3.5. KAM theorem

3.5.1. Rational approximants of irrational numbers

How to approximate an irrational number by rational numbers?

example: $\pi = 3.1415926\ldots$

$$\frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000} \qquad \text{slow convergence}$$

$$\frac{3}{1} = 3.0$$

$$\frac{22}{7} = 3.1429\ldots$$

$$\frac{333}{106} = 3.14151\ldots$$

$$\frac{355}{113} = 3.1415929\ldots \qquad \text{fast convergence}$$

Continued fraction expansion: $\sigma \in \mathbb{R}$

$$\sigma = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [a_0, a_1, a_2, \dots] \quad a_{i \geq 1} \in \mathbb{N} \quad (a_0 \in \mathbb{Z})$$

How to determine a_i?

- rational example: $\sigma = 6.4 \implies a_0 = 6$

$$\sigma = 6 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}} \implies \underbrace{\frac{1}{\sigma - 6}}_{2.5} = a_1 + \cfrac{1}{a_2 + \ddots}$$

$$\implies a_1 = 2$$
$$a_2 = 2$$
$$a_{i \geq 3} = \infty$$

$$\implies 6.4 = [6, 2, 2]$$

- irrational example: $\sigma = \pi = [3, 7, 15, 1, 292, \dots]$

rational approximants:

$$[3] = 3$$

$$[3,7] = 3 + \frac{1}{7} = \frac{22}{7}$$

$$[3,7,15] = \dots = \frac{333}{106}$$

$$[3,7,15,1] = \frac{355}{113}$$

$$[3,7,15,1,292] = \frac{103993}{33102} \qquad \text{error} < 10^{-9}$$

Quality of approximations:

$\exists$ approximation: $\quad \forall \sigma \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists \frac{p}{q} \in \mathbb{Q}: \quad \left|\sigma - \frac{p}{q}\right| < \varepsilon$

$\Leftrightarrow$ rational numbers are dense within real numbers

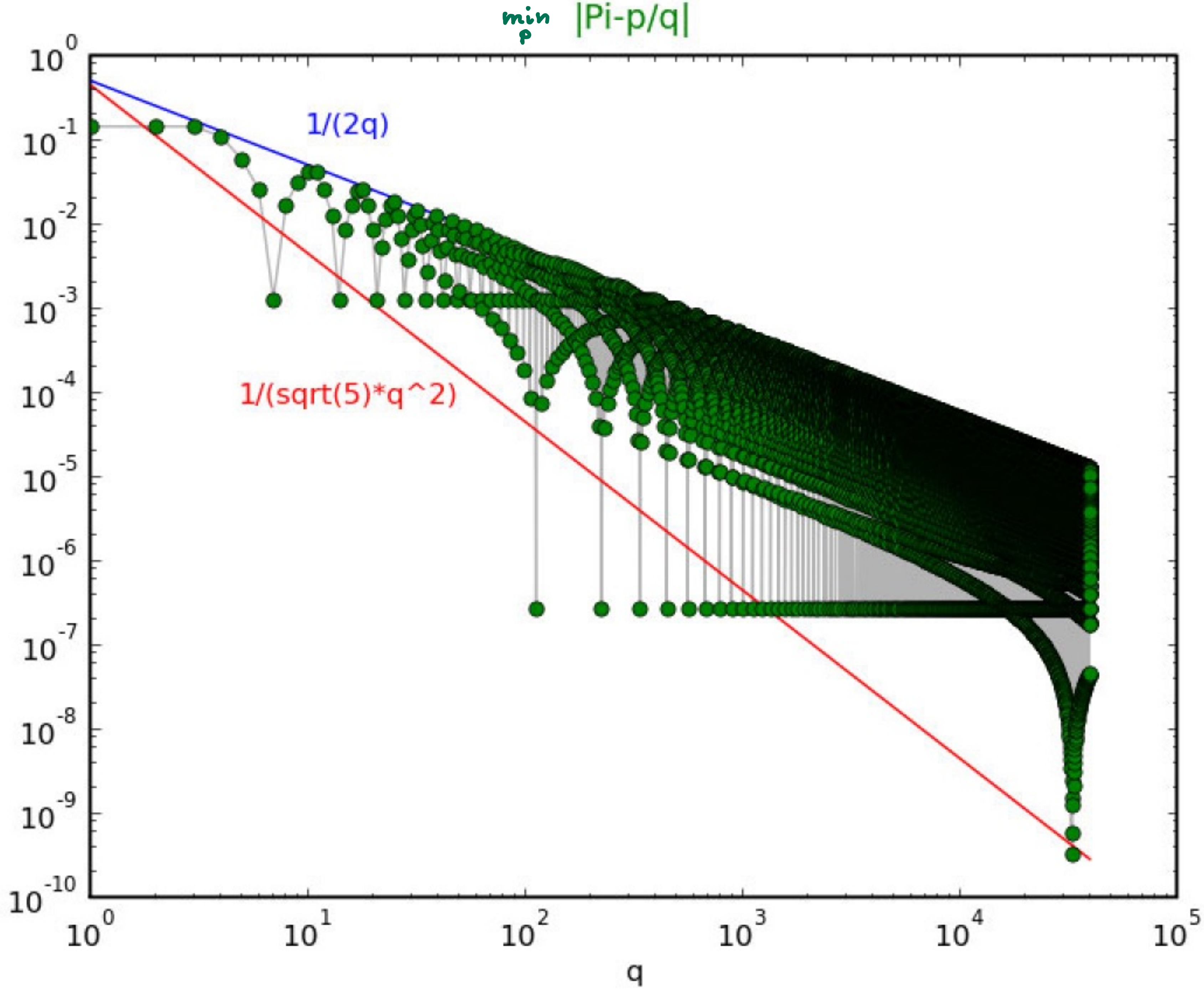
'good' approximation: $\quad \forall \sigma \in \mathbb{R} \quad \forall q \in \mathbb{N} \quad \exists p \in \mathbb{Z}: \quad \left|\sigma - \frac{p}{q}\right| \leq \frac{1}{2q}$

