3.4. Time-periodic systems

$H(\vec{q}, \vec{p}, t+\tau) = H(\vec{q}, \vec{p}, t)$ period τ

extended phasespace: $\vec{q}, \vec{p}, t$ with $2N+1$ dimensions

simplest non-integrable system: $N=1$ with periodic driving (1.5 d.o.f.)

$N=2$ autonomous

Stroboscopic Poincaré section: consider times $t \bmod \tau = t_0$

Poincaré map T: $\vec{x}_0 \to \vec{x}(\vec{x}_0(t_0), t_0+\tau)$

example: driven pendulum (see video 1.1.: 21:00 – 28:30)

general remarks:

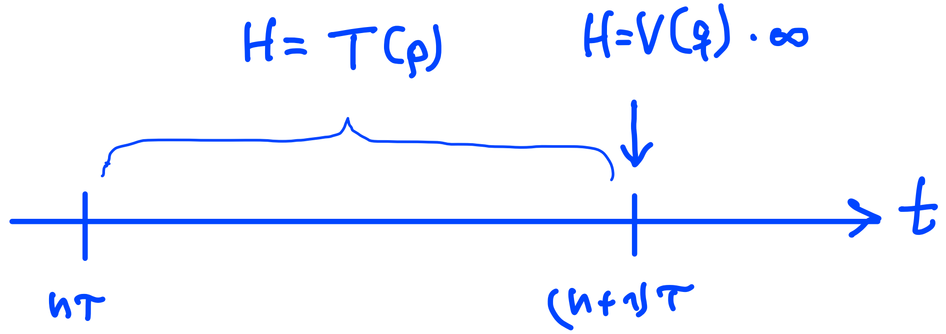
- fixed point in Poincaré map $\leftrightarrow$ periodic orbit for flow
- stable/unstable manifolds:

  1D manifold in Poincaré map $\leftrightarrow$ 2D manifold for flow

3.4.1. Kicked systems

motivation: no diff. eq. has to be solved $\Rightarrow$ numerically fast (also q.m.)

$$H(q,p,t) = \underset{\uparrow \text{kinetic energy}}{T(p)} + \underset{\uparrow \text{kick potential}}{V(q)} \; \underset{\uparrow \text{period}}{\tau} \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$



Time evolution:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{dT}{dp} \quad \Rightarrow \quad q(t) = q(t_0) + \int_{t_0}^{t} dt' \left.\frac{dT}{dp}\right|_{p=p(t')} \qquad (1)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{dV}{dq}\,\tau \sum_n \delta(t - n\tau) \quad \Rightarrow \quad p(t) = p(t_0) - \tau \int_{t_0}^{t} dt' \left.\frac{dV}{dq}\right|_{q(t')} \sum_n \delta(t' - n\tau) \qquad (2)$$

between kicks: (2) $\Rightarrow$ p constant

(1) $\Rightarrow$ q changes

during kick: (1) $\Rightarrow$ q constant

(2) $\Rightarrow$ p changes

Poincaré map right after kick:

$$\boxed{\begin{aligned} q_{n+1} &= q_n + \tau T'(p_n) \\ p_{n+1} &= p_n - \tau V'(q_{n+1}) \end{aligned}} \quad = p_n - \tau V'\left(q_n + \tau T'(p_n)\right)$$

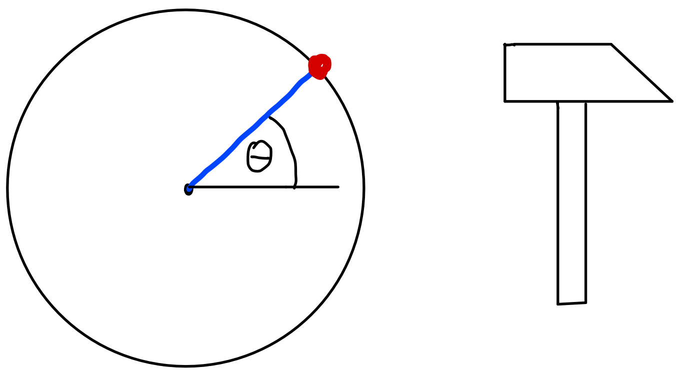
Conservation of area: use linearized map (Jacobi matrix)

