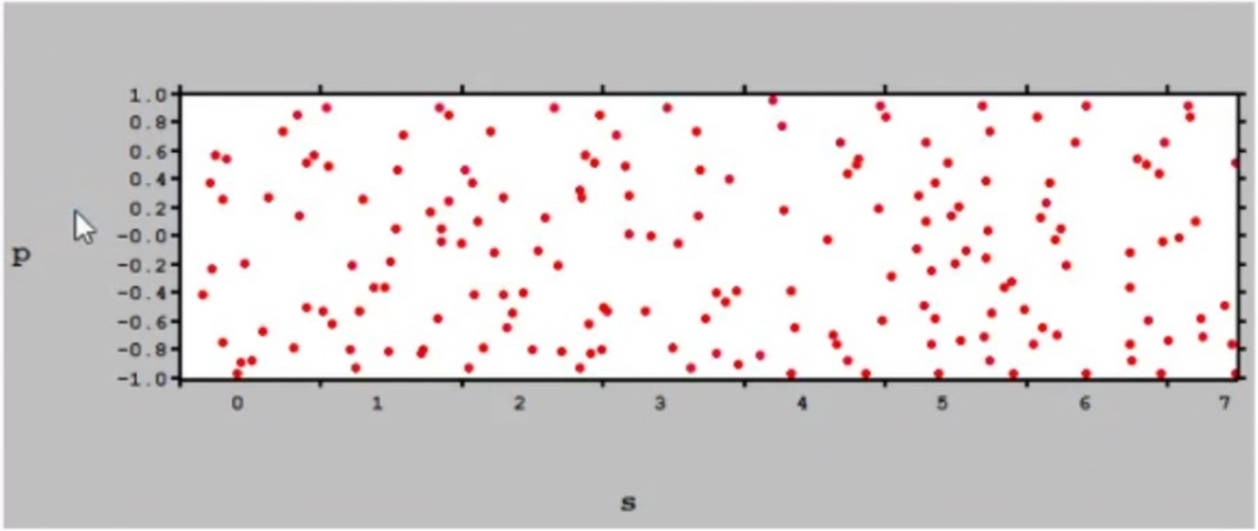




- non-integrable
- reflection at circle is defocussing
- chaotic dynamics (exceptions of measure zero)
- ergodic