'best' approximation: $\quad \forall \sigma \in \mathbb{R} \quad \exists \left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}} \subseteq \mathbb{Q} \quad : \quad \left|\sigma - \frac{p_n}{q_n}\right| < \frac{1}{\sqrt{5}\, q_n^2}$

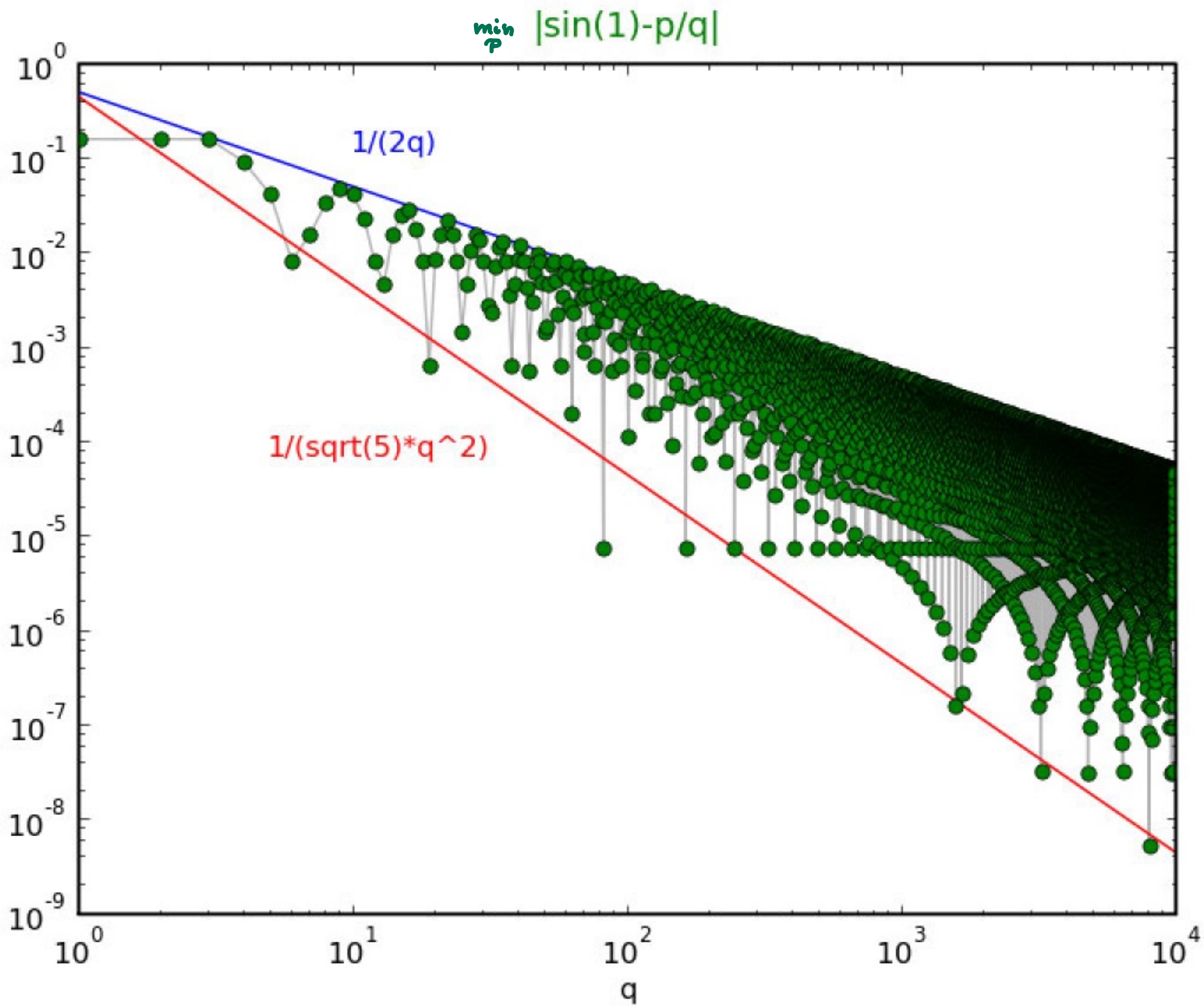
$\uparrow$

infinite series

given by cont. frac. expansion



$\min_p$ |Pi-p/q|
1/(2q)
1/(sqrt(5)*q^2)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^1
10^2
10^3
10^4
10^5
q


