3.6. Poincaré - Birkhoff theorem

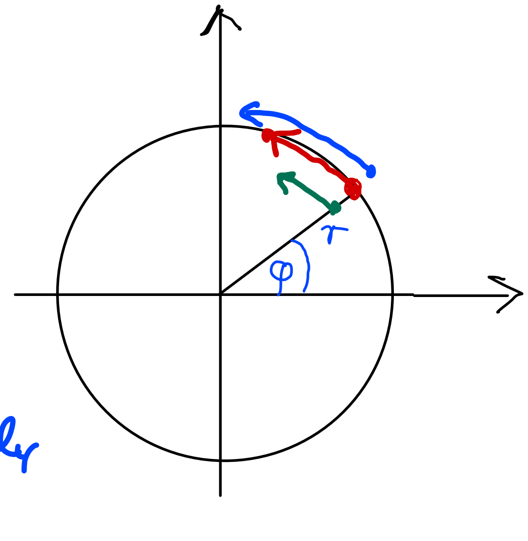
Motivation: What happens to resonant tori?

Def.: Twist map M_0 :

$$r_{n+1} = r_n$$

$$\varphi_{n+1} = \varphi_n + 2\pi R(r_n) \mod 2\pi$$

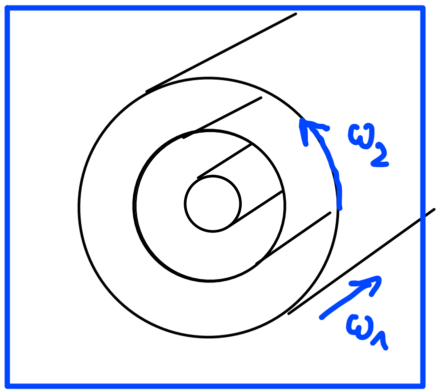
$\frac{dR}{dr} > 0$ monotonously increasing



properties:

- area preserving: $\det \begin{pmatrix} 1 & 0 \\ 2\pi R' & 1 \end{pmatrix} = 1$ ✓

- corresponds qualitatively to Poincaré section of torus of $N=2$ integrable system



$$\frac{\omega_2(\vec{I})}{\omega_1(\vec{I})} \hat{=} R(r_n)$$

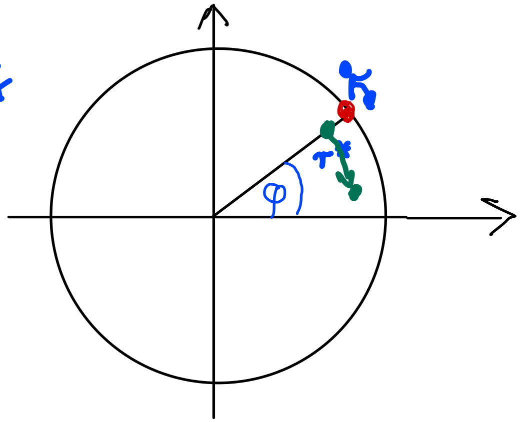
- resonant tori: $R(r^*) = \frac{p}{q} \in \mathbb{Q}$

All points on torus r^* are fixed points of M_0^q.

proof: $r_{n+q} = r_n = r^*$

$$\varphi_{n+q} = \varphi_n + q \cdot 2\pi \underbrace{R(r^*)}_{p/q} = \varphi_n \mod 2\pi \quad \checkmark \quad \forall \varphi_0$$

M_0^q



- $r = r^* + \delta$: M_0^q anti clockwise
- $r = r^* - \delta$: M_0^q clockwise

Perturbed twist map M:
$$r_{n+1} = r_n + \varepsilon f(r_n, \varphi_n)$$
$$\varphi_{n+1} = \varphi_n + 2\pi R(r_n) + \varepsilon g(r_n, \varphi_n) \mod 2\pi$$

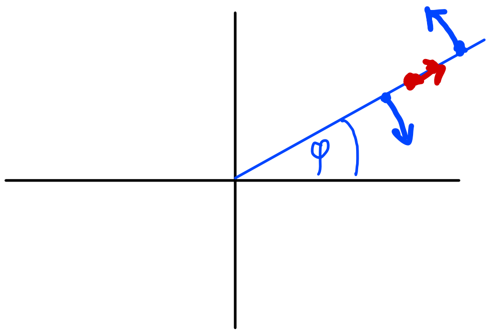
$\varepsilon \ll 1$, f, g s.t. M area preserving

What happens to fixed points of M_0^q under M^q?

Procedure to find fixed points of M^q:

• for which points (r, φ) remains φ unchanged?

