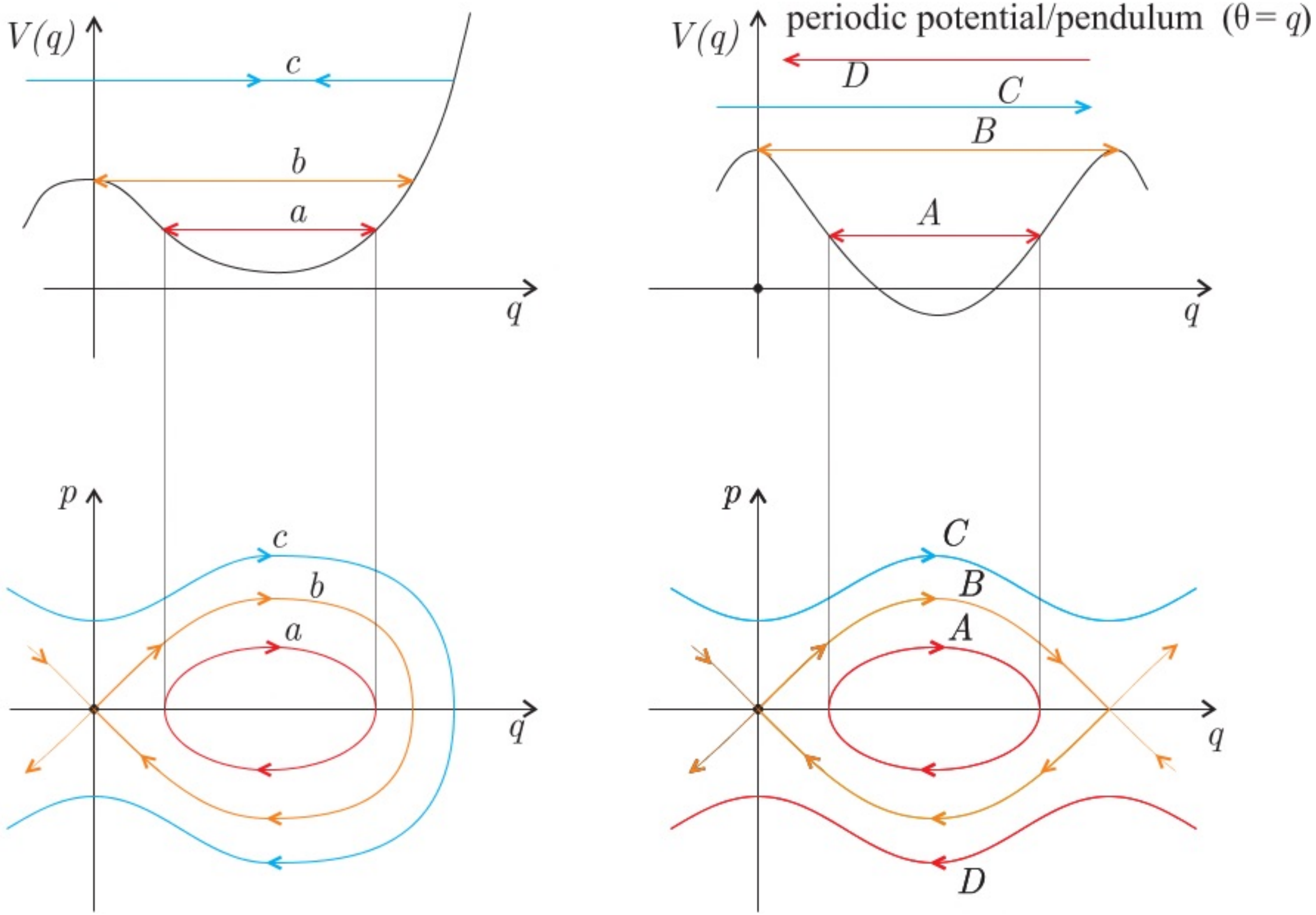
4 Chaotic dynamics

4. Chaotic dynamics

4.1. Where does it emerge?

Start from integrable example : $H = \frac{p^2}{2} + V(q)$

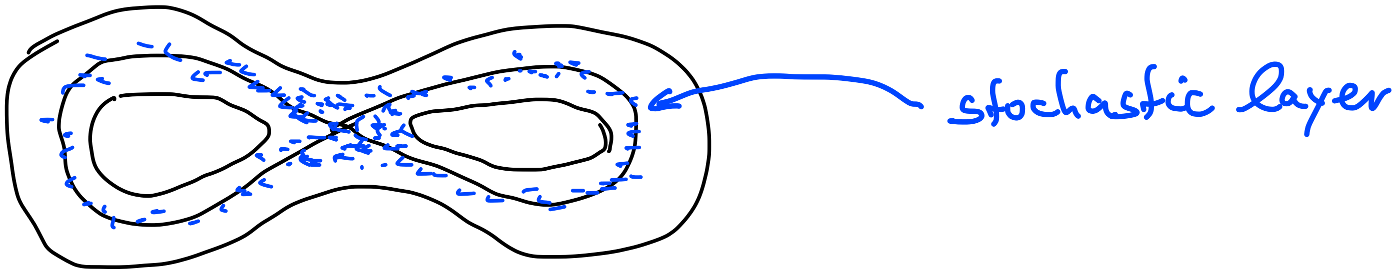


Add non-integrable perturbation:

- second d.o.f.
- time-periodic driving: $\varepsilon H_1(q,t)$

Where is strongest influence on integrable dynamics?

- close to separatrix:
  - at hyp. fixed point: type of motion can change
  - integrable motion has very long periods
    $\Rightarrow$ perturbations are uncorrelated $\Rightarrow$ „irregular" dynamics



stroboscopic Poincaré section