min p |sin(1)-p/q|
1/(2q)
1/(sqrt(5)*q^2)
q
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
10^-9
10^1
10^2
10^3
10^4

most irrational number:

smaller a_i $\Rightarrow$ slower convergence

$\Rightarrow$ $a_i \equiv 1$ gives slowest convergence

$$\sigma = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}} = \frac{1}{1+\sigma}$$

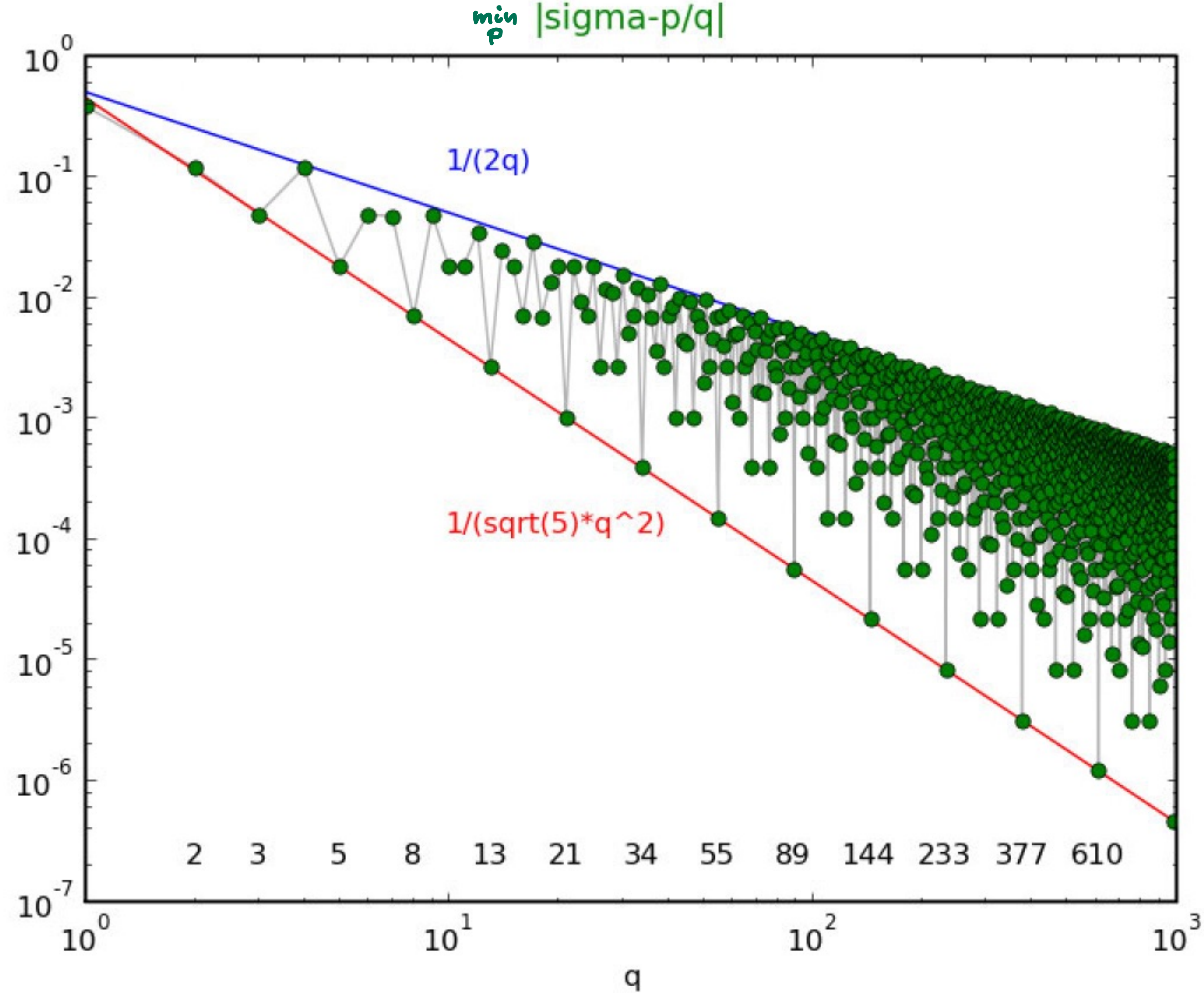
$\Rightarrow$ this irrational number is hardest to approximate by rationals

$$\Rightarrow \sigma(1+\sigma) = 1$$

$\Rightarrow$ 'most irrational'

$$\Rightarrow \sigma = \frac{\sqrt{5}-1}{2} = 0.618\ldots$$

golden mean



min p |sigma-p/q|
1/(2q)
1/(sqrt(5)*q^2)
2 3 5 8 13 21 34 55 89 144 233 377 610
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^0 10^1 10^2 10^3
q

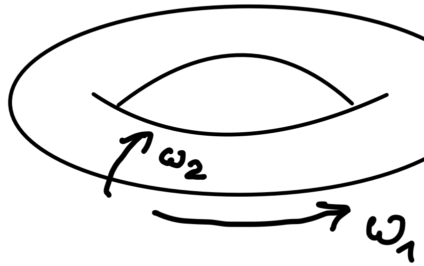
3.5.2. Nonintegrable Perturbation

Let $H(\vec{I},\vec{\varphi}) = H_0(\vec{I}) + \varepsilon H_1(\vec{I},\vec{\varphi})$

with a) H_0 integrable:

$$\dot{\vec{I}} = -\frac{\partial H_0}{\partial \vec{\varphi}} = 0$$

$$\dot{\vec{\varphi}} = \frac{\partial H_0}{\partial \vec{I}} = \vec{\omega}(\vec{I})$$



b) $\varepsilon \ll 1$

Is H integrable?

reasonable answers:

i) Yes, but difficult to find transformation to action-angle coordinates

ii) No, all constants of motion $\vec{I}$ destroyed

correct answer:

KAM theorem

(Kolmogorov 1954, Arnold 1963, Moser 1962)

Majority of tori of H_0 still exist for H (but slightly deformed)

remark: measure of surviving tori $\xrightarrow[\varepsilon \to 0]{}$ full measure

idea of proof:

- choose sufficiently irrational torus of H_0
- search for torus of H with same frequencies

assumptions:

- ε sufficiently small
- H analytic (Moser: 333 times differentiable)
- $\omega_i(\vec{I})$ change independently with $\vec{I}$
- $\dfrac{\omega_i(\vec{I})}{\omega_j(\vec{I})}$ change independently with $\vec{I}$

- Arnold, App. 8
- Tabor
- Arrowsmith/Place

remarks:

1. KAM makes no statement about
$\underbrace{\text{resonant (commensurate, rational)}}_{\exists m_1,\dots,m_n \in \mathbb{Z} \text{ with } \vec{m}\cdot\vec{\omega}=0 \text{ and } \vec{m}\neq 0}$ tori of H_0

How many resonant tori exist?

$N=2$: $\frac{\omega_1(\vec{I})}{\omega_2(\vec{I})}$ changes with $\vec{I}$ $\Rightarrow$ resonant tori are dense, but of zero measure

2. Even $\overbrace{\text{non-resonant (incommensurate, irrational)}}^{\cancel{\exists} m_1,\dots,m_n \in \mathbb{Z} \text{ with } \vec{m}\cdot\vec{\omega}=0 \text{ and } \vec{m}\neq 0}$ tori
show in perturbation theory 'problem of small divisors'

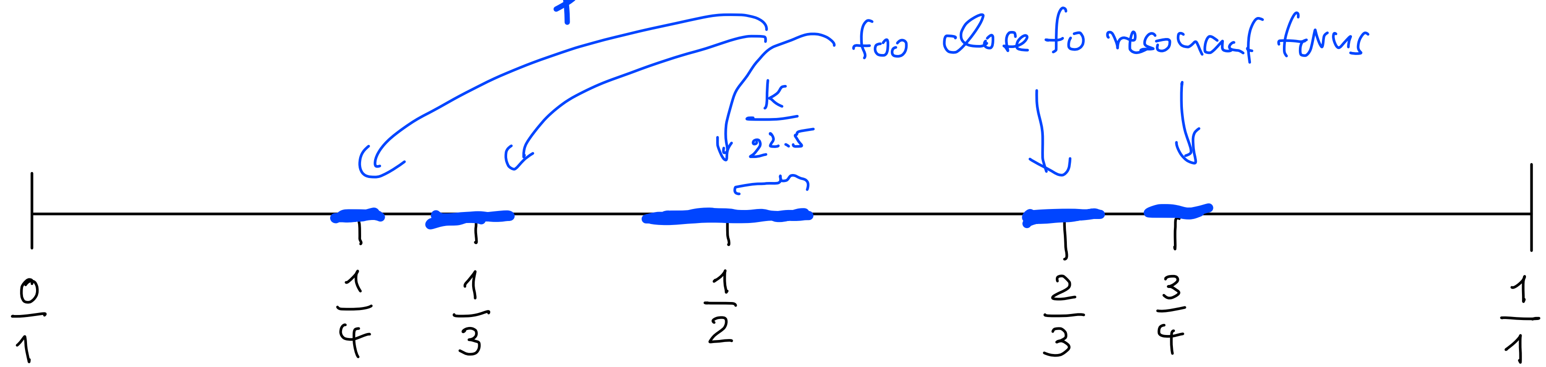
$\vec{m}\cdot\vec{\omega}$ arbitrarily small $\Rightarrow$ convergence?

solution: KAM with superconvergent perturbation theory

3. KAM-theorem is about sufficiently non-resonant tori:

$N=2$: $\forall p,q \quad \left|\frac{\omega_1}{\omega_2} - \frac{p}{q}\right| > \frac{K(\varepsilon)}{q^{2.5}}$ with some $K(\varepsilon)$ with $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$

e.g. in $[0,1]$:



How much is taken out? less than $\sum_{q=1}^{\infty} q \, \frac{2K(\varepsilon)}{q^{2.5}} = 2K(\varepsilon) \underbrace{\sum_{q=1}^{\infty} \frac{1}{q^{1.5}}}_{\text{const}} \xrightarrow[\varepsilon\to 0]{} 0$

$\Rightarrow$ surviving tori have positive measure $\xrightarrow[\varepsilon\to 0]{}$ full measure

$\Rightarrow$ (destroyed tori have positive measure $\xrightarrow[\varepsilon\to 0]{}$ zero measure)

quiz: What is wrong?

Theorem (Arnold and Avez (1968), Theorem 21.7). If H_1 is small enough, then for almost all $\boldsymbol{\omega}^*$ there exists an invariant torus $T(\boldsymbol{\omega}^*)$ of the perturbed system such that $T(\boldsymbol{\omega}^*)$ is "close to" $T_0(\boldsymbol{\omega}^*)$.

4. Which tori survive for increasing perturbation ε?

tori with most irrational $\frac{\omega_1}{\omega_2}$

standard map: $\frac{\omega_1}{\omega_2} \in [-\frac{1}{2}, \frac{1}{2}]$ most irrational: $(1-\sigma) = 0.381966\ldots = [0, 2, 1, 1, 1, \ldots]$