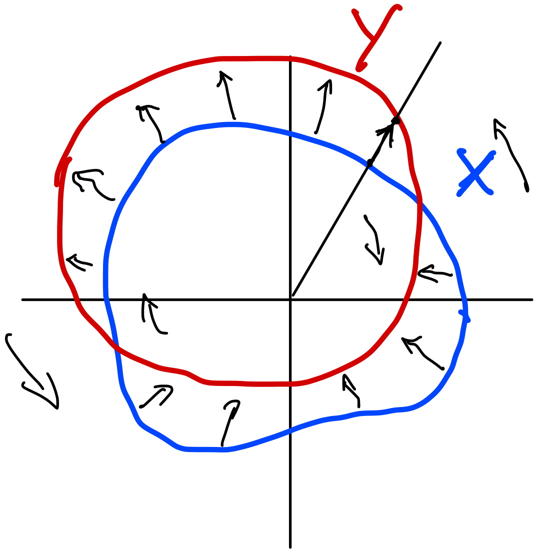
M^q



$\forall \varphi \ \exists\, r(\varphi)$ with $M^q(r,\varphi) = (r', \varphi)$

$\Rightarrow$ curve X

M^q



• curve $Y := M^q X$

• intersections of X and Y are fixed points x_f

$x_f \in X \Rightarrow \varphi$ unchanged

$x_f \in Y \Rightarrow r$ unchanged

• area conservation, $\varepsilon \ll 1$

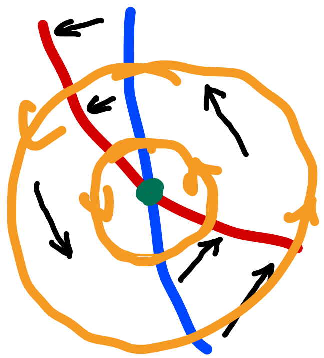
$\Rightarrow \exists$ intersections

• How many? $2k$ $(k = 1, 2, 3, ..)$ fixed points (unless tangency)

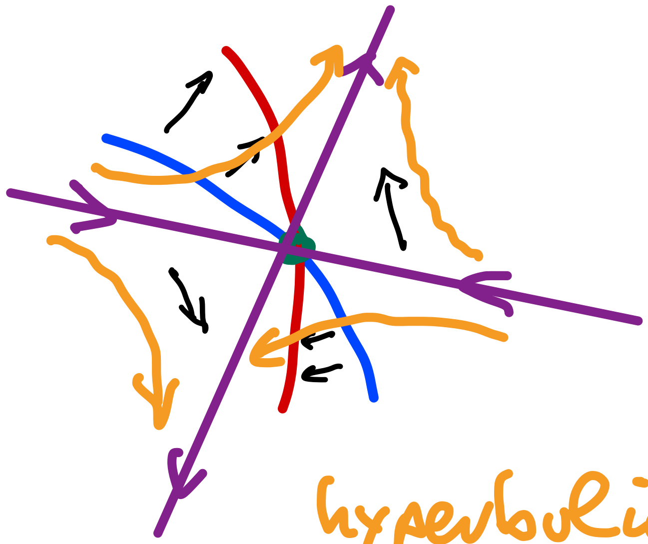
$M^q(M^n x_f) = M^n M^q x_f = M^n x_f$ ✓

• $2kq$ fixed points, as $M^1, M^2, \dots, M^{q-1}$ gives new fixed point

• fixed points are alternating elliptic and hyperbolic (Poincaré-Birkhoff 1927)

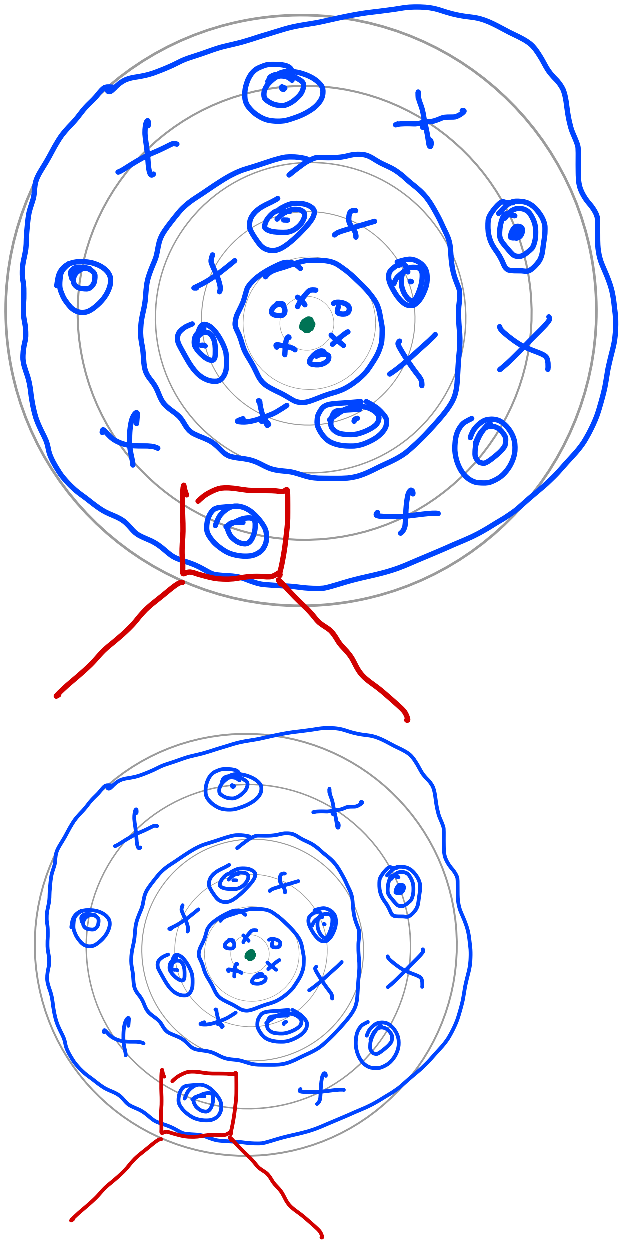


elliptic



hyperbolic

Generic phase space of integrable system + small perturbation



- KAM: most irrational tori still exist
- P.-B.: all rational tori break up
  into chains of elliptic + hyperbolic f.p.
- very different from linearized system
  (Hartman-Grobman theorem does not apply)

magnification:

- neighborhood of elliptic f.p. of M^q

  $\hat{=}$ integrable system + small perturbation

$\Rightarrow$ self-similarity on all scales!