$$\det \begin{pmatrix} \frac{\partial q_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{pmatrix} = \det \begin{pmatrix} 1 & \tau T''(p_n) \\ -\tau V''(q_{n+1}) & 1 - \tau V''(q_{n+1})\, \tau T''(p_n) \end{pmatrix}$$

$$= 1 \checkmark$$

3.4.2 Kicked rotor / Chirikov standard map

Physical model: kicked rotor (kicks from right with period τ)



dimensionless units:

- angle θ, angular momentum p, period $\tau = 1$
- kinetic energy: $T(p) = \frac{p^2}{2}$
- kick potential: $V(\theta) = K \cos\theta$

Poincaré map right after kick:

$$\boxed{\begin{aligned} \theta_{n+1} &= \theta_n + p_n \\ p_{n+1} &= p_n + K \sin\theta_{n+1} \end{aligned}}$$

Chirikov standard map

remark: limiting case of many dynamical systems

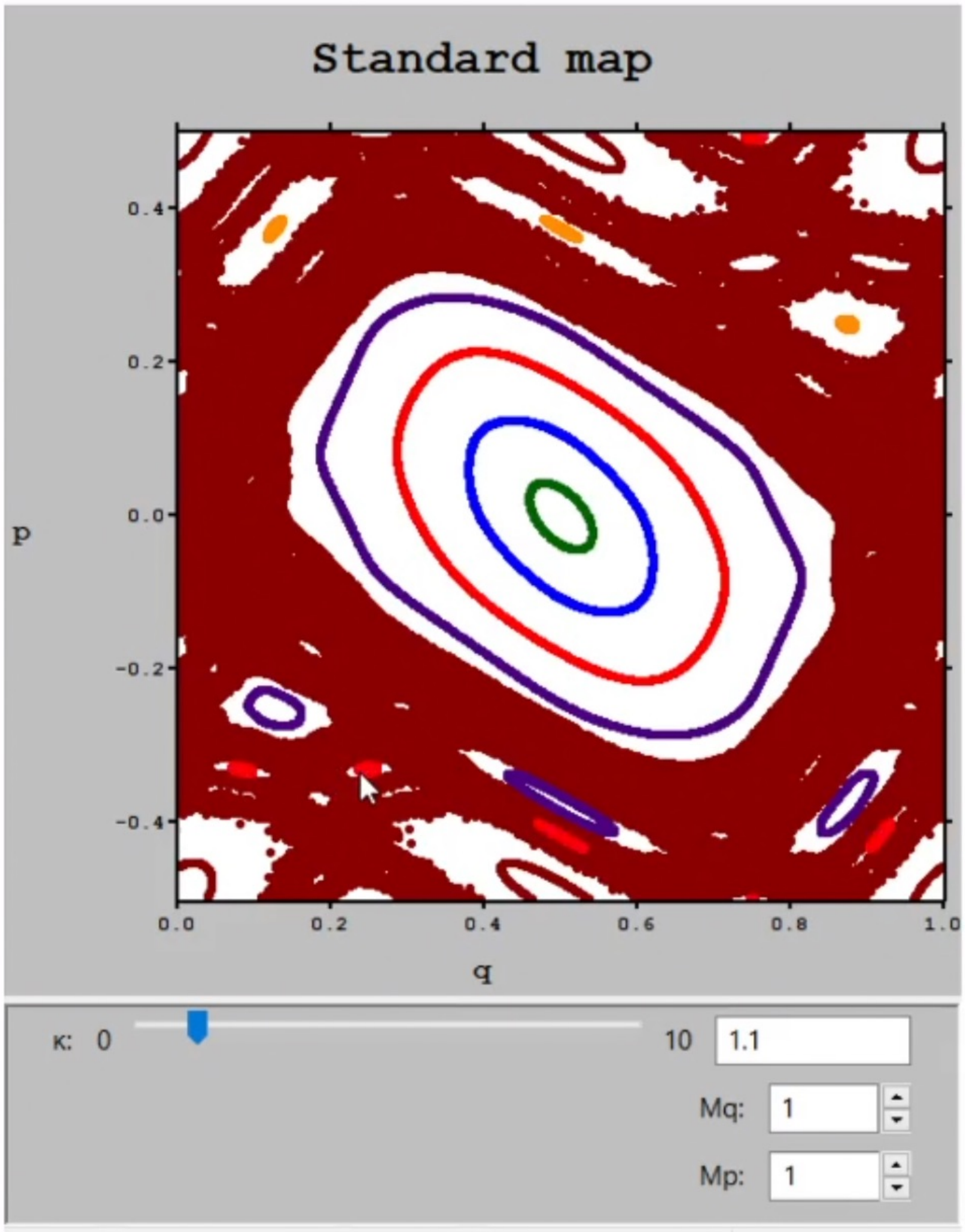
4 variants:

a) plane $\theta, p \in (-\infty, \infty)$

b) cylinder in p direction
$\theta \bmod 2\pi$

c) cylinder in θ direction
$p \bmod 2\pi$

d) torus: $\theta, p \bmod 2\pi$



$$\boxed{\begin{aligned} \theta_{n+1} &= \theta_n + p_n \\ p_{n+1} &= p_n + K \sin \theta_{n+1} \end{aligned}}$$

$$\downarrow$$

$$\boxed{\begin{aligned} q_{n+1} &= q_n + p_n \\ p_{n+1} &= p_n + \frac{K}{2\pi} \sin 2\pi q_{n+1} \end{aligned}}$$

$$q \in [0, 1]$$

$$p \in [-0.5, 0.5]$$

Chirikov standard map

$$\boxed{\begin{aligned} \theta_{n+1} &= \theta_n + p_n \\ p_{n+1} &= p_n + K \sin\theta_{n+1} \end{aligned}}$$

4 variants:

a) plane $\theta, p \in (-\infty, \infty)$

b) cylinder in p direction
θ mod 2π

c) cylinder in θ direction
p mod 2π

d) torus: θ, p mod 2π

all variants identical

when taken mod 2π in θ and p

- dynamics 2π periodic in θ
- dynamics 2π periodic in p ?

$$(\theta_0, p_0 + 2\pi) \to (\theta_1 + 2\pi, p_1 + 2\pi)$$

Fixed points: (on torus)

$$\theta_{n+1} \overset{!}{=} \theta_n \Rightarrow p_f = 0 \mod 2\pi \Rightarrow p_f = 0$$

$$p_{n+1} \overset{!}{=} p_n \Rightarrow K\sin\theta_f = 0 \mod 2\pi \Rightarrow K\sin\theta_f = 2\pi s, \; s \in \mathbb{Z}$$

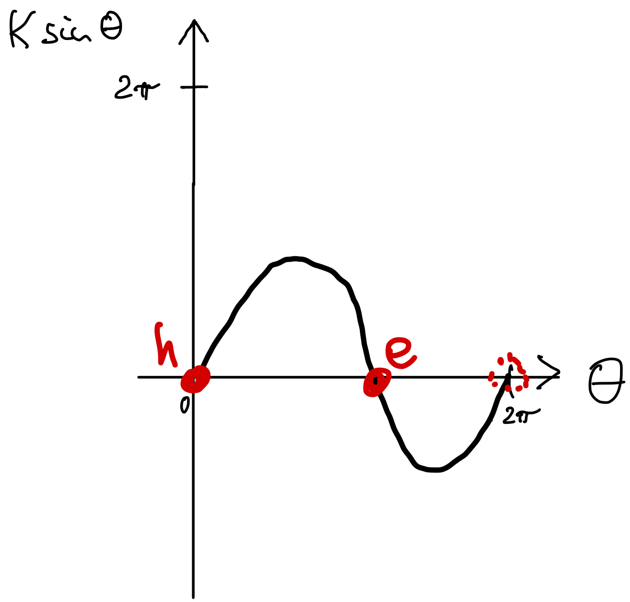
$$\Rightarrow \theta_f = \arcsin\frac{2\pi s}{K}, \; s \in \mathbb{Z} \text{ with } \left|\frac{2\pi s}{K}\right| \leq 1 \Leftrightarrow |K| \geq 2\pi|s|$$