• Bunimovich stadium billiard





- non-integrable
- reflection at circle is defocussing
- chaotic dynamics (exceptions of measure zero)
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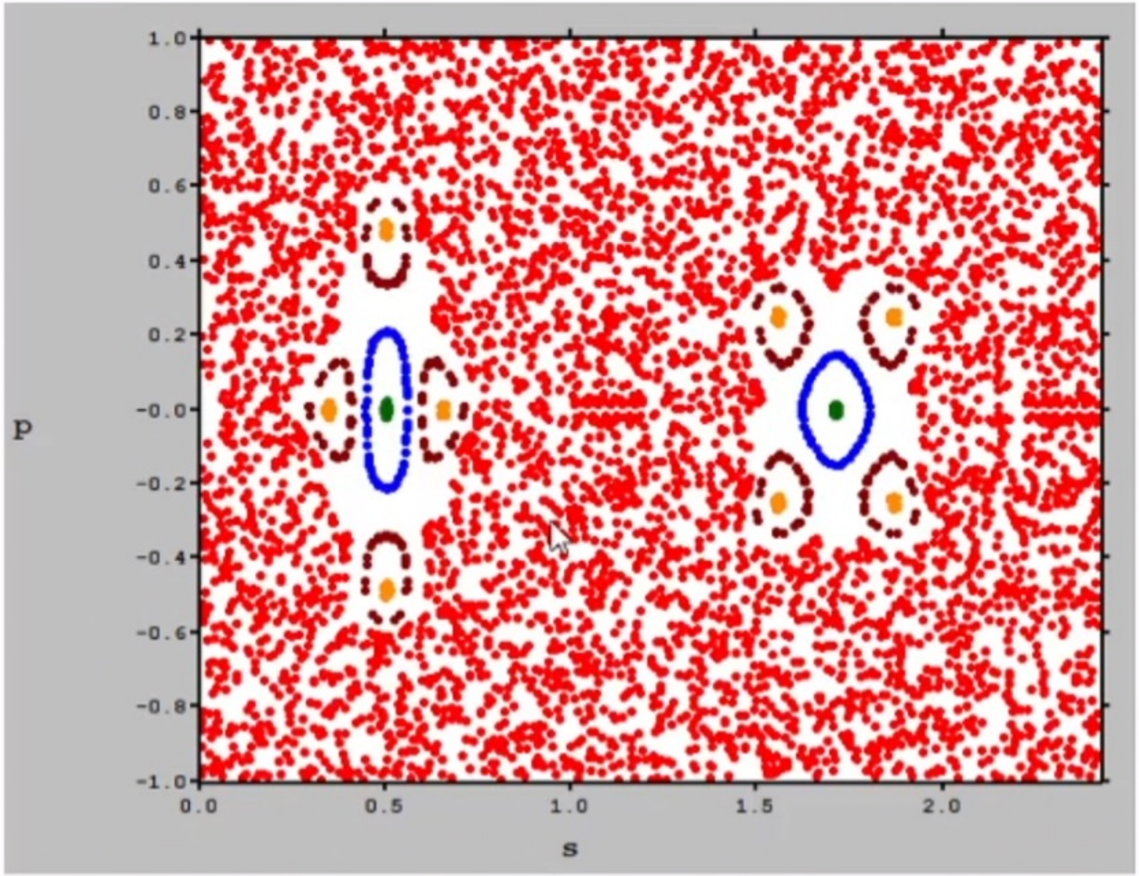
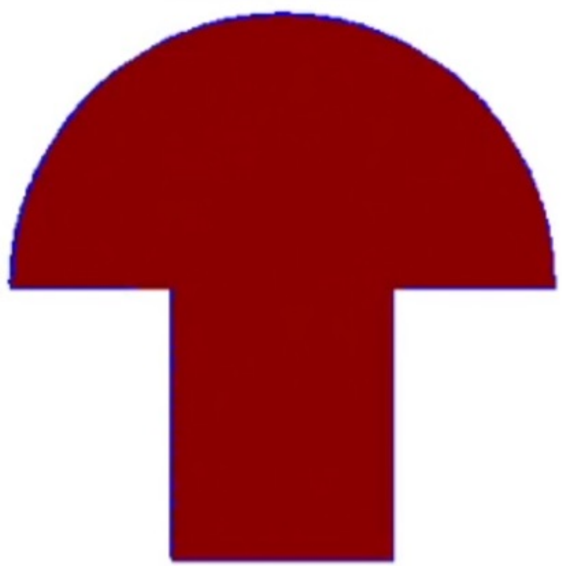
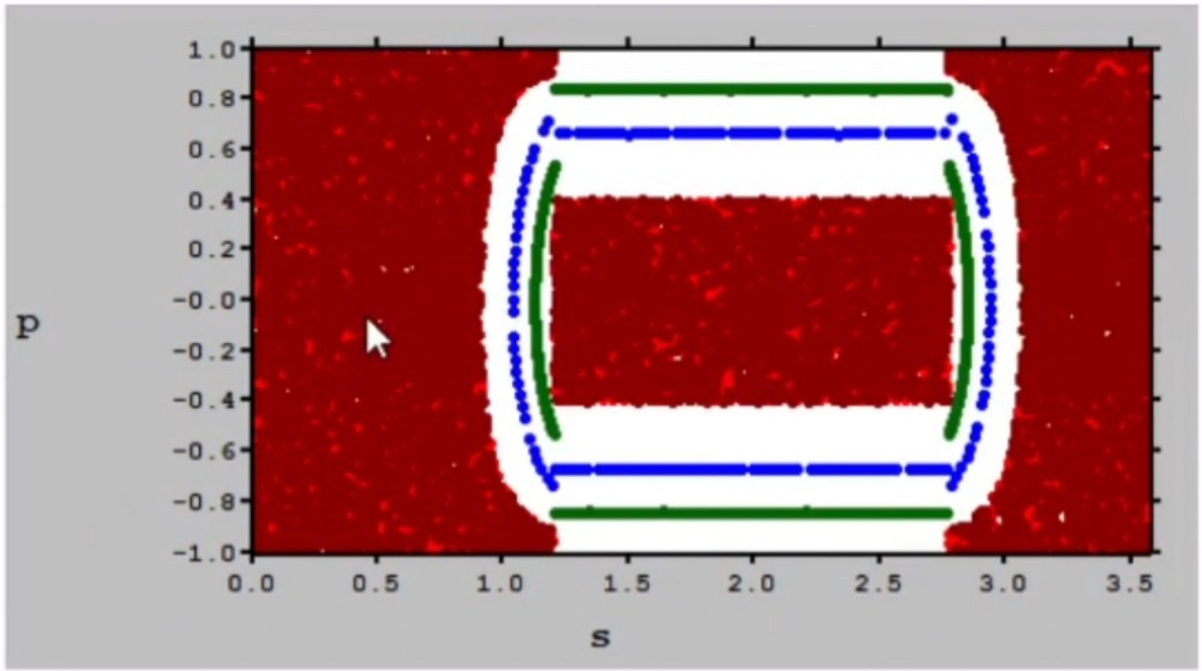
• Cosine billiard



- non-integrable
- phase space regions with → regular dynamics, → chaotic dynamics } mixed phase space
- „generic" Hamiltonian system for $N=2$

- Mushroom billiard





- non-integrable
- phase space regions with → regular dynamics / → chaotic dynamics } mixed phase space
- sharply divided phase space, non-generic