- far away from separatrix:
  - most tori survive (KAM)

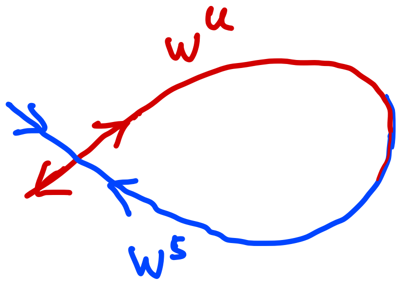
Which condition of KAM thm. is not fulfilled at separatrix?

4.2. Homoclinic tangle

How do stable and unstable invariant manifolds of hyp. fixed point connect?

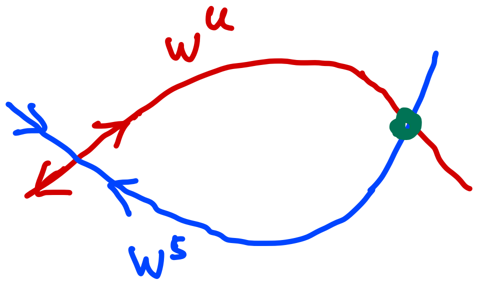
2D Poincaré map:

a) coincide



still separatrix
$\Rightarrow$ no chaotic dynamics

b) intersect



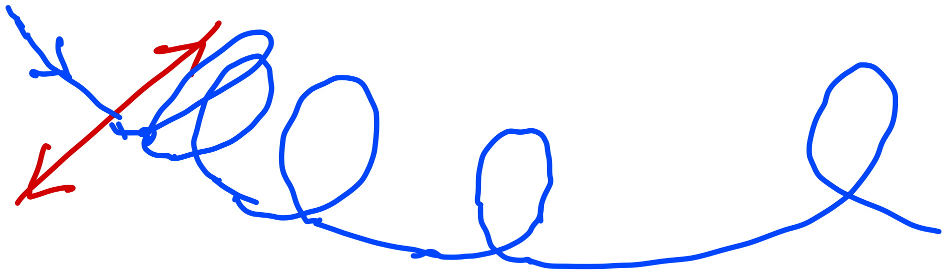
$\exists$ transverse homoclinic point

$\Uparrow$ Melnikov method

$\Downarrow$ chaos

W^s and W^u cannot intersect themselves

show for W^s by contradiction:



Assume one self-intersection

$\Downarrow$

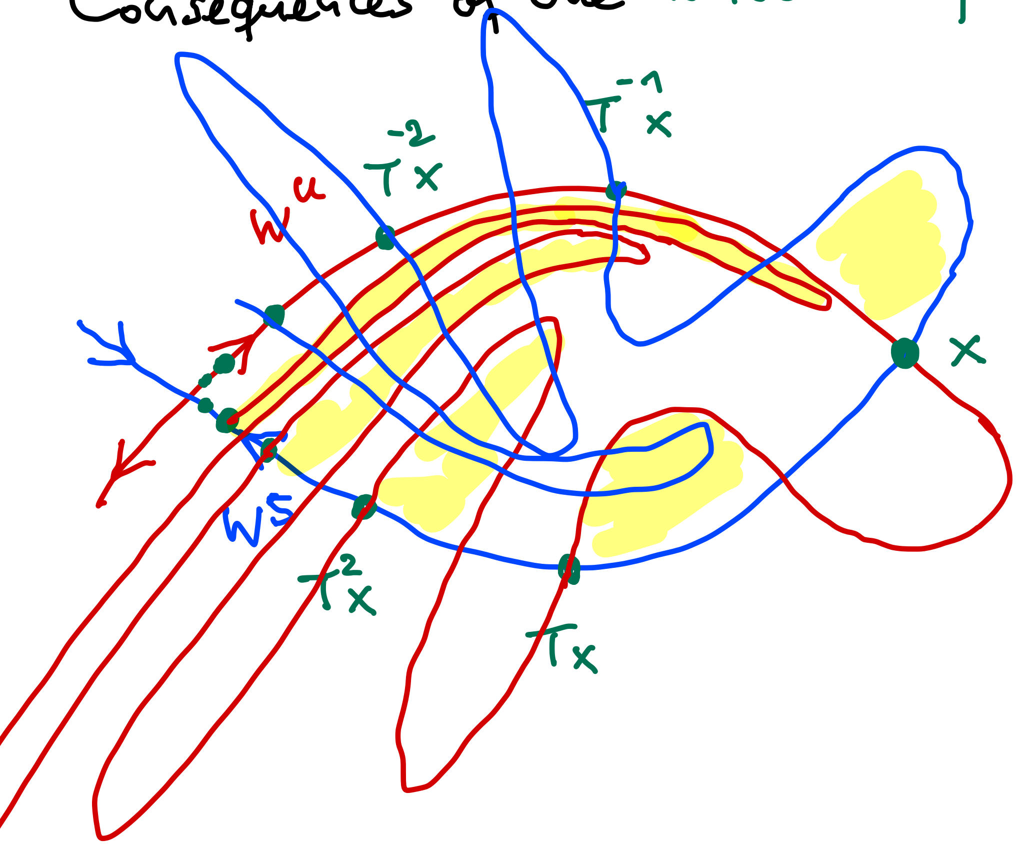
further self-intersections closer to hyp. f.p.

$\Downarrow$

contradiction: invariant manifold thm.

m.f. tangential for lin. and nonlin. system

Consequences of one homoclinic point:



homoclinic tangle

1. all images and pre-images are homoclinic p.
2. h.p. converge exponentially towards hyp. f.p.
3. h.p. are not periodic $\Rightarrow$ ∞-many
4. manifold between two h.p. is mapped to line between images of h.p. (orientation is preserved if $\det J = +1$)
5. Loop areas are conserved
   W^u must not intersect itself
6. distance between $T^n x$ and $T^{n+1} x$ decreases exponentially
   $\Rightarrow$ length of loop increases exponentially
   $\Rightarrow$ sensitive dependence on initial conditions for two points in loop
7. W^s for $t \to -\infty$ shows same behavior