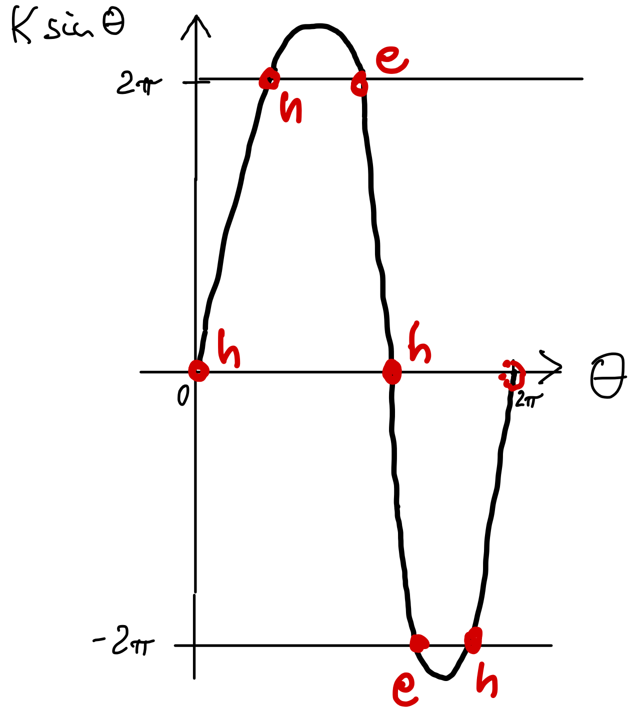
$$\theta_f = \pi - \arcsin\frac{2\pi s}{K}$$

$\Rightarrow \quad s = 0 : \quad \theta_f = 0, \pi$

$s = \pm 1$ and $K \geq 2\pi$: $\quad \theta_f = \arcsin\left(\pm \frac{2\pi}{K}\right)$

and $\quad \theta_f = \pi - \arcsin\left(\pm \frac{2\pi}{K}\right)$





Dynamics near fixed point: stability elliptic or hyperbolic?

use linearized map at fixed point (θ_f, p_f):

$$\theta_{n+1} = \theta_n + p_n$$
$$p_{n+1} = p_n + K \sin\theta_{n+1}$$

$$\begin{pmatrix} \frac{\partial\theta_{n+1}}{\partial\theta_n} & \frac{\partial\theta_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial\theta_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{pmatrix}_{\theta_f, p_f} = \begin{pmatrix} 1 & 1 \\ \underbrace{K\cos\theta_f}_{=:\alpha} & \underbrace{1+K\cos\theta_f}_{1+\alpha} \end{pmatrix}$$

eigenvalues: $\gamma_{1/2} = 1 + \frac{\alpha}{2} \pm \sqrt{\alpha\left(1+\frac{\alpha}{4}\right)}$

hyperbolic f.p. $(\gamma \in \mathbb{R})$: $\alpha\left(1+\frac{\alpha}{4}\right) > 0$

elliptic f.p. $(\gamma = e^{\pm i\omega})$: $\alpha\left(1+\frac{\alpha}{4}\right) < 0 \Rightarrow \underbrace{\alpha < 0 \text{ and } 1+\frac{\alpha}{4} > 0}_{-4<\alpha<0}$

$\alpha > -4$