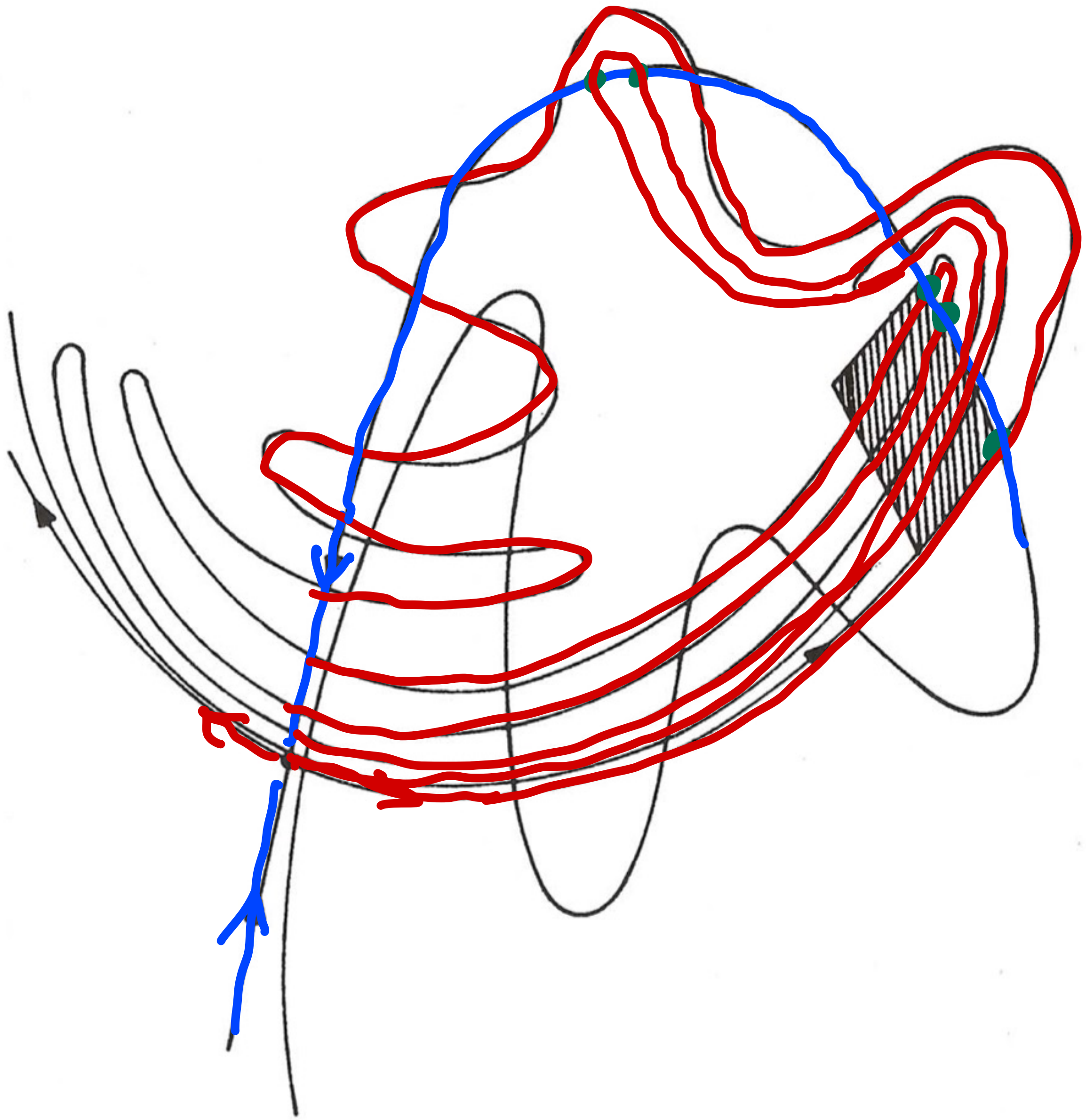


Fig. 3.6. The stable manifold has multiple intersections with the unstable manifold.

4.3. Melnikov method

aim: proof of existence of one homoclinic point

(⇒ homoclinic tangle, see 4.2.)

approach:
- define distance of W^s, W^u = Melnikov function
- simple zero ⇔ transverse homoclinic point
- approximate Melnikov function with dynamics of